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A NEW CLASS OF GENERALIZED POLYNOMIALS ASSOCIATED WITH MILNE-THOMSON-BASED POLY-BERNOULLI POLYNOMIALS

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Abstract. Motivated by their importance and potential for applications in certain problems in number theory, combinatorics, classical and numerical analysis, and other field of applied mathematics, a variety of polynomials and numbers with their variants and extensions have recently been introduced and investigated. In this sequel, we modify the known generating functions of polynomials, due to both Milne-Thomson and Dere and Simsek, to introduce a new class of generalized polynomials and present some of their involved properties. As obvious special cases of the newly introduced polynomials, we also called power sum-Laguerre-Hermite polynomials and generalized Laguerre and poly-Bernoulli polynomials and present some of their involved identities and formulas. The results presented here, being very general, are pointed out to be specialized to yield a number of known and new identities involving relatively simple and familiar polynomials.

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1. INTRODUCTION

Throughout this presentation, we use the following standard notions $\mathbb{N} = \{1,2,3,\ldots\}$, $\mathbb{N}_0 = \{0,1,2,\ldots\} = \mathrm{N} \cup \{0\}$, $\mathbb{Z}^- = \{-1,-2,\ldots\}$. Also as usual $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers and $\mathbb{C}$ denotes the set of complex numbers.

The two variable Laguerre polynomials (2-VLP) $L_n(x;y)$ are defined by the following generating function [1,5]:

$$\frac{1}{1-yt}\exp\left(\frac{-xt}{1-yt}\right) = \sum_{n=0}^{\infty} L_n(x;y)\frac{t^n}{n!} \qquad (\mid yt \mid < 1).$$

Also, equivalently, the polynomials $L_n(x,y)$ are given by [4]

$$e^{yt}C_0(xt)=\sum_{n=0}^{\infty}L_n(x;y)\frac{t^n}{n!}, \tag{1.1}$$

where $C_0(x)$ denotes the 0^{th} order Tricomi function. The n^{th} order Tricomi functions $C_n(x)$ are defined by [4]

$$\exp\left(t-\frac{x}{t}\right)=\sum_{n=0}^{\infty}C_n(x)t^n \qquad (t\in\mathbb{C}\setminus\{0\},\ x\in\mathbb{C}).$$

We have

$$C_n(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^r}{r!(n+r)!}. \tag{1.2}$$

The Tricomi functions $C_n(x)$ are associated with the Bessel function of the first kind $J_n(x)$ [3, 5]

$$C_n(x)=x^{-\frac{n}{2}}J_n(2\sqrt{x}).$$

From (1.1) and (1.2), we note that

$$L_n(x,y)=n!\sum_{k=0}^{n}\frac{(-1)^k y^{n-k}x^k}{(k!)^2(n-k)!}=y^nL_n(x,y), \tag{1.3}$$

where $L_n(x)$ are the ordinary Laguerre polynomials [25]. We have

$$L_n(x,0)=\frac{(-1)^n x^n}{n!}, \qquad L_n(0,y)=y^n \qquad L_n(x,1)=L_n(x).$$

For $s\in\mathbb{Z}$, the polylogarithm function is defined by a power series in z as

$$\mathrm{Li}_s(z)=\sum_{j=1}^{\infty}\frac{j^n}{j^s}=z+\frac{z^2}{2^s}+\frac{z^3}{3^s}, \qquad (|\,z\,|<1).$$

It is notice that

$$\mathrm{Li}_1(z)=\sum_{j=1}^{\infty}\frac{z^j}{j}=-\log(1-z).$$

In (2015), Jolany *et al.* [7] introduced the generalized poly-Bernoulli polynomials are defined by means of the following generating function:

$$\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{xt}=\sum_{n=0}^{\infty}B_n^{(k)}(x;a,b)\frac{t^n}{n!}; \qquad |\,t\,|<\frac{2\pi}{|\ln a+\ln b\,|}.$$

In the case when $x=0$, $B_n^{(k)}(a,b)=B_n^{(k)}(0;a,b)$ are called the generalized poly-Bernoulli numbers [8].

Milne-Thomson [27] defined polynomials $\Phi_n^{(\alpha)}(x)$ of degree n and order α by the following generating function:

$$f(t,\alpha)e^{xt+g(t)}=\sum_{n=0}^{\infty}\Phi_n^{(\alpha)}(x)\frac{t^n}{n!},$$

where $f(t,\alpha)$ is a function of t and $\alpha \in \mathbb{Z}$ and $g(t)$ is a function of t. Then, by choosing some explicit functions of $f(t,\alpha)$ and $g(t)$, he [27] presented several interesting properties for certain polynomials such as Bernoulli polynomials and Hermite polynomials.

Dere and Simsek [6] made a slight modification of the Milne-Thomson polynomials $\Phi_n^{(\alpha)}(x)$ to give polynomials $\Phi_n^{(\alpha)}(x,\nu)$ of degree n and order α by means of the following generating function:

$$G_1(t,x;\alpha,\nu) = f(t,\alpha)e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} \Phi_n^{(\alpha)}(x;\nu)\frac{t^n}{n!}, \tag{1.4}$$

where $f(t,\alpha)$ and $h(t,\nu)$ are functions of t and $\alpha \in \mathbb{Z}$ and t and $\nu \in \mathbb{N}_0$, respectively, which are analytic in a neighborhood of $t = 0$.

Observe that $\Phi_n^{(\alpha)}(\nu) = \Phi_n^{(\alpha)}(0;\nu)$, (see for details [26,27]). In particular, choosing $f(t,\alpha) = \left(\frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{1-e^{-t}}\right)^{\alpha}$ in (1.4), we obtain the following following polynomials given by the generating function:

$$G_2(t,x;k,\nu) = \frac{\mathrm{Li}_k(1-\mathrm{e}^{-t})}{1-e^{-t}}e^{xt+h(t,\nu)} = \sum_{n=0}^{\infty} B_n^{(k)}(x;\nu)\frac{t^n}{n!}, \tag{1.5}$$

Note that the polynomials $B_n^{(k)}(x;\nu)$ are related to both Bernoulli polynomials and Hermite polynomials. For example, if $h(t,0) = 0$ in (1.5), we get

$$B_n^{(k)}(x;0) = B_n^{(k)}(x),$$

where $B_n^{(k)}(x)$ are called the poly-Bernoulli polynomials [2].

The two-variable Hermite-Kampé de Fériet polynomials $H_n(x,y)$ [2,3] are generated by

$$e^{xt+yt^2} = \sum_{n=0}^{\infty} H_n(x,y)\frac{t^n}{n!}. \tag{1.6}$$

Note that

$$H_n(x,y) = \sum_{r=0}^{\left[\frac{n}{2}\right]} n!\frac{y^r x^{n-2r}}{r!(n-2r)!} \qquad \text{(see [9, 10, 14, 16])},$$

It is clear that

$$H_n(2x,-1) = H_n(x), \qquad H_n(x,0) = x^n.$$

Dere and Simsek [6] generalized the polynomials $H_n(x,y)$ in (1.6) to define two variable Hermite polynomials $H_n^{(l)}(x,y)$ by the following generating function:

$$e^{xt+yt^l} = \sum_{n=0}^{\infty} H_n^{(l)}(x,y)\frac{t^n}{n!}; \qquad (l \in \mathbb{N}\setminus\{0\}).$$

Taking $h(t,y) = yt^2$ in (1.5), we get the generalized Hermite poly-Bernoulli polynomials of two variables ${}_HB^{(k)}(x,y)$ introduced by Pathan and Khan [21]:

$$\frac{\mathrm{Li_k}(1-\mathrm{e^{-t}})}{1-e^{-t}}e^{xt+yt^2)} = \sum_{n=0}^{\infty} {}_HB_n^{(k)}(x,y)\frac{t^n}{n!}. \tag{1.7}$$

In this paper, we introduce generalized Laguerre-based poly-Bernoulli polynomials by using Milne-Thomson polynomials and investigate some of their properties such as explicit summation formulas, addition formulas, implicit formula and symmetry identities.

2. Milne-Thomson based poly-Bernoulli polynomials

In this section, we introduce generalized Laguerre-based poly-Bernoulli polynomials by using Milne-Thomson polynomials. By replacing x by y and ν by z in (1.4) and $f(t,\alpha) = \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}}C_0(xt)$. First, we start with the following definition as.

Definition 1. Let $a,b \in \mathbb{R}^+$ with $a \neq b$. The generalized poly-Bernoulli polynomials $B_n^{(k)}(x,y,z;a;b,e)$ are defined by the following generating function:

$$\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}}e^{yt+h(t,z)}C_0(xt) = \sum_{n=0}^{\infty} B_n^{(k)}(x,y,z;a,b,e)\frac{t^n}{n!},$$
$$\left(x,y,z \in \mathbb{R};\ |\, t \,| < \frac{2\pi}{|\ln a + \ln b\,|}\right). \tag{2.1}$$

In particular, taking $h(t,z) = zt^2$ in (2.1), we get the following.

Definition 2. Let $a,b > 0,\ a \neq b$. The generalized Laguerre-based poly-Bernoulli polynomials ${}_LB_n^{(k)}(x,y,z;a,b)$ are defined by

$$\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}}\exp(yt+zt^2)C_0(xt) = \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y,z;a,b)\frac{t^n}{n!},$$
$$\left(x,y,z \in \mathbb{R};\ |\, t \,| < \frac{2\pi}{|\ln a + \ln b\,|}\right). \tag{2.2}$$

Remark 1. On setting $z = 0$ in (2.2) reduces to the generalized Laguerre-based poly-Bernoulli polynomials in two variables defined as

$$\frac{\mathrm{Li_k}(1+(ab)^{-t})}{b^t - a^{-t}}e^{yt}C_0(xt) = \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y;a,b)\frac{t^n}{n!}, \qquad \left(|\, t \,| < |\, \frac{2\pi}{(\ln a + \ln b)\,|}\right).$$

Remark 2. Taking $k = 1,\ a = 1,\ b = e$ and $z = 0$ in (2.2) reduces to the known Laguerre Bernoulli polynomials defined as

$$\left(\frac{t}{e^t - 1}\right)e^{yt}C_0(xt) = \sum_{n=0}^{\infty} {}_LB_n(x,y)\frac{t^n}{n!}; \qquad (|\, t \,| < |\, 2\pi).$$

Remark 3. Letting $x = 0$ in (2.2) reduces to the known generalized Hermite poly-Bernoulli polynomials defined by [19]

$$\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}} e^{yt+zt^2} = \sum_{n=0}^{\infty} {}_HB_n^{(k)}(y,z;a,b)\frac{t^n}{n!}, \qquad \left(|\, t \,|<|\, \frac{2\pi}{(\ln a + \ln b)\,|}\right).$$

By using generalized Laguerre-based poly-Bernoulli polynomials ${}_LB_n^{(k)}(x,y,z;a,b)$ defined by (2.2), we have the following properties which are stated as theorems below.

Theorem 1. *For $n \geq 0$, we have*

$$ {}_LB_n^{(k)}(x,y,z;a,b) = \sum_{m=0}^{n} \binom{n}{m} {}_HB_m^{(k)}(0,z;a,b) L_{n-m}(x,y). \tag{2.3}$$

Proof. Using (1.3), (1.7) and (2.2), we have

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_LH^{B_n^{(k)}}(x,y,z;a,b)\frac{t^n}{n!} &= \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}} \exp(yt+zt^2)C_0(xt) \\
&= \sum_{m=0}^{\infty} {}_HB_m^{(k)}(0,z;a,b)\frac{t^m}{m!} \sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \binom{n}{m} {}_HB_m^{(k)}(0,z;a,b) L_{n-m}(x,y) \right) \frac{t^n}{n!}.
\end{aligned}$$

By comparing the coefficients of t^n in the above equation, we obtain the result (2.3). □

Theorem 2. *For $n \geq 0$, we have*

$$ {}_LB_n^{(k)}(x,y+u,z+v;a,b) = \sum_{m=0}^{n} \binom{n}{m} {}_LB_{n-m}^{(k)}(x,y,z;a,b) H_m(u,v). \tag{2.4}$$

$$ {}_LB_n^{(k)}(x,y+u,z;a,b) = \sum_{m=0}^{n} \binom{n}{m} {}_LB_{n-m}^{(k)}(x,y;a,b) H_m(u,z). \tag{2.5}$$

Proof. On replacing y by $y+u$ and z by $z+v$ in (2.2), we get

$$\begin{aligned}
\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y+u,z+v;a,b)\frac{t^n}{n!} &= \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t - a^{-t}} \exp(yt+zt^2)C_0(xt)\exp\left(ut+vt^2\right) \\
&= \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y,z;a,b)\frac{t^n}{n!} \sum_{m=0}^{\infty} H_m(u,v)\frac{t^m}{m!},
\end{aligned}$$

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y+u,z+v;a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty} \left[\sum_{m=0}^{n} \binom{n}{m} {}_LB_{n-m}^{(k)}(x,y,z;a,b) H_m(u,v) \right] \frac{t^n}{n!}.$$

Now equating the coefficients of the like powers of t in the above equation, we get the result (2.4).

Again, from (2.2), we have

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y+u,z;a,b)\frac{t^n}{n!} = \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}\exp(yt+zt^2)C_0(xt)\exp(ut)$$

$$= \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{yt}C_0(xt)\exp(ut+zt^2)$$

$$= \sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y;a,b)\frac{t^n}{n!}\sum_{m=0}^{\infty} H_m(u,z)\frac{t^m}{m!}.$$

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,y+u,z;a,b)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left[\sum_{m=0}^{n}\binom{n}{m} {}_LB_{n-m}^{(k)}(x,y;a,b)H_m(u,z)\right]\frac{t^n}{n!}. \tag{2.6}$$

On comparing the coefficients of equal powers of t^n in (2.6), we acquire the desired result (2.5). □

Theorem 3. *Let $p,q>0$ and $n\geq 0$, we have*

$$ {}_LB_n^{(k)}(x,py,qz;a,b) = \sum_{m=0}^{n}\binom{n}{m} {}_LB_{n-m}^{(k)}(x,(p-1)y,(q-1)z;a,b)H_m(y,z). \tag{2.7}$$

Proof. Rewriting the generating function (2.2), we have

$$\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,py,qz;a,b)\frac{t^n}{n!} = \frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{yt+zt^2}e^{(p-1)yt+(q-1)zt^2}C_0(xt)$$

$$= \left(\sum_{n=0}^{\infty} {}_LB_n^{(k)}(x,(p-1)y,(q-1)z;a,b)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} H_m(y,z)\frac{t^m}{m!}\right) \tag{2.8}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m} {}_LB_{n-m}^{(k)}(x,(p-1)y,(q-1)z;a,b)H_m(y,z)\right)\frac{t^n}{n!}.$$

On comparing the coefficients of $\frac{t^n}{n!}$ in (2.8), we obtain at the desired result (2.7). □

3. Summation formulae for generalized Laguerre-based poly-Bernoulli polynomials.

In this section, we derive some implicit summation formulae for generalized Laguerre-based poly-Bernoulli polynomials ${}_LB_n^{(k)}(x,y,z;a,b)$. We start with the following theorem as.

Theorem 4. *The following relation holds true:*

$$ {}_LB_{l+p}^{(k)}(x,v,z;a,b) = \sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(v-y)^{m+n} {}_LB_{l+p-m-n}^{(k)}(x,y,z;a,b). \tag{3.1}$$

Proof. Replace t by $t+u$ in (2.2) and rewrite the generating function (2.2), we have

$$\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-(\mathrm{t+u})})}{b^t-a^{-(t+u)}}\exp(z(t+u)^2)C_0(x(t+u))$$
$$=\exp(-y(t+u))\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,y,z;a,b)\frac{t^lu^p}{l!p!}. \quad (3.2)$$

Upon setting y by v in (3.2), it is not difficult to show that

$$\exp((v-y)(t+u))\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,y,z;a,b)\frac{t^lu^p}{l!p!}=\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,v,z;a,b)\frac{t^lu^p}{l!p!}. \quad (3.3)$$

Expanding the exponential part from the above equation (3.3) with the help of result [19], we get

$$\begin{aligned}&\sum_{N=0}^{\infty}\frac{[(v-y)(t+u)]^N}{N!}\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,y,z;a,b)\frac{t^lu^p}{l!p!}\\&=\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,v,z;a,b)\frac{t^lu^p}{l!p!}.\\&\sum_{m,n=0}^{\infty}\frac{(v-y)^{m+n}t^mu^n}{m!n!}\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,y,z;a,b)\frac{t^lu^p}{l!p!}\\&=\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(x,v,z;a,b)\frac{t^lu^p}{l!p!}.\end{aligned} \quad (3.4)$$

On replacing l with $l-m$ and p with $p-n$ in L.H.S. of (3.4), yields

$$\begin{aligned}&\sum_{l,p=0}^{\infty}\sum_{m,n=0}^{l,p}\frac{(v-y)^{m+n}}{m!n!}{}_LB_{l+p-m-n}^{(k)}(x,y,z;a,b)\frac{t^lu^p}{(l-m)!(p-n)!}\\&=\sum_{l,p=0}^{\infty}{}_LB_{l+p}^{(k)}(v,x,z;a,b)\frac{t^lu^p}{l!p!}.\end{aligned} \quad (3.5)$$

On comparing the coefficients of equal powers of t and u form (3.5), we obtain the result (3.1). □

Remark 4. For $l=0$ in (3.1) reduces to

$${}_LB_p^{(k)}(x,v,z;a,b)=\sum_{n=0}^{p}\binom{p}{n}(v-y)^n{}_LB_{p-n}^{(k)}(x,y,z;a,b).$$

Remark 5. On taking $v=0$ in (3.1), we get

$${}_LB_{l+p}^{(k)}(x,z;a,b)=\sum_{m,n=0}^{l,p}\binom{l}{m}\binom{p}{n}(-y)^{m+n}{}_LB_{l+p-m-n}^{(k)}(x,y,z;a,b).$$

Theorem 5. *The following relation holds true:*

$$ {}_LB_n^{(k)}(x,y+1,z;a,b)=\sum_{m=0}^{n-2j}\sum_{j=0}^{[\frac{n}{2}]}\binom{n-2j}{m}{}_LB_{n-2j-m}^{(k)}(x,y;a,b)\frac{n!}{j!(n-2j)!}z^j. \quad (3.6)$$

Proof. On replacing y by $y+1$, (2.2) converts as

$$\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y+1,z;a,b)\frac{t^n}{n!}=\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{(y+1)t+zt^2}C_0(xt).$$

$$\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y+1,z;a,b)\frac{t^n}{n!}=\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{yt}C_0(xt)e^{t+zt^2}$$

$$=\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y;a,b)\frac{t^n}{n!}\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{j=0}^{\infty}z^j\frac{t^{2j}}{j!}.$$

$$\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y+1,z;a,b)\frac{t^n}{n!}=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}{}_LB_{n-m}^{(k)}(x,y;a,b)\frac{t^n}{n!}\sum_{j=0}^{\infty}z^j\frac{t^{2j}}{j!}.$$

$$L.H.S=\sum_{n=0}^{\infty}\left[\sum_{j=0}^{[\frac{n}{2}]}\sum_{m=0}^{n-2j}\binom{n-2j}{m}z^j{}_LB_{n-2j-m}^{(k)}(y;a,b)\frac{n!}{j!(n-2j)!}\right]\frac{t^n}{n!}.$$

On comparing of the coefficients of $\frac{t^n}{n!}$, we obtain at the desired result (3.6). □

Theorem 6. *The following relation holds true:*

$$ {}_LB_n^{(k)}(x,y+1,z;a,b)\frac{t^n}{n!}=\sum_{m=0}^{n}\binom{n}{m}{}_LB_m^{(k)}(x,y,z;a,b). \quad (3.7)$$

Proof. We start with the following relation

$$\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y+1,z;a,b)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y,z;a,b)\frac{t^n}{n!}$$

$$=\frac{\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}})}{b^t-a^{-t}}e^{yt+zt^2}C_0(xt)(e^t-1)$$

$$=\sum_{m=0}^{\infty}{}_LB_m^{(k)}(x,y,z;a,b)\frac{t^m}{m!}\sum_{n=0}^{\infty}\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y,z;a,b)\frac{t^n}{n!}$$

$$=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}{}_LB_m^{(k)}(x,y,z;a,b)\frac{t^n}{n!}-\sum_{n=0}^{\infty}{}_LB_n^{(k)}(x,y,z;a,b)\frac{t^n}{n!}.$$

On comparing the coefficients of equal powers of t^n produce the result (3.7). □

4. SYMMETRY IDENTITIES FOR THE GENERALIZED LAGUERRE-BASED POLY-BERNOULLI POLYNOMIALS

Recently, Khan *et al.* [11–13, 15, 17, 18] and Pathan and Khan [19–24] have established some interesting symmetry identities for various polynomials. Here, we present certain symmetry identities for the generalized Laguerre poly-Bernoulli polynomials ${}_LB_n^{(k)}(x,y,z;a,b)$ in the following form.

Theorem 7. *The following symmetry identities hold true:*

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}{}_LB_{n-m}^{(k)}(bx_1,by,b^2z;A,B){}_LB_m^{(k)}(ax_2,ay,a^2z;A,B)$$
$$=\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^{m}{}_LB_{n-m}^{(k)}(ax_1,ay,a^2z;A,B){}_LB_m^{(k)}(bx_2,by,b^2z;A,B). \quad (4.1)$$

Proof. Consider the following function

$$A(t)=\left[\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))^2}{(B^{at}-A^{-at})(B^{bt}-A^{-bt})}\right]e^{abyt+a^2b^2zt^2}C_0(abx_1t)C_0(abx_2t).$$

We see that the function $A(t)$ is symmetric in a and b and we can expand $A(t)$ into series in two ways to obtain

$$A(t)=\left(\sum_{n=0}^{\infty}{}_LB_n^{(k)}(bx_1,by,b^2z;A,B)\frac{(at)^n}{n!}\right)\left(\sum_{m=0}^{\infty}{}_LB_m^{(k)}(ax_2,ay,a^2z;A,B)\frac{(bt)^m}{m!}\right).$$

$$A(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}a^{n-m}b^{m}{}_LB_{n-m}^{(k)}(bx_1,by,b^2z;A,B)\right. \quad (4.2)$$
$$\left.\times{}_LB_m^{(k)}(ax_2,ay,a^2z;A,B)\right)\frac{t^n}{n!}.$$

Similarly, we have

$$A(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}b^{n-m}a^{m}{}_LB_{n-m}^{(k)}(ax_1,ay,a^2z;A,B)\right.$$
$$\left.\times{}_LB_m^{(k)}(bx_2,by,b^2z;A,B)\right)\frac{t^n}{n!}. \quad (4.3)$$

On comparing the coefficients of t^n in (4.2) and (4.3), we obtain the result (4.1). □

Remark 6. If $b=1$, Theorem 7 reduces to

$$\sum_{m=0}^{n}\binom{n}{m}a^{n-m}{}_LB_{n-m}^{(k)}(x_1,y,z;A,B){}_LB_m^{(k)}(ax_2,ay,a^2z;A,B)$$
$$=\sum_{m=0}^{n}\binom{n}{m}a^{m}{}_LB_{n-m}^{(k)}(ax_1,ay,a^2z;A,B){}_LB_m^{(k)}(x_2,y,z;A,B).$$

Theorem 8. *The following symmetry identities hold true:*

$$\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}b^m a^{n-m}{}_LB_n^{(k)}(bx_1,by+\frac{b}{a}i+j,b^2z;A,B){}_LB_m^{(k)}(ax_2,av;A,B)$$
$$=\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}a^m b^{n-m}{}_LB_n^{(k)}(ax_1,ay+\frac{a}{b}i+j,a^2z;A,B)\times{}_LB_m^{(k)}(bx_2,bv;A,B). \tag{4.4}$$

Proof. Let

$$B(t)=\left[\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))^2}{(B^{at}-A^{-at})(B^{bt}-A^{-bt})}\right]\frac{(e^{abt}-1)^2e^{ab(y+v)t+a^2b^2zt^2}}{(e^{at}-1)(e^{bt}-1)}C_0(abx_1t)C_0(abx_2t)$$
$$=\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))}{(B^{at}-A^{-at})}e^{abyt+a^2b^2zt^2}C_0(abx_1t)\left(\frac{(e^{abt}-1)}{(e^{bt}-1)}\right)\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))}{(B^{bt}-A^{-bt})}\times e^{abvt}C_0(abx_2t)\left(\frac{(e^{abt}-1)}{(e^{at}-1)}\right)$$
$$=\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))}{(B^{at}-A^{-at})}e^{abyt+a^2b^2zt^2}C_0(abx_1t)\times\sum_{i=0}^{a-1}e^{bti}\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))}{(B^{bt}-A^{-bt})}e^{abvt}C_0(abx_2t)\sum_{j=0}^{b-1}e^{atj}$$
$$=\frac{(\mathrm{Li_k}(1-(\mathrm{ab})^{-\mathrm{t}}))}{(B^{at}-A^{-at})}e^{a^2b^2zt^2}C_0(abx_1t)\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}e^{(by+\frac{b}{a}i+j)at}\times\sum_{m=0}^{\infty}{}_LB_m^{(k)}(ax_2,av;A,B)\frac{(bt)^m}{m!}$$
$$=\sum_{n=0}^{\infty}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}{}_LB_n^{(k)}(bx_1,by+\frac{b}{a}i+j,b^2z;A,B)\frac{(at)^n}{n!}\sum_{m=0}^{\infty}{}_LB_m^{(k)}(ax_2,av;A,B)\frac{(bt)^m}{m!}.$$

$$B(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{a-1}\sum_{j=0}^{b-1}b^m a^{n-m}{}_LB_n^{(k)}(bx_1,by+\frac{b}{a}i+j,b^2z;A,B)\times{}_LB_m^{(k)}(ax_2,av;A,B)\right)\frac{t^n}{n!}. \tag{4.5}$$

On the other hand,

$$B(t)=\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\sum_{i=0}^{b-1}\sum_{j=0}^{a-1}a^m b^{n-m}{}_LB_n^{(k)}(ax_1,ay+\frac{a}{b}i+j,a^2z;A,B)\times{}_LB_m^{(k)}(bx_2,bv;A,B)\right)\frac{t^n}{n!}. \tag{4.6}$$

Comparing the coefficients of t^n in (4.5) and (4.6), we get the result (4.4). □

REFERENCES

[1] A. A. Atash, "On the two-variable generalized Laguerre polynomials." *Open J. Math. Sci.*, vol. 3, no. 1, pp. 152–158, 2019, doi: 10.30538/oms2019.0058.

[2] A. Bayad and Y. Hamahata, "Polylogarithms and poly-Bernoulli polynomials." *Kyushu J. Math.*, vol. 65, no. 11, pp. 15–24, 2011, doi: 10.2206/kyushujm.65.15.

[3] G. Dattoli, "Generalized polynomials, operational identities and their applications." *JCAM*, vol. 118, pp. 111–123, 2000, doi: 10.1016/S0377-0427(00)00283-1.

[4] G. Dattoli and A. Torre, *Theory and applications of generalized Bessel function.* Aracne: Aracne,, 1996.

[5] G. Dattoli, A. Torre, and A. M. Mancho, "The generalized Laguerre polynomials, the associated Bessel functions and applications to propagation problems." *Radiat Phys. Chem.*, vol. 59, no. 3, pp. 229–237, 2000, doi: 10.1016/S0969-806X(00)00273-5.

[6] R. Dere and Y. Simsek, "Bernoulli type polynomials on umbral algebra." *Russ. J. Math. Phy.*, vol. 23, no. 1, pp. 1–6, 2015, doi: 10.48550/arXiv.1110.1484.

[7] H. Jolany and R. B. Corcino, "Explicit formula for generalization of poly-Bernoulli numbers and polynomials with a,b,c parameters." *J. Class. Anal.*, vol. 6, no. 2, pp. 119–135, 2015, doi: 10.48550/arXiv.1109.1387.

[8] M. Kaneko, "Poly-Bernoulli numbers." *J. Théor Nombres Bordeaux*, vol. 9, pp. 221–228, 1997.

[9] W. A. Khan, "Some properties of the generalized Apostol type Hermite-based polynomials." *Kyung. Math. J.*, vol. 55, pp. 597–614, 2015, doi: 10.5666/KMJ.2015.55.3.597.

[10] W. A. Khan, "A note on Hermite-based poly-Euler and multi poly-Euler polynomials," *Palest. J. Math.*, vol. 6, pp. 204–214, 2017.

[11] W. A. Khan, "A new class of higher-order hypergeometric Bernoulli polynomials associated with Hermite polynomials." *BSPM*, vol. 40, pp. 1–14, 2022, doi: 10.5269/bspm.51845.

[12] W. A. Khan, "A note on q-analogue of degenerate Catalan numbers associated p-adic integral on $\mathbb{Z}_p$." *Symmetry*, vol. 14, no. 6, pp. 1–10, 2022, doi: 10.3390/sym14061119.

[13] W. A. Khan, "A note on q-analogues of degenerate Catalan-Daehee numbers and polynomials." *Journal of Mathematics*, 2022, doi: 10.1155/2022/9486880.

[14] W. A. Khan, "On generalized Lagrange-based Apostol type and related polynomials." *Kragujevac J. Math.*, vol. 46, no. 6, pp. 865–882, 2022, doi: 10.46793/KgJMat2206.865K.

[15] W. A. Khan, R. Ali, K. A. H. Alzobydi, and N. Ahmad, "A new family of degenerate poly-Genocchi polynomials with its certain properties." *JFS*, vol. 4, pp. 1–18, 2021, doi: 10.1155/2021/6660517.

[16] W. A. Khan, K. S. Nisar, M. Acikgoz, U. Duran, and H. A. Abusufian, "On unified Gould-Hopper based Apostol-type polynomials." *J. Math. Computer Sci.*, vol. 24, pp. 287–298, 2022, doi: 10.22436/jmcs.024.04.01.

[17] W. A. Khan and S. K. Sharma, "A new class of Hermite-based higher-order central Fubini polynomials." *IJACM*, vol. 87, pp. 1–14, 2020, doi: 10.1007/s40819-020-00840-3.

[18] W. A. Khan and D. Srivastava, "On the generalized Apostol-type Frobenius-Genocchi polynomials." *Filomat*, vol. 33, no. 7, pp. 1967–1977, 2019, doi: 10.2298/FIL1907967K.

[19] M. A. Pathan and W. A. Khan, "Some implicit summation formulas and symmetric identities for the generalized Hermite-Bernoulli polynomials." *Mediterr. J. Math.*, vol. 12, pp. 675–695, 2015, doi: 10.1007/s00009-014-0423-0.

[20] M. A. Pathan and W. A. Khan, "A new class of generalized polynomials associated with Hermite and Euler polynomials." *Mediterr. J. Math.*, vol. 13, pp. 913–328, 2016, doi: 10.1007/s00009-015-0551-1.

[21] M. A. Pathan and W. A. Khan, "A new class of generalized polynomials associated with Hermite and poly-Bernoulli polynomials." *Miskolc Math. J.*, vol. 22, no. 1, pp. 317–330, 2021, doi: 10.18514/MMN.2021.1684.

[22] M. A. Pathan and W. A. Khan, "On three families of extended Laguerre-based Apostol type polynomials." *Proyecciones J. Math.*, vol. 40, no. 2, pp. 291–312, 2021, doi: 10.22199/issn.0717-6279-2021-02-0019.

[23] M. A. Pathan and W. A. Khan, "On a class of generalized Humbert-Hermite polynomials via generalized fibonacci polynomials." *TJM*, vol. 46, pp. 929–945, 2022, doi: 10.55730/1300-0098.3133.

[24] M. A. Pathan and W. A. Khan, "On λ-Changhee-Hermite polynomials." *Analysis*, vol. 42, no. 2, pp. 1–13, 2022, doi: 10.1515/anly-2021-1012.

[25] E. D. Rainville, *Special functions.* New York: Chelsea Publishing Company, 1971.

[26] Y. Simsek, "Formulas for Poisson-Charlier, Hermite, Milne-Thomson and other type polynomials by their generating functions and p-adic integral approach." *Rev. R. Acad. Cienc. Exactas Fs. Nat. Ser. A Mat. RACSAM*, vol. 113, pp. 931–948, 2019, doi: 10.1007/s13398-018-0528-6.

[27] L. M. Thomson, "Two classes of generalized polynomials." *Proc. London Math. Soc.*, vol. 35, no. 1, pp. 514–522, 1933, doi: 10.1112/plms/s2-35.1.514.

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