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HYERS-ULAM-RASSIAS STABILITY OF SOME SEQUENTIAL NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH CAPUTO-HADAMARD FRACTIONAL DERIVATIVE

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Abstract. In this article, we employ a fixed point theory to investigate the stability in the sense of Hyers-Ulam-Rassias of some sequential neutral functional differential equations with Caputo-Hadamard fractional derivative. We present two examples to illustrate our main results. In this way, we generalize several earlier outcomes.

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1. INTRODUCTION

Functional equations grow in an exponential way in many applications like networks (see e.g. [11, 14]). Particular cases of functional equations are functional differential equations that are also of central importance in many disciplines such as control theory, neural networks, epidemiology, etc. [1]. Fractional derivatives are capable of describing hereditary and memory effects in many processes and materials like e.g., bioengineering, biology, aerodynamics, and chemistry. So the study of neutral functional differential equations in presence of fractional derivatives constitutes an important area of research. For more details, see the text [31].

Recently, there has been a considerable progress in fractional calculus, and many other problems of fractional differential equations, see e.g. [21], and Podlubny [23]. It should be noted that, much of the work on the topic involves Caputo and Riemann-Liouville type fractional derivatives (see e.g. [4,9,26,29,30]). Besides the mentioned fractional derivatives, there is another fractional derivative introduced by Hadamard in 1892 (see [16]). The new derivative introduced by Hadamard is known as Hadamard derivative and differs from above mentioned derivatives in the presence of logarithmic function of arbitrary exponent in its kernel. It should be remarked that Jarad et.al. in [19] modified the fractional derivative of the mentioned Hadamard type into a more suitable one with physically interpretable initial conditions comparable to the Caputo setting and named it Caputo-Hadamard fractional derivative. For somehow detailed explanation of Hadamard fractional derivative and integral, the reader is advised to see the articles [12].

Stability theory arose as a consequence of the famous talk delivered by Ulam at a conference held in Wisconsin University in 1940 (see e.g. [17]). Ulam's type of stability is useful, because it guarantees that there is a close exact solution. Nowadays, it becomes a research trend in many applications (see e.g. [18] for more references and details). The famous stability question presented by Ulam can be rewritten as follows:

If $\mathfrak{G}$ is a group and $(\mathfrak{G}^*, \rho)$ is a metric group. Is it true that for $\varepsilon > 0$, there exist a $\delta > 0$ such that if $F : \mathfrak{G} \to \mathfrak{G}^*$ satisfies

$$\rho(F(ab), F(a)F(b)) < \delta$$

for all $a, b \in \mathfrak{G}$, then a homomorphism $g : \mathfrak{G} \to \mathfrak{G}^*$ exists such that

$$\rho(F(a), g(a)) < \varepsilon$$

for all $a \in \mathfrak{G}$?

The open problem mentioned above has been handled by many mathematicians as follows. In 1941, D. H. Hyers introduced a positive answer to it in case of Banach spaces. Since then, the stability problem is called Ulam-Hyers or Hyers-Ulam stability problem. The most important result after Hyers seems to be that introduced in 1978 by Rassias (see [25]). The idea of Rassias is simply a generalization of the result of Hyers by considering the stability in the case of unbounded Cauchy differences. The result obtained by Rassias can be rewritten as follows (see [25]):

Theorem 1. *Consider two Banach spaces $\mathfrak{B}, \mathfrak{B}^*$, and a continuous mapping f from $\mathbb{R}$ into $\mathfrak{B}^*$. Suppose that there exists $\omega \geq 0$ and $\vartheta \in [0, 1)$ such that*

$$\|f(b_1 + b_2) - f(b_1) - f(b_2)\| \leq \omega(\|b_1\|^\vartheta + \|b_2\|^\vartheta),\ b_1, b_2 \in \mathfrak{B}^* \setminus \{0\}.$$

Then there exists a unique solution $f^* : \mathfrak{B} \to \mathfrak{B}^*$ of the Cauchy equation with

$$\|f(b_1) - f^*(b_1)\| \leq \frac{2\omega\|b_1\|^\vartheta}{|2 - 2^\vartheta|},\ b_1 \in \mathfrak{B}^* \setminus \{0\}.$$

Since then, the stability problem is known as the Ulam-Hyers-Rassias or the Hyers-Ulam-Rassias stability (see also [10]).

During the last three decades, stability of differential equations has been a focus of scientific investigations by many researchers see e.g. [6, 22]. As a consequence of the interesting results presented in this direction, many articles devoted to this subject have been written (see e.g. [8, 20, 24]. In 2019, Shikhare and Kucche (see [28]), employed weakly Picard operator to investigate the stability of some kind of equations in Banach Spaces in the sense of Hyers-Ulam. Furthermore, they obtained stability in the sense of Ulam-Hyers-Rassias for such kind of equations via Pachpatte's integral inequalities. Also Shah and Zada in 2019 (see [27]) used a fixed point approach to investigate the stability of impulsive Volterra integral equation. In 2020, the authors in [5] investigated the stability of some general differential equation using fixed point approach. See also [15], where the authors studied the stability of some Caputo fractional differential equations using fixed point approach. Note that in the case of the Riemann-Louisville fractional derivative only the case without any delays is studied (see e.g. [7]). The authors in [2] investigated the Hyers-Ulam stability of the nonlinear fractional stochastic neutral differential equations system. In [3], the authors studied the Ulam stability of Caputo type fractional stochastic neutral differential equations.

The article is divided into three sections. In the next section we recall some preliminaries, section 3 shows the main results: stability results in Hyers-Ulam-Rassias sense, in section 4, we used two examples to illustrate our results, and in section 5 we conclude our work.

2. Preliminaries

We use this section to introduce notations, some definitions, and recall some well-known results. We start to recall the notions of the Hadamard fractional integral and derivative respectively as follows. Throughout the paper, we use $\mathbb{R}$ to denote the set of real numbers and $C([a,b],\mathbb{R})$ to denote the set of continuous functions from an interval $[a,b]$ into the set $\mathbb{R}$.

Definition 1. [21] The Hadamard fractional integral of order λ for a function h is defined as

$$I^{\lambda}h(t) = \frac{1}{\Gamma(\lambda)} \int_1^t \left(\log \frac{t}{\nu}\right)^{\lambda-1} \frac{h(\nu)}{\nu} d\nu, \quad \lambda > 0,$$

provided the integral exists.

Definition 2. ([21]) The Hadamard derivative of fractional order $\lambda \in (0,1)$ for a function $h : [1,\infty) \to \mathbb{R}$ is defined as

$$D^{\lambda}h(t) = \frac{1}{\Gamma(1-\lambda)} \left(t\frac{d}{dt}\right) \int_1^t \left(\log \frac{t}{\nu}\right)^{-\lambda} \frac{h(\nu)}{\nu} d\nu,$$

with $\log(\cdot) = \log_e(\cdot)$.

Now, we recall the notion of the Caputo-Hadamard fractional derivative as follows.

Definition 3. [21] The Caputo-Hadamard derivative of fractional order $\lambda \in (0,1)$ for a function $h : [1,\infty) \to \mathbb{R}$ is defined as

$${}^{C}D^{\lambda}h(t) = D^{\lambda}\left[h(t) - h(1)\right].$$

Now, the concept of contractive operator (see e.g. [13])

Definition 4. ([13]) Let $(X,\varkappa)$ be a generalized complete metric space. The mapping $\Pi : X \to X$ is called a contraction if there exists a constant q_1 with $0 < q_1 < 1$ such that whenever $\varkappa(x_1,x_2) < \infty$ one has

$$\varkappa(\Pi x_1,\Pi x_2) \leq q_1\varkappa(x_1,x_2).$$

The following is an important results that will play an important role in our analysis. We recall it from [13].

Theorem 2. *Suppose (F,d) is a complete metric space and $L : F \to F$ is a contraction (for some $\delta \in [0,1]$), $d(L(x),L(y)) \leq \delta d(x,y)$ for all $x,y \in F$. Suppose that there exists $v \in F$, $\chi > 0$ and $d(v,L(v)) \leq \chi$. Then, there exists a unique $k \in F$ such that $k = L(k)$. Moreover,*

$$d(v,k) \leq \frac{\chi}{1-\delta}.$$

The purpose of the article is to study the stability of the solution of the following Caputo-Hadamard sequential fractional order neutral functional differential equations

$$D^{\upsilon_1}[D^{\upsilon_2}y(\sigma) - f_2(\sigma,y_\sigma)] = f_1(\sigma,y_\sigma), \quad \sigma \in I := [1,b], \tag{2.1}$$

with initial conditions $y(\sigma) = \phi(\sigma), \quad \sigma \in [1-\kappa,1]$, $D^{\upsilon_2}y(1) = \eta \in \mathbb{R}$, where $\phi \in C([1-\kappa,1],\mathbb{R})$, $0 < \upsilon_1,\upsilon_2 < 1$, $f_1, f_2 : I \times C([-\kappa,0],\mathbb{R}) \to \mathbb{R}$ are given functions.

For any function y defined on $[1-\kappa,b]$ and any $t \in I$, we denote by y_σ the element of $C_\kappa := C([-\kappa,0],\mathbb{R})$ defined by $y_\sigma(s) = y(\sigma+s), \quad s \in [-\kappa,0]$, with norm $\|y_\sigma\| = \sup\{y(\sigma+s); -\kappa \leq s \leq 0\}$.

3. MAIN RESULTS

Here, we apply the Banach fixed point theory to study stability of (2.1) in the sense of Hyers-Ulam-Rassias. The following theorem is our main theorem, where we show that under some conditions functions that satisfy (2.1) approximately (in some sense) are close (in some way) to the solutions of (2.1). In other words, we investigate the Hyers-Ulam-Rassias stability of (2.1).

Theorem 3. *Let $f_1 : I \times \mathbb{R} \to \mathbb{R}$, $f_2 : I \times \mathbb{R} \to \mathbb{R}$ are some continuous functions satisfying the following conditions*

$$|f_i(\sigma,\phi_1) - f_i(\sigma,\phi_2)| \leq l_i\|\phi_1 - \phi_2\|,$$

for all $\sigma \in I, \phi_1,\ \phi_2 \in C_\kappa$ and for some $l_i > 0$ $i = 1,2$.

If $z \in C^2([1-\kappa,b],\mathbb{R})$ satisfies

$$|D^{\upsilon_1}[D^{\upsilon_2}z(\sigma) - f_2(\sigma,z_\sigma)] - f_1(\sigma,z_\sigma)| \leq \varepsilon\gamma(\sigma),$$

for all $\sigma \in [1,b]$, where $\varepsilon > 0$ and $\gamma(\sigma)$ is a positive, nondecreasing, continuous function, then there exists a unique solution z^* of (2.1) with $z^*(\sigma) = z(\sigma), \quad \sigma \in$

$[1-\kappa,1]$, *such that*

$$|z(\sigma)-z^*(\sigma)| \leq \left(\frac{1}{1-\left(\frac{l_1}{\mu^{\upsilon_1+\upsilon_2}}+\frac{l_2}{\mu^{\upsilon_1}}\right)}\right)\frac{Mb^{\mu}}{\Gamma(\upsilon_1+\upsilon_2+1)}\,\varepsilon\gamma(\sigma), \qquad \forall\sigma\in[1,b],$$

where $M=\sup\limits_{s\in[1,b]}\left(\frac{(\log s)^{\upsilon_1+\upsilon_2}}{s^{\mu}}\right)$ *and* μ *is a positive constant such that*

$$\left(\frac{l_1}{\mu^{\upsilon_1+\upsilon_2}}+\frac{l_2}{\mu^{\upsilon_1}}\right)<1.$$

Proof. Consider the metric d on $E=C([1-\kappa,b],\mathbb{R})$ by

$$d(y_1,y_2)=\inf\left\{k\in[0,\infty):\frac{|y_1(\sigma)-y_2(\sigma)|}{\psi(\sigma)}\leq k\tilde{\gamma}(\sigma),\forall\sigma\in[1-\kappa,b]\right\},$$

with $\psi(\sigma)=\sigma^{\mu}$ for $\sigma\in[1,b]$ and $\psi(\sigma)=1$ for $\sigma\in[1-\kappa,1]$, and $\widetilde{\gamma}(\sigma)=\gamma(\sigma)$ for $\sigma\in[1,b]$ and $\widetilde{\gamma}(\sigma)=\gamma(1)$ for $\sigma\in[1-\kappa,1]$.

We consider the operator $\mathcal{B}:E\to E$ such that $(\mathcal{B}y)(\sigma)=z(\sigma)$, for $\sigma\in[1-\kappa,1]$, and

$$\begin{aligned}(\mathcal{B}y)(\sigma)&=z(1)+(D^{\upsilon_2}z(1)-f_2(1,z(1)))\frac{(\log\sigma)^{\upsilon_2}}{\Gamma(\upsilon_2+1)}\\&+\frac{1}{\Gamma(\upsilon_1)}\int_1^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\upsilon_1-1}\frac{f_2(s,y_s)}{s}ds\\&+\frac{1}{\Gamma(\upsilon_1+\upsilon_2)}\int_1^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\upsilon_1+\upsilon_2-1}\frac{f_2(s,y_s)}{s}ds.\end{aligned}$$

Let $y_1,y_2\in E$, we have $(\mathcal{B}y_1)(\sigma)-(\mathcal{B}y_2)(\sigma)=0$, for all $\sigma\in[1-\kappa,1]$.

For $\sigma\in[1,b]$, we get

$$\begin{aligned}&\left|(\mathcal{B}y_1)(\sigma)-(\mathcal{B}y_2)(\sigma)\right|\\&\leq\frac{1}{\Gamma(\upsilon_1)}\int_1^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\upsilon_1-1}\left|\frac{f_2(s,y_{1s})-f_2(s,y_{2s})}{s}\right|ds\\&\quad+\frac{1}{\Gamma(\upsilon_1+\upsilon_2)}\int_1^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\upsilon_1+\upsilon_2-1}\left|\frac{f_1(s,y_{1s})-f_1(s,y_{2s})}{s}\right|ds\\&\leq\frac{l_2}{\Gamma(\upsilon_1)}\int_1^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\upsilon_1-1}\frac{\|y_{1s}-y_{2s}\|}{s}ds\\&\quad+\frac{l_1}{\Gamma(\upsilon_1+\upsilon_2)}\int_1^{\sigma}\left(\log\frac{\sigma}{s}\right)^{\upsilon_1+\upsilon_2-1}\frac{\|y_{1s}-y_{2s}\|}{s}ds.\end{aligned}$$

For $s\in[1,\sigma]$, there is $q\in[-\kappa,0]$ such that $\|y_{1s}-y_{2s}\|=|y_1(s+q)-y_2(s+q)|$. Therefore,

$$\begin{aligned}\|y_{1s}-y_{2s}\| &= \frac{|y_1(s+q)-y_2(s+q)|}{\psi(s+q)\widetilde{\gamma}(s+q)}\psi(s+q)\widetilde{\gamma}(s+q)\\ &\le d(y_1,y_2)\psi(s+q)\widetilde{\gamma}(s+q)\le d(y_1,y_2)\psi(s)\gamma(s).\end{aligned}$$

Therefore,

$$\begin{aligned}\Big|(\mathcal{B}y_1)(\sigma)-(\mathcal{B}y_2)(\sigma)\Big| &\le \frac{l_2}{\Gamma(\upsilon_1)}d(y_1,y_2)\gamma(\sigma)\int_1^\sigma\left(\log\frac{\sigma}{s}\right)^{\upsilon_1-1}\frac{s^\mu}{s}ds\\ &\quad+\frac{l_1}{\Gamma(\upsilon_1+\upsilon_2)}d(y_1,y_2)\gamma(\sigma)\int_1^\sigma\left(\log\frac{\sigma}{s}\right)^{\upsilon_1+\upsilon_2-1}\frac{s^\mu}{s}ds.\end{aligned}$$

For $\lambda>0$, by using the change of variable $u=\mu\log\sigma-\mu\log s$, we get

$$\int_1^\sigma\left(\log\frac{\sigma}{s}\right)^{\lambda-1}\frac{s^\mu}{s}ds=\int_0^{\mu\log\sigma}\left(\frac{u}{\mu}\right)^{\lambda-1}\sigma^\mu\frac{e^{-u}}{\mu}du\le\frac{\sigma^\mu}{\mu^\lambda}\Gamma(\lambda).$$

Then,

$$\begin{aligned}\Big|(\mathcal{B}y_1)(\sigma)-(\mathcal{B}y_2)(\sigma)\Big| &\le l_2d(y_1,y_2)\gamma(\sigma)\frac{\sigma^\mu}{\mu^{\upsilon_1}}+l_1d(y_1,y_2)\gamma(\sigma)\frac{\sigma^\mu}{\mu^{\upsilon_1+\upsilon_2}}\\ &\le\left(\frac{l_1}{\mu^{\upsilon_1+\upsilon_2}}+\frac{l_2}{\mu^{\upsilon_1}}\right)d(y_1,y_2)\gamma(\sigma)\sigma^\mu\end{aligned}$$

Then, $\mathcal{B}$ is contractive (see Definition 4). For $\sigma\in[1-\kappa,1]$, we have

$$(\mathcal{B}z)(\sigma)-z(\sigma)=0.$$

We have

$$|D^{\upsilon_1}[D^{\upsilon_2}z(\sigma)-f_2(\sigma,z_\sigma)]-f_1(\sigma,z_\sigma)|\le\varepsilon\gamma(\sigma),\forall\sigma\in[1,b].$$

By using Lemma 2.3 in [1], we get

$$\begin{aligned}|z(\sigma)-\mathcal{B}z(\sigma)| &\le\frac{\varepsilon}{\Gamma(\upsilon_1+\upsilon_2)}\int_1^\sigma\left(\log\frac{\sigma}{s}\right)^{\upsilon_1+\upsilon_2-1}\frac{\gamma(s)}{s}ds\\ &\le\frac{\varepsilon\gamma(\sigma)}{\Gamma(\upsilon_1+\upsilon_2)}\int_1^\sigma\left(\log\frac{\sigma}{s}\right)^{\upsilon_1+\upsilon_2-1}\frac{1}{s}ds\\ &\le\frac{\varepsilon\gamma(\sigma)}{\Gamma(\upsilon_1+\upsilon_2+1)}(\log\sigma)^{\upsilon_1+\upsilon_2},\forall\sigma\in[1,b].\end{aligned}$$

Hence

$$\begin{aligned}\frac{|z(\sigma)-\mathcal{B}z(\sigma)|}{\sigma^\mu} &\le\frac{\varepsilon}{\Gamma(\upsilon_1+\upsilon_2+1)}\gamma(\sigma)\frac{(\log\sigma)^{\upsilon_1+\upsilon_2}}{\sigma^\mu}\\ &\le\frac{\varepsilon M}{\Gamma(\upsilon_1+\upsilon_2+1)}\gamma(\sigma),\qquad\forall\sigma\in[1,b],\end{aligned}$$

then

$$d(z,\mathcal{B}z) \leq \varepsilon \frac{M}{\Gamma(\upsilon_1+\upsilon_2+1)}.$$

Using Theorem 2, there exists a solution z^* of (2.1) such that

$$d(z,z^*) \leq \varepsilon \left(\frac{1}{1-\left(\frac{l_1}{\mu^{\upsilon_1+\upsilon_2}}+\frac{l_2}{\mu^{\upsilon_1}}\right)} \right) \frac{M}{\Gamma(\upsilon_1+\upsilon_2+1)},$$

so that

$$|z(\sigma)-z^*(\sigma)| \leq \left(\frac{1}{1-\left(\frac{l_1}{\mu^{\upsilon_1+\upsilon_2}}+\frac{l_2}{\mu^{\upsilon_1}}\right)} \right) \frac{Mb^{\mu}}{\Gamma(\upsilon_1+\upsilon_2+1)} \varepsilon\gamma(\sigma),$$

for all $\sigma \in [1,b]$. □

Remark 1. It should be remarked that in our analysis, we proved the existence as a part of the stability results in Theorem 3. Moreover, we get rid off the essential condition in Theorem 3.2 in [1].

The following theorem represents the stability of (2.1) in the sense of Ulam-Hyers.

Theorem 4. *Let $f_1 : I\times\mathbb{R} \to \mathbb{R}$, $f_2 : I\times\mathbb{R} \to \mathbb{R}$ are some continuous functions satisfying the following conditions*

$$|f_i(\sigma,\phi_1)-f_i(\sigma,\phi_2)| \leq l_i\|\phi_1-\phi_2\|,$$

for all $\sigma \in I, \phi_1,\ \phi_2 \in C_\kappa$ and for some $l_i > 0$ $i = 1,2$.

If $z \in C^2([1-\kappa,b],\mathbb{R})$ satisfies

$$|D^{\upsilon_1}[D^{\upsilon_2}z(\sigma)-f_2(\sigma,z_\sigma)]-f_1(\sigma,z_\sigma)| \leq \varepsilon,$$

for all $\sigma \in [1,b]$, where $\varepsilon > 0$, then there exists a unique solution z^* of (2.1) with $z^*(\sigma) = z(\sigma)$, $\quad \sigma \in [1-\kappa,1]$, such that

$$|z(\sigma)-z^*(\sigma)| \leq \left(\frac{1}{1-\left(\frac{l_1}{\mu^{\upsilon_1+\upsilon_2}}+\frac{l_2}{\mu^{\upsilon_1}}\right)} \right) \frac{Mb^{\mu}}{\Gamma(\upsilon_1+\upsilon_2+1)} \varepsilon, \forall\sigma \in [1,b],$$

where $M = \sup_{s\in[1,b]} \left(\frac{(\log s)^{\upsilon_1+\upsilon_2}}{s^{\mu}} \right)$ and μ is a positive constant such that

$$\left(\frac{l_1}{\mu^{\upsilon_1+\upsilon_2}}+\frac{l_2}{\mu^{\upsilon_1}} \right) < 1.$$

Proof. The proof is similar to Theorem 3. □

4. EXAMPLES

Two illustrative examples are given to show the validity of our results.

Example 1. Consider the equation (2.1) for $\kappa = 0.5$, $\upsilon_1 = 0.5$, $\upsilon_2 = 0.3$ $b = 2$, $f_1(\sigma,\phi) = \sigma^3 \cos(\phi(-\kappa))$ and $f_2(\sigma,\phi) = \sigma^2 \sin(\phi(-\kappa))$.

We have

$$|f_1(\sigma,\phi_1) - f_1(\sigma,\phi_2)| \leq 8\|\phi_1 - \phi_2\|,\ \forall\, \sigma \in [1,2],\ \phi_1,\phi_2 \in C_{0.5},$$

and

$$|f_2(\sigma,\phi_1) - f_2(\sigma,\phi_2)| \leq 4\|\phi_1 - \phi_2\|,\ \forall\, \sigma \in [1,2],\ \phi_1,\phi_2 \in C_{0.5},$$

Then $l_1 = 8$ and $l_2 = 4$.

Suppose that $z \in C^2([0.5,2],\mathbb{R})$ satisfies

$$|D^{0.5}[D^{0.3}z(\sigma) - f_2(\sigma,z_\sigma)] - f_1(\sigma,z_\sigma)| \leq \sigma,$$

for all $\sigma \in [1,2]$.

Here, $\gamma(\sigma) = \sigma$ and $\varepsilon = 1$. Using Theorem 3 there is a solution z^* of the fractional differential equations and $K > 0$ such that

$$|z(\sigma) - z^*(\sigma)| \leq K\sigma,\ \forall\, \sigma \in [1,2].$$

Example 2. Consider the equation (2.1) for $\kappa = 0.5$, $\upsilon_1 = 0.6$, $\upsilon_2 = 0.3$ $b = 5$, $f_1(\sigma,\phi) = \sin^2(\phi(-\kappa))$ and $f_2(\sigma,\phi) = \sigma\cos(\phi(-\kappa))$.

We have

$$|f_1(\sigma,\phi_1) - f_1(\sigma,\phi_2)| \leq 2\|\phi_1 - \phi_2\|,\ \forall\, \sigma \in [1,5],\ \phi_1,\phi_2 \in C_{0.5},$$

and

$$|f_2(\sigma,\phi_1) - f_2(\sigma,\phi_2)| \leq 5\|\phi_1 - \phi_2\|,\ \forall\, \sigma \in [1,2],\ \phi_1,\phi_2 \in C_{0.5},$$

Then $l_1 = 2$ and $l_2 = 5$.

Suppose that $z \in C^2([0.5,5],\mathbb{R})$ satisfies

$$|D^{0.6}[D^{0.3}z(\sigma) - f_2(\sigma,z_\sigma)] - f_1(\sigma,z_\sigma)| \leq 0.01,$$

for all $\sigma \in [1,5]$.

Here, $\varepsilon = 0.01$. Using Theorem 4 there is a solution z^* of the fractional differential equations and $K > 0$ such that

$$|z(\sigma) - z^*(\sigma)| \leq 0.01K,\ \forall\, \sigma \in [1,5].$$

5. CONCLUSION

We managed to use a version of Banach fixed point theorem to prove that under certain conditions, functions that satisfy some sequential neutral functional differential equations with Caputo-Hadamard fractional derivative approximately, are close in some sense to the exact solutions of such problems. In other words, we present stability results for some fractional differential equations in Ulam-Hyers-Rassias sense. In order to show the validity of our results, we presented two examples. Potential future work could be to invent a new method to obtain such stability results or to investigate the stability of a much more complicated fractional differential equations.

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