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COMPLEMENTS OF NABLA AND DELTA HARDY-COPSON TYPE INEQUALITIES AND THEIR APPLICATIONS

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Abstract. In this paper the classical nabla and delta Hardy-Copson type inequalities, which are derived for $\zeta > 1$, are complemented to the new case $\zeta < 0$. These complements have exactly the same forms as the aforementioned classical inequalities except that the exponent ζ is not greater than one but it is less than zero. The obtained inequalities are not only novel but also unify the continuous and discrete cases for which the case $\zeta < 0$ has not been considered so far either. Moreover one of the applications of Hardy-Copson type inequalities, which is to find nonoscillation criteria for the half linear differential/dynamic/difference equations, are presented by using complementary delta Hardy-Copson type inequalities.

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1. INTRODUCTION

The theory of inequalities containing series or integrals has been given a great importance due to their effective usage in differential equations and their applications after the celebrated discrete and continuous inequalities that Hardy have been obtained. In 1920, when Hardy [24] tried to find a simple and elementary proof of Hilbert's inequality [32]

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m c_n}{m+n} \le \pi\left(\sum_{m=1}^{\infty} a_m^2\right)^{1/2}\left(\sum_{n=1}^{\infty} c_n^2\right)^{1/2},$$

where $a_m, c_n \ge 0$ and $\sum_{m=1}^{\infty} a_m^2$ and $\sum_{n=1}^{\infty} c_n^2$ are convergent, he showed the following pioneering discrete inequality

$$\sum_{j=1}^{\infty}\left(\frac{1}{j}\sum_{i=1}^{m} c(i)\right)^{\zeta} \le \left(\frac{\zeta}{\zeta-1}\right)^{\zeta}\sum_{j=1}^{\infty} c^{\zeta}(j), \quad c(j)\ge 0,\ \zeta>1 \tag{1.1}$$

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and pioneering continuous inequality for a nonnegative function Γ and for a real constant $\zeta > 1$, as

$$\int_0^\infty \left(\frac{1}{t}\int_0^t \Gamma(s)\mathrm{d}s\right)^\zeta \mathrm{d}t \le \left(\frac{\zeta}{\zeta-1}\right)^\zeta \int_0^\infty \Gamma^\zeta(t)\mathrm{d}t, \tag{1.2}$$

where $\int_0^\infty \Gamma^\zeta(t)\mathrm{d}t < \infty$. In fact, Hardy only stated inequality (1.2) in [24] but did not prove it. After that in 1925, the proof of inequality (1.2), which depends on the calculus of variations, was shown by Hardy in [25].

The constant $\left(\frac{\zeta}{\zeta-1}\right)^\zeta$ that appears in the above inequalities also has been found as the best possible one, since if it is replaced by a smaller constant then inequalities (1.1) and (1.2) is not fulfilled anymore for the involved sequences and functions, respectively.

Then Hardy et al. [26, Theorem 330] developed inequality (1.2) and derived the following integral inequality for a nonnegative function Γ as

$$\int_0^\infty \frac{\Psi^\zeta(t)}{t^\theta}\mathrm{d}t \le \left|\frac{\zeta}{\theta-1}\right|^\zeta \int_0^\infty \frac{\Gamma^\zeta(t)}{t^{\theta-\zeta}}\mathrm{d}t, \quad \zeta > 1, \tag{1.3}$$

where $\Psi(t) = \begin{cases} \int_0^t \Gamma(s)\mathrm{d}s, & \text{if } \theta > 1, \\ \int_t^\infty \Gamma(s)\mathrm{d}s, & \text{if } \theta < 1. \end{cases}$

The exhibition of the results which contain the improvements, generalizations and applications of the discrete and continuous Hardy inequalities can be found in the books [8, 26, 32, 33, 37] and references therein.

Since various generalizations and numerous variants of the discrete Hardy inequality (1.1) exist in the literature, all of which can not be covered here, we only focus on the extensions which have been established by Copson [16, Theorem 1.1, Theorem 2.1]. This is why we name our inequalities as Hardy-Copson type inequalities. The discrete Hardy inequality (1.1) or Copson's discrete inequalities were generalized in [10, 15, 20, 34–36] and references therein.

Similar to the discrete Hardy inequality (1.1), the continuous versions (1.2) or (1.3) have attracted many mathematicians' interests and expansions of these continuous inequalities have appeared in the literature. The first continuous refinements were obtained by Copson [17, Theorem 1, Theorem 3] and after these results many papers devoted to continuous analogues and continuous improvements of the discrete Hardy-Copson inequalities, see [9, 27, 38, 40, 42].

Following the development of the time scale concept [7, 11, 12, 21, 22], the analysis of dynamic inequalities have become a popular research area and most classical inequalities have been extended to an arbitrary time scale. The surveys [1, 52] and the monograph [4] can be used to see these extended dynamic inequalities for delta approach. Although the nabla dynamic inequalities are less attractive compared to the delta ones, some of the nabla dynamic inequalities can be found in [6, 13, 23, 30, 31, 39, 41].

The growing interest to Hardy-Copson type inequalities take place in the time scale calculus as well and delta unifications of these inequalities are established in the books [5] and in the articles [3, 19, 43, 44, 46, 47, 50, 51, 55] for $\zeta > 1$ whereas their reverse versions can be found in [18, 45, 46, 48] for $0 < \zeta < 1$. The nabla unifications of Hardy-Copson type inequalities for $\zeta > 1$ can be seen in [28, 29].

All of the abovementioned articles contain Hardy-Copson type inequalities obtained for $\zeta > 1$. For our further purposes, we will show the delta and nabla Hardy-Copson type inequalities [28, 51] established for $\zeta > 1$ and used in the sequel.

The delta time scale generalizations of the foregoing inequalities in an arbitrary time scale are given in the next four theorems for nonnegative functions z and h and for

$$G(t) = \int_t^\infty z(s)\Delta s,\ H(t) = \int_a^t z(s)h(s)\Delta s,\ \overline{G}(t) = \int_a^t z(s)\Delta s,\ \overline{H}(t) = \int_t^\infty z(s)h(s)\Delta s. \tag{1.4}$$

A delta unification of the discrete inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3] and continuous inequality obtained by Saker et al. [51, Corollary 2.2] or Kayar and Kaymakçalan [28, Remark 3.22] is stated as follows.

Theorem 1 ([51]). *Let z, h, G and H be defined as in (1.4). If $\zeta > 1$, $\eta \geq 0$, $\eta + \theta < 1$, then we have*

$$\int_a^\infty \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}\Delta t. \tag{1.5}$$

The following theorem establishes a delta unification of the discrete inequality obtained by Copson [16, Theorem 2.1] or Bennett [10, Corollary 4] or Leindler [35, Proposition 2] and its continuous counterpart obtained by Copson [17, Theorem 3] as well as its continuous generalization obtained in [40, Theorem 1] or [42, Theorem 3]. Moreover the continuous inequality (1.3) obtained by Hardy et al. [26, Theroem 330] is unified.

Theorem 2 ([51]). *Let $z, h, \overline{G}$ and $\overline{H}$ be defined as in (1.4). If $\zeta > 1$, $\eta \geq 0$, $\eta + \theta < 1$, then we have*

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^\sigma(t)]^{\eta+\theta}}\Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[\overline{G}^\sigma(t)]^{\eta+\theta-\zeta}}\Delta t. \tag{1.6}$$

A delta unification of the discrete inequality obtained by Bennett [10, Corollary 6] or Leindler [35, Proposition 4] and continuous inequality obtained by Saker et al. [51, Corollary 2.3] or Kayar and Kaymakçalan [28, Remark 3.11] is stated as follows.

Theorem 3 ([51]). *Let z, h, G and $\overline{H}$ be defined as in (1.4). For $\dfrac{G^{\sigma}(t)}{G(t)} \geq \dfrac{1}{K} > 0$, if $\zeta > 1$, $\eta \geq 0$, $\eta + \theta > 1$, then we have*

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[K^{\eta+\theta-1} \frac{\eta+\zeta}{\eta+\theta-1} \right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[G(t)]^{\eta+\theta-\zeta}} \Delta t. \tag{1.7}$$

The following theorem establishes a delta unification of the discrete inequality obtained by Hardy [24, Theroem B] as inequality (2) and the discrete inequality obtained by Copson [16, Theorem 1.1] or Bennett [10, Corollary 3] or Leindler [35, Proposition 1] and their continuous counterparts obtained by Hardy [24, Theroem B] as inequality (4) and Hardy et al. [26, Theroem 330] and Copson [17, Theorem 1] as well as their continuous generalizations obtained in [40, Theorem 1] or [42, Theorem 1].

Theorem 4 ([28,51]). *Let $z, h, \overline{G}$ and H be defined as in (1.4). For $\dfrac{\overline{G}(t)}{\overline{G}^{\sigma}(t)} \geq \dfrac{1}{J} > 0$, if $\zeta > 1$, $\eta \geq 0$, $\eta + \theta > 1$, then we have*

$$\int_a^{\infty} \frac{z(t)[H^{\sigma}(t)]^{\eta+\zeta}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta}} \Delta t \leq \left[J^{\eta+\theta-1} \frac{\eta+\zeta}{\eta+\theta-1} \right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H^{\sigma}(t)]^{\eta}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta-\zeta}} \Delta t. \tag{1.8}$$

There are more results about delta Hardy-Copson type inequalities different than the previous ones. For example, in [46] it seems that delta Hardy-Copson type inequalities were obtained for $\zeta > 1$ as well as for $0 < \zeta < 1$ and $\zeta < 0$. However there are many errors in this article even for the case $\zeta > 1$. Therefore the obtained inequalities are not correct in [46]. Another result was given in [3] by using a different method in the proof of the theorems. In this article, it seems that delta Hardy-Copson type inequalities were obtained for $\zeta > 1$ as well as $0 < \zeta < 1$. However for $0 < \zeta < 1$, since the obtained inequalities are in reverse directions of those derived for the case $\zeta > 1$, they can not be considered as new delta Hardy-Copson type inequalities. In fact, they already have been established in [48] as delta Bennett-Leindler type inequalities. Therefore for $\zeta < 0$, finding delta Hardy-Copson type inequalities by preserving the directions of the inequalities established for $\zeta > 1$ is still an open problem.

The construction of nabla time scale calculus, which has been introduced simultaneously with delta time scale calculus, can be found in [7, 11, 12, 21, 22].

Contrary to delta case, nabla Hardy-Copson type inequalities have not been considered until 2021. The first results of this case obtained by Kayar and Kaymakçalan in [28].

The nabla time scale generalizations of the foregoing inequalities in an arbitrary time scale are given in the next four theorems for nonnegative functions z and h and for

$$\begin{aligned} G(t) &= \int_t^\infty z(s)\nabla s, \quad H(t) = \int_a^t z(s)h(s)\nabla s, \\ \overline{G}(t) &= \int_a^t z(s)\nabla s, \quad \overline{H}(t) = \int_t^\infty z(s)h(s)\nabla s. \end{aligned} \tag{1.9}$$

A nabla unification of the discrete inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3] and continuous inequality obtained by Saker et al. [51, Corollary 2.2] or Kayar and Kaymakçalan [28, Remark 3.22] is stated as follows.

Theorem 5 ([28]). *Let z,h,G and H be defined as in (1.9). If $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta < 1$ are real constants, then we have*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G^\rho(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{1.10}$$

The following theorem establishes a nabla unification of the discrete inequality obtained by Copson [16, Theorem 2.1] or Bennett [10, Corollary 4] or Leindler [35, Proposition 2] and its continuous counterpart obtained by Copson [17, Theorem 3] as well as its continuous generalization obtained in [40, Theorem 1] or [42, Theorem 3]. Moreover the continuous inequality (1.3) obtained by Hardy et al. [26, Theroem 330] is unified.

Theorem 6 ([28]). *Let $z,h,\overline{G}$ and $\overline{H}$ be defined as in (1.9). If $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta < 1$ are real constants, then we have*

$$\int_a^\infty \frac{z(t)[\overline{H}^\rho(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^\rho(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{1.11}$$

A nabla unification of the discrete inequality obtained by Bennett [10, Corollary 6] or Leindler [35, Proposition 4] and continuous inequality obtained by Saker et al. [51, Corollary 2.3] or Kayar and Kaymakçalan [28, Remark 3.11] is stated as follows.

Theorem 7 ([28]). *Let z,h,G and $\overline{H}$ be defined as in (1.9). Suppose that there exists $M > 0$ such that $\dfrac{G^\rho(t)}{G(t)} \leq M$ for $t \in (a,\infty)_{\mathbb{T}}$. If $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$ are real constants, then we have*

$$\int_a^\infty \frac{z(t)[\overline{H}^\rho(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}} \nabla t \leq \left[M^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^\rho(t)]^\eta}{[G^\rho(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{1.12}$$

The following theorem establishes a nabla unification of the discrete inequality obtained by Hardy [24, Theroem B] as inequality (2) and the discrete inequality obtained by Copson [16, Theorem 1.1] or Bennett [10, Corollary 3] or Leindler [35, Proposition 1] and their continuous counterparts obtained by Hardy [24, Theroem B] as inequality (4) and Hardy et al. [26, Theroem 330] and Copson [17, Theorem 1] as well their continuous generalizations obtained in [40, Theorem 1] or [42, Theorem 1].

Theorem 8 ([28]). *Let $z,h,\overline{G}$ and H be defined as in (1.9). Suppose that there exists $L>0$ such that $\dfrac{\overline{G}(t)}{\overline{G}^{\rho}(t)} \leq L$ for $t \in (a,\infty)_{\mathbb{T}}$. If $\zeta>1,\ \eta \geq 0$ and $\eta+\theta>1$ are real constants, then we have*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\nabla t \leq \left[L^{\eta+\theta-1}\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta}\int_a^\infty \frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{1.13}$$

The reason Hardy-Copson type inequalities have generalized in many directions is that they have wide applications in operator theory and geometry [8,33], in partial differential equations, such as Fokker-Planck equations and equations modelling fluid dynamics [37], in theory of qualitative behaviour of differential equations, such as finding necessary and sufficient conditions for the existence of positive solutions of half linear equations [9,43,49], oscillation theory [14,55] and references therein.

Although delta and nabla Hardy-Copson type inequalities for the case $\zeta>1$ have been deeply analyzed, the case $\zeta<0$ has been investigated neither via nabla and delta approaches nor for continuous and discrete cases. Hence the main contribution of this article is to extend aforementioned Hardy-Copson type inequalities obtained for $\zeta>1$ to the case $\zeta<0$ by using nabla and delta time scale calculi without changing the directions of the inequalities derived for $\zeta>1$. By means of application, we use complementary delta Hardy-Copson type inequalities to obtain necessary and sufficient conditions for the nonoscillation of the related half linear dynamic equations.

Our results are inspired from the papers [28] and [51] which contain nabla and delta Hardy-Copson type inequalities for the case $\zeta>1$. We notice that when $\zeta<0$ and $\eta\geq 0$, six choices appear contrary to the three cases in [28] and [51] for $\zeta>1$ and $\eta \geq 0$. The cases $0\leq \eta+\theta<1,\ \eta+\zeta>1$ and $\eta+\theta>1,\ \eta+\zeta>1$ were considered in [28] and [51] whereas the case $\eta+\theta\leq 0,\ \eta+\zeta>1$ was not investigated. $\zeta<0$ provides new three cases, namely $\eta+\theta\leq 0,\ 0<\eta+\zeta<1$ and $0\leq\eta+\theta<1,\ 0<\eta+\zeta<1$ and $\eta+\theta>1,\ 0<\eta+\zeta<1$. Therefore in the light of these six options, delta and nabla Hardy-Copson type inequalities, which are presented in Theorem 1-Theorem 8 and obtained for $\zeta>1$, are extended to the case $\zeta<0$. These inequalities are established in the same directions of those stated in the above mentioned theorems. Furthermore discrete and continuous Hardy-Copson type inequalites proven for $\zeta>1$ are expanded to $\zeta<0$. The application of the delta Hardy-Copson type inequalities provides that the nonoscillation criteria obtained for half linear dynamic equation yield results concerning half linear differential and half linear difference equations.

The organization of this paper can be seen as follows. The nabla time scale calculus and its main properties are introduced in Section 2. The delta version can be obtained similarly. The contribution of Section 3, which includes the main result, is to extend the recently developed results, which are established for $\zeta > 1$ and presented in [28,51], to the case $\zeta < 0$ by using the properties of nabla and delta derivatives and integrals. Then the special cases of nabla and delta Hardy-Copson type inequalities, which are continuous and discrete inequalities, are stated. The final section serves as applications of the obtained inequalities in oscillation theory of the half linear dynamic equations.

2. Preliminaries

This section is devoted to present the main definitions and theorems of nabla time scale calculus. We refer the reader to [7,11] for the concept of time scale calculus in delta and nabla senses.

If $\mathbb{T} \neq \varnothing$ is a closed subset of $\mathbb{R}$, then $\mathbb{T}$ is called a time scale. The backward jump operator ρ is defined as $\rho(t) := \sup(-\infty,t)_{\mathbb{T}}$, for $t \in \mathbb{T}$, provided $\sup\varnothing = \inf\mathbb{T}$. The backward graininess function $\nu\colon \mathbb{T} \to \mathbb{R}_0^+$ is defined by $\nu(t) := t - \rho(t)$, for $t \in \mathbb{T}$.

The ∇-derivative of $\Gamma\colon \mathbb{T} \to \mathbb{R}$ at the point $t \in \mathbb{T}_\kappa = \mathbb{T}/[\inf\mathbb{T},\sigma(\inf\mathbb{T}))$ denoted by $\Gamma^\nabla(t)$ is the number enjoys the property that for all $\varepsilon > 0$, there exists a neighborhood $V \subset \mathbb{T}$ of $t \in \mathbb{T}_\kappa$ such that

$$|\Gamma(s) - \Gamma(\rho(t)) - \Gamma^\nabla(t)(s-\rho(t))| \leq \varepsilon|s-\rho(t)|$$

for all $s \in V$.

Some important features of the nabla derivative can be listed below.

Lemma 1 ([7,11]). *Suppose that $\Lambda : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}_\kappa$.*

(1) *If Λ is nabla differentiable at t, then Λ is continuous at t.*
(2) *If Λ is continuous at a left scattered point t, then Λ is nabla differentiable at t with $\Lambda^\nabla(t) = \dfrac{\Lambda(t)-\Lambda(\rho(t))}{\nu(t)}$.*
(3) *Λ is nabla differentiable at a left dense point t if and only if the limit $\Lambda^\nabla(t) = \lim\limits_{s\to t}\dfrac{\Lambda(t)-\Lambda(s)}{t-s}$ exists as a finite number.*
(4) *If Λ is nabla differentiable at t, then $\Lambda^\rho(t) = \Lambda(t) - \nu(t)\Lambda^\nabla(t)$.*

A function $\Gamma : \mathbb{T} \to \mathbb{R}$ is ld-continuous if it is continuous at each left-dense points in $\mathbb{T}$ and $\lim\limits_{s\to t^+}\Gamma(s)$ exists as a finite number for all right-dense points in $\mathbb{T}$. The set $C_{ld}(\mathbb{T},\mathbb{R})$ denotes the class of real, ld-continuous functions defined on a time scale $\mathbb{T}$. If $\Gamma \in C_{ld}(\mathbb{T},\mathbb{R})$, then there exists a function $\overline{\Gamma}(t)$ such that $\overline{\Gamma}^\nabla(t) = \Gamma(t)$ and the nabla integral of Γ is defined by $\displaystyle\int_a^b \Gamma(s)\nabla s = \overline{\Gamma}(b) - \overline{\Gamma}(a)$.

Some of the properties of the nabla integral are gathered next.

Lemma 2 ([7,11]). *Let $t_1,t_2,t_3 \in \mathbb{T}$ with $t_1 < t_3 < t_2$ and $a,b \in \mathbb{R}$. If $\Lambda,\Gamma\colon \mathbb{T} \to \mathbb{R}$ are ld-continuous, then*

1) $\int_{t_1}^{t_2}[a\Lambda(s)+b\Gamma(s)]\nabla s = a\int_{t_1}^{t_2}\Lambda(s)\nabla(s)+b\int_{t_1}^{t_2}\Gamma(s)\nabla s$ *and* $\int_{t_1}^{t_1}\Lambda(s)\nabla(s)=0,$

2) $\int_{t_1}^{t_3}\Lambda(s)\nabla s+\int_{t_3}^{t_2}\Lambda(s)\nabla s=\int_{t_1}^{t_2}\Lambda(s)\nabla s=-\int_{t_2}^{t_1}\Lambda(s)\nabla s,$

3) *integration by parts formula holds:*

$$\int_{t_1}^{t_2}\Lambda(s)\Gamma^{\nabla}(s)\nabla s=\Lambda(t_2)\Gamma(t_2)-\Lambda(t_1)\Gamma(t_1)-\int_{t_1}^{t_2}\Lambda^{\nabla}(s)\Gamma(\rho(s))\nabla s.$$

Lemma 3 (Hölder's inequality, [39]). *Let* $t_1,t_2\in\mathbb{T}$. *For* $\Lambda,\Gamma\in C_{ld}([t_1,t_2]_{\mathbb{T}},\mathbb{R})$ *and for the conjugate numbers* $\kappa,\varpi>1$ *satisfying* $\frac{1}{\kappa}+\frac{1}{\varpi}=1$, *Hölder's inequality*

$$\int_{t_1}^{t_2}|\Lambda(s)\Gamma(s)|\nabla s\le\left[\int_{t_1}^{t_2}|\Lambda(s)|^{\kappa}\nabla s\right]^{1/\kappa}\left[\int_{t_1}^{t_2}|\Gamma(s)|^{\varpi}\nabla s\right]^{1/\varpi}\ \textit{holds true.}$$

If $0<\kappa<1$ *or* $\kappa<0$ *with* $\frac{1}{\kappa}+\frac{1}{\varpi}=1$, *then the reverse Hölder's inequality*

$$\int_{t_1}^{t_2}|\Lambda(s)\Gamma(s)|\nabla s\ge\left[\int_{t_1}^{t_2}|\Lambda(s)|^{\kappa}\nabla s\right]^{1/\kappa}\left[\int_{t_1}^{t_2}|\Gamma(s)|^{\varpi}\nabla s\right]^{1/\varpi}\tag{2.1}$$

is satisfied.

Lemma 4 (Chain rule for the nabla derivative, [23]). *If* $\Lambda:\mathbb{R}\to\mathbb{R}$ *is continuously differentiable and* $\Gamma:\mathbb{T}\to\mathbb{R}$ *is nabla differentiable, then* $\Lambda\circ\Gamma$ *is nabla differentiable and*

$$(\Lambda\circ\Gamma)^{\nabla}(s)=\Gamma^{\nabla}(s)\left[\int_0^1\Lambda'(\Gamma(\rho(s))+h\nu(s)\Gamma^{\nabla}(s))\mathrm{d}h\right].$$

3. HARDY-COPSON TYPE INEQUALITIES

In the sequel, we will obtain several Hardy-Copson type inequalities for non-negative, ld-continuous, ∇-differentiable and locally nabla integrable functions z and h and for the functions $G,H,\overline{G}$ and $\overline{H}$ defined in (1.9).

The next theorem, which is proven for $\zeta<0$, $\eta\ge0$ and $\eta+\theta\le0$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta>1$, $\eta\ge0$ and $0\le\eta+\theta<1$. These previous Hardy-Copson type inequalities are listed as follows:

(a) The discrete inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3].
(b) The continuous inequality obtained by Saker et al. [51, Corollary 2.2] or Kayar and Kaymakçalan [28, Remark 3.22].
(c) The delta inequality (1.5) in Theorem 1 obtained by Saker et al. [51, Theorem 2.2].

(d) The nabla inequality (1.10) in Theorem 5 obtained by Kayar and Kaymakçalan [28, Theorem 3.19].

Theorem 9. *For the functions z, h, G and H defined as in (1.9), suppose that there exists $L_1 > 0$ such that $\frac{G^\rho(t)}{G(t)} \leq L_1$ for $t \in (a,\infty)_{\mathbb{T}}$. Let $\zeta < 0,\ \eta \geq 0$ and $\eta + \theta \leq 0$ be real constants.*

(1) *If $\eta + \zeta > 1$, then we have*

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\rho(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \nabla t \tag{3.1}$$

and

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{L_1^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\rho(t)]^\eta}{[G^\rho(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{3.2}$$

(2) *If $0 < \eta + \zeta < 1$, then we have*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}} \nabla t \tag{3.3}$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G^\rho(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{3.4}$$

Proof. Although we try to obtain nabla Hardy-Copson type inequalities, the same methodology used in the proof of [31, Theorem 3.1] and [31, Theorem 3.4], which present nabla Bennett-Leindler type inequalities, works for the proofs of this theorem. To be clear, we sketch the proof of inequality (3.1). If the same steps are followed in the proof of [31, Theorem 3.1] for the function H and [31, Theorem 3.4] for the function G, we can show that

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \geq \frac{\eta+\zeta}{1-\eta-\theta} \int_a^\infty \frac{z(t)h(t)[H^\rho(t)]^{\eta+\zeta-1}}{[G(t)]^{\eta+\theta-1}} \nabla t. \tag{3.5}$$

The difference of the proofs appear after this step. By applying reverse Hölder inequality (2.1) to the right hand side of inequality (3.5) with the constants $\zeta < 0$ and $0 < \frac{\zeta}{\zeta-1} < 1$, we get

$$\left[\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t\right]^{1/\zeta} \geq \frac{\eta+\zeta}{1-\eta-\theta} \left[\int_a^\infty \frac{z(t)h(t)[H^\rho(t)]^{\eta}}{[G(t)]^{\eta+\theta-\zeta}} \nabla t\right]^{1/\zeta}. \tag{3.6}$$

Then raising both sides of inequality (3.6) to $\zeta < 0$ changes the direction of this inequality, which implies the desired result (3.1). The proofs of inequalities (3.2)-(3.4) can be obtained by following the same method as above.

□

Remark 1. The nabla Hardy-Copson type inequalities (3.1)-(3.2) obtained for $\zeta < 0,\ \eta \geq 0$ and $\eta+\theta \leq 0$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.19] for $\zeta > 1,\ \eta \geq 0$ and $0 \leq \eta+\theta < 1$.

Note that the condition $\eta+\zeta > 1$ is automatically satisfied in [28, Theorem 3.19] while the other one, $0 < \eta+\zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.3)-(3.4), which are obtained for $\zeta < 0,\ \eta \geq 0,\ \eta+\theta \leq 0$ and $0 < \eta+\zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 1. *From the inequalities* (3.1)-(3.4) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* G^{ρ}, G, H^{ρ}, H *presented in* (1.9) *by* $G, G^{\sigma}, H, H^{\sigma}$ *defined in* (1.4)*, respectively.*

Let the functions z, h, G *and* H *be defined as in* (1.4)*. Suppose that there exists* $M_1 > 0$ *such that* $\dfrac{G(t)}{G^{\sigma}(t)} \leq M_1$ *for* $t \in (a,\infty)_{\mathbb{T}}$*. In this case for* $\zeta < 0,\ \eta \geq 0$ *and* $\eta+\theta \leq 0$*, nabla Hardy-Copson type inequalities* (3.1)-(3.4) *become novel delta Hardy-Copson type inequalities, two of which obtained from* (3.2) *and* (3.4) *can be written as follows, respectively. For* $\eta+\zeta > 1$*, we have*

$$\int_a^{\infty} \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{M_1^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[G(t)]^{\eta+\theta-\zeta}} \Delta t$$

and for $0 < \eta+\zeta < 1$*, we have*

$$\int_a^{\infty} \frac{z(t)[H^{\sigma}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H^{\sigma}(t)]^{\eta}}{[G(t)]^{\eta+\theta-\zeta}} \Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.1)-(3.2) *obtained for* $\zeta < 0,\ \eta \geq 0$ *and* $\eta+\theta \leq 0$ *are complements of the delta Hardy-Copson type inequalities given in [51, Theorem 2.2] for* $\zeta > 1,\ \eta \geq 0$ *and* $0 \leq \eta+\theta < 1$*.*

If $\eta = 0$*, then the delta Hardy-Copson type inequalities in [47, Theorem 2.9] obtained for* $\zeta > 1$ *and* $0 \leq \theta < 1$ *are extended to the cases* $\zeta < 0$ *and* $\theta \leq 0$ *by the delta variant of nabla Hardy-Copson type inequalities* (3.1)-(3.2)*.*

Note that the condition $\eta+\zeta > 1$ is automatically satisfied in [51, Theorem 2.2] and [47, Theorem 2.9] while the other one, $0 < \eta+\zeta < 1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.3)-(3.4), which are obtained for $\zeta < 0,\ \eta \geq 0,\ \eta+\theta \leq 0$ and $0 < \eta+\zeta < 1$ for the first time, offer novelties in the current literature by this corollary.

Remark 2. If the time scale is the set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_1 = 1$ in (3.1)-(3.4). Hence for $0 < \eta+\zeta$, inequalities (3.1)-(3.4) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.1) reduces to the following inequality as

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\mathrm{d}t \le \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}\mathrm{d}t, \tag{3.7}$$

where $\zeta<0,\ \eta\ge 0$ and $\eta+\theta\le 0$ and the functions G and H is defined as

$$G(t)=\int_t^\infty z(s)\mathrm{d}s \quad \text{and} \quad H(t)=\int_a^t z(s)h(s)\mathrm{d}s. \tag{3.8}$$

For the continuous case, when $\zeta>1,\ \eta\ge 0$ and $0\le\eta+\theta<1$, the first Hardy-Copson type inequality was established in [51, Corollary 2.2] and [28, Remark 3.22] for the given aforementioned functions G and H. By this remark, these results are extended to the cases $\zeta<0,\ \eta\ge 0$ and $\eta+\theta\le 0$ by the novel continuous Hardy-Copson type inequality (3.7).

Remark 3. If the time scale is the set of natural numbers, then for all $t\in\mathbb{N}$, the backward jump operator results in $\rho(t)=t-1$ in (3.1)-(3.4).

Using $\int_t^\infty z(s)\nabla s=\sum_{k=t+1}^\infty z(k)$, we have $G^\rho(t)=G(t-1)=\sum_{k=t}^\infty z(k)$, where $G(t)=\sum_{k=t+1}^\infty z(k)$. Moreover $H(t)=\sum_{k=a+1}^t z(k)h(k)$. Let us assume that there exist $L_1>0$ such that $\frac{G(t-1)}{G(t)}\le L_1$. For $a=0,\ \zeta<0,\ \eta\ge 0$ and $\eta+\theta\le 0$, in the set of natural numbers, inequalities (3.1)-(3.4) become novel discrete Hardy-Copson type inequalities, two of which obtained from (3.2) and (3.4) can be written as follows, respectively. For $\eta+\zeta>1$, we have

$$\sum_{t=1}^\infty \frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \le \left[\frac{L_1^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^\zeta \sum_{t=1}^\infty \frac{z(t)h^\zeta(t)[H(t-1)]^\eta}{[G(t-1)]^{\eta+\theta-\zeta}}$$

and for $0<\eta+\zeta<1$, we have

$$\sum_{t=1}^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}} \le \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \sum_{t=1}^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[G(t-1)]^{\eta+\theta-\zeta}}.$$

The discrete Hardy-Copson type inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3] for $\zeta>1,\ \eta=0,\ 0\le\theta<1$ and the ones obtained in [28, Remark 3.23] for $\zeta>1,\ \eta\ge 0,\ 0\le\eta+\theta<1$, are extended to the cases $\zeta<0,\ \eta\ge 0,\ \eta+\theta\le 0$ by Theorem 9 and particularly by this remark.

Note that the condition $\eta+\zeta>1$ is automatically satisfied in [10, Corollary 5], [35, Proposition 3] and [28, Remark 3.23] while the other one, $0<\eta+\zeta<1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta<0,\ \eta\ge 0,\ \eta+\theta\le 0$ and $0<\eta+\zeta<1$ for the first time, offer novelties in the current literature by this remark.

The next theorem, which is proven for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta+\theta < 1$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta+\theta < 1$. These previous Hardy-Copson type inequalities are listed as follows:

(a) The discrete inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3].

(b) The continuous inequality obtained by Saker et al. [51, Corollary 2.2] or Kayar and Kaymakçalan [28, Remark 3.22].

(c) The delta inequality (1.5) in Theorem 1 obtained by Saker et al. [51, Theorem 2.2].

(d) The nabla inequality (1.10) in Theorem 5 obtained by Kayar and Kaymakçalan [28, Theorem 3.19].

Theorem 10. *For the functions z, h, G and H defined as in (1.9), suppose that there exists $L_2 > 0$ such that $1 \geq \dfrac{G(t)}{G^{\rho}(t)} \geq L_2$ for $t \in (a,\infty)_{\mathbb{T}}$. Let $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta+\theta < 1$ be real constants.*

(1) *If $\eta+\zeta > 1$, then we have*

$$\int_a^{\infty} \frac{z(t)[H^{\rho}(t)]^{\eta+\zeta}}{[G^{\rho}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{L_2(\eta+\zeta)}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H^{\rho}(t)]^{\eta}}{[G^{\rho}(t)]^{\eta+\theta-\zeta}} \nabla t \tag{3.9}$$

and

$$\int_a^{\infty} \frac{z(t)[H^{\rho}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H^{\rho}(t)]^{\eta}}{[G(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{3.10}$$

(2) *If $0 < \eta+\zeta < 1$, then we have*

$$\int_a^{\infty} \frac{z(t)[H(t)]^{\eta+\zeta}}{[G^{\rho}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[G^{\rho}(t)]^{\eta+\theta-\zeta}} \nabla t \tag{3.11}$$

and

$$\int_a^{\infty} \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[G(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{3.12}$$

Proof. The combination of the techniques used in the proof of [31, Theorem 3.1] and in the proof of Theorem 9 work for the proof of this theorem. □

Remark 4. The nabla Hardy-Copson type inequalities (3.9)-(3.10) obtained for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta+\theta < 1$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.19] for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta+\theta < 1$.

Note that the condition $\eta+\zeta > 1$ is automatically satisfied in [28, Theorem 3.19] while the other one, $0 < \eta+\zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.11)-(3.12), which are obtained for $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta+\theta < 1$ and $0 < \eta+\zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 2. *From the inequalities* (3.9)-(3.12) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* G^{ρ}, G, H^{ρ}, H *presented in* (1.9) *by* $G, G^{\sigma}, H, H^{\sigma}$ *defined in* (1.4)*, respectively.*

Let the functions z, h, G *and* H *be defined as in* (1.4)*. Suppose that there exists* $M_2 > 0$ *such that* $1 \leq \frac{G(t)}{G^{\sigma}(t)} \leq \frac{1}{M_2}$ *for* $t \in (a, \infty)_{\mathbb{T}}$*. In this case for* $\zeta < 0$, $\eta \geq 0$ *and* $0 \leq \eta + \theta < 1$, *nabla Hardy-Copson type inequalities* (3.9)-(3.12) *become novel delta Hardy-Copson type inequalities, two of which obtained from* (3.9) *and* (3.11) *can be written as follows, respectively. For* $\eta + \zeta > 1$, *we have*

$$\int_a^{\infty} \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{M_2(\eta+\zeta)}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[G(t)]^{\eta+\theta-\zeta}} \Delta t$$

and for $0 < \eta + \zeta < 1$, *we have*

$$\int_a^{\infty} \frac{z(t)[H^{\sigma}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H^{\sigma}(t)]^{\eta}}{[G(t)]^{\eta+\theta-\zeta}} \Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.9)-(3.10) *obtained for* $\zeta < 0$, $\eta \geq 0$ *and* $0 \leq \eta + \theta < 1$ *are complements of the delta Hardy-Copson type inequalities given in [51, Theorem 2.2] for* $\zeta > 1$, $\eta \geq 0$ *and* $0 \leq \eta + \theta < 1$.

If $\eta = 0$, *then the delta Hardy-Copson type inequalities in [47, Theorem 2.9] obtained for* $\zeta > 1$ *and* $0 \leq \theta < 1$ *are extended to the cases* $\zeta < 0$ *and* $\theta < 1$ *by the delta variant of Hardy-Copson type inequalities* (3.9)-(3.10).

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [51, Theorem 2.2] and [47, Theorem 2.9] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.11)-(3.12), which are obtained for $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this corollary.

Remark 5. If the time scale is the set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_2 = 1$ in (3.9)-(3.12). Hence for $0 < \eta + \zeta$, inequalities (3.9)-(3.12) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.9) reduces to the following inequality as

$$\int_a^{\infty} \frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \mathrm{d}t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[G(t)]^{\eta+\theta-\zeta}} \mathrm{d}t, \tag{3.13}$$

where $\eta \geq 0$, $\zeta < 0$ and $0 \leq \eta + \theta < 1$ and the functions G and H is defined as in (3.8).

For the continuous case, when $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, the first Hardy-Copson type inequality was established in [51, Corollary 2.2] and [28, Remark 3.22] for the given aforementioned functions G and H. By this remark, these results are extended to the cases $\zeta<0,\ \eta\geq 0$ and $\eta+\theta\leq 0$ by the novel continuous Hardy-Copson type inequality (3.13).

Remark 6. If the time scale is the set of natural numbers, then for all $t\in\mathbb{N}$, the backward jump operator results in $\rho(t)=t-1$ in (3.9)-(3.12). Let us assume that there exist $L_2>0$ such that $1\leq\frac{G(t-1)}{G(t)}\leq\frac{1}{L_2}$. For $a=0,\ \zeta<0,\ \eta\geq 0$ and $0\leq\eta+\theta<1$, in the set of natural numbers, inequalities (3.9)-(3.12) become novel discrete Hardy-Copson type inequalities, two of which obtained from (3.9) and (3.11) can be written as follows, respectively. For $\eta+\zeta>1$, we have

$$\sum_{t=1}^{\infty}\frac{z(t)[H(t-1)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}}\leq\left[\frac{L_2(\eta+\zeta)}{1-\eta-\theta}\right]^{\zeta}\sum_{t=1}^{\infty}\frac{z(t)h^{\zeta}(t)[H(t-1)]^{\eta}}{[G(t-1)]^{\eta+\theta-\zeta}}$$

and for $0<\eta+\zeta<1$, we have

$$\sum_{t=1}^{\infty}\frac{z(t)[H(t)]^{\eta+\zeta}}{[G(t-1)]^{\eta+\theta}}\leq\left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta}\sum_{t=1}^{\infty}\frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[G(t-1)]^{\eta+\theta-\zeta}},$$

where the series G and H are defined as in Remark 3.

The discrete Hardy-Copson type inequality obtained by Bennett [10, Corollary 5] or Leindler [35, Proposition 3] for $\zeta>1,\ \eta=0,\ 0\leq\theta<1$ and the ones obtained in [28, Remark 3.23] for $\zeta>1,\ \eta\geq 0,\ 0\leq\eta+\theta<1$, are extended to the cases $\zeta<0,\ \eta\geq 0,\ 0\leq\eta+\theta<1$ by Theorem 10 and particularly by this remark.

Note that the condition $\eta+\zeta>1$ is automatically satisfied in [10, Corollary 5], [35, Proposition 3] and [28, Remark 3.23] while the other one, $0<\eta+\zeta<1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta<0,\ \eta\geq 0,\ 0\leq\eta+\theta<1$ and $0<\eta+\zeta<1$ for the first time, offer novelties in the current literature by this remark.

The next theorem, which is proven for $\zeta<0,\ \eta\geq 0$ and $\eta+\theta\leq 0$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta>1,\ \eta\geq 0$ and $0\leq\eta+\theta<1$. These previous Hardy-Copson type inequalities are listed as follows:

(a) The discrete inequality obtained by Copson [16, Theorem 2.1] and the discrete inequality obtained by Bennett [10, Corollary 4] or Leindler [35, Proposition 2].

(b) The continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330], the continuous inequality obtained by Copson [17, Theorem 3] and the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (9) in [42, Theorem 3].

(c) The delta inequality (1.6) in Theorem 2 obtained by Saker et al. [51, Theorem 2.1].

(d) The nabla inequality (1.11) in Theorem 6 obtained by Kayar and Kaymakçalan [28, Theorem 3.13].

Theorem 11. *For the functions $z, h, \overline{G}$ and $\overline{H}$ defined as in (1.9), suppose that there exists $L_3 > 0$ such that $1 \le \frac{\overline{G}(t)}{\overline{G}^{\rho}(t)} \le L_3$ for $t \in (a,\infty)_{\mathbb{T}}$. Let $\zeta < 0,\ \eta \ge 0$ and $\eta + \theta \le 0$ be real constants.*

(1) *If $\eta + \zeta > 1$, then we have*

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}} \nabla t \le \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-\zeta}} \nabla t \tag{3.14}$$

and

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \le \left[\frac{L_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{3.15}$$

(2) *If $0 < \eta + \zeta < 1$, then we have*

$$\int_a^{\infty} \frac{z(t)[\overline{H}^{\rho}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}} \nabla t \le \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}^{\rho}(t)]^{\eta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-\zeta}} \nabla t \tag{3.16}$$

and

$$\int_a^{\infty} \frac{z(t)[\overline{H}^{\rho}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \le \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}^{\rho}(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{3.17}$$

Proof. The combination of the techniques used in the proof of [31, Theorem 3.9] and in the proof of Theorem 9 work for the proof of this theorem. □

Remark 7. The nabla Hardy-Copson type inequalities (3.14)-(3.15) obtained for $\zeta < 0,\ \eta \ge 0$ and $\eta + \theta \le 0$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.13] for $\zeta > 1,\ \eta \ge 0$ and $0 \le \eta + \theta < 1$.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Theorem 3.13] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.16)-(3.17), which are obtained for $\zeta < 0,\ \eta \ge 0,\ \eta + \theta \le 0$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 3. *From the inequalities (3.14)-(3.17) obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing $\overline{G}^{\rho}, \overline{G}, \overline{H}^{\rho}, \overline{H}$ presented in (1.9) by $\overline{G}, \overline{G}^{\sigma}, \overline{H}, \overline{H}^{\sigma}$ defined in (1.4), respectively.*

Let the functions $z, h, \overline{G}$ and $\overline{H}$ be defined as in (1.4). Suppose that there exists $M_3 > 0$ such that $\frac{\overline{G}^{\sigma}(t)}{\overline{G}(t)} \le M_3$ for $t \in (a,\infty)_{\mathbb{T}}$. In this case for $\zeta < 0,\ \eta \ge 0$ and $\eta + \theta \le 0$, nabla Hardy-Copson type inequalities (3.14)-(3.17) become novel delta

Hardy-Copson type inequalities, two of which obtained from (3.15) *and* (3.17) *can be written as follows, respectively. For* $\eta+\zeta>1$*, we have*

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[\overline{G}^\sigma(t)]^{\eta+\theta}}\Delta t \le \left[\frac{M_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^\sigma(t)]^\eta}{[\overline{G}^\sigma(t)]^{\eta+\theta-\zeta}}\Delta t$$

and for $0<\eta+\zeta<1$*, we have*

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^\sigma(t)]^{\eta+\theta}}\Delta t \le \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[\overline{G}^\sigma(t)]^{\eta+\theta-\zeta}}\Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.14)-(3.15) *obtained for* $\zeta<0$, $\eta\ge 0$ *and* $\eta+\theta\le 0$ *are complements of the delta Hardy-Copson type inequalities given in [51, Theorem 2.1] for* $\zeta>1$, $\eta\ge 0$ *and* $0\le\eta+\theta<1$.

If $\eta=0$*, then the delta Hardy-Copson type inequalities in [47, Theorem 2.5] obtained for* $\zeta>1$ *and* $0\le\theta<1$ *are extended to the cases* $\zeta<0$ *and* $\theta\le 0$ *by the delta variant of nabla Hardy-Copson type inequalities* (3.14)-(3.15).

Note that the condition $\eta+\zeta>1$ is automatically satisfied in [51, Theorem 2.1] and [47, Theorem 2.5] while the other one, $0<\eta+\zeta<1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.16)-(3.17), which are obtained for $\zeta<0$, $\eta\ge 0$, $\eta+\theta\le 0$ and $0<\eta+\zeta<1$ for the first time, offer novelties in the current literature by this corollary.

Remark 8. If the time scale is the set of real numbers, then for all $t\in\mathbb{R}$, the backward jump operator results in $\rho(t)=t$ and $L_3=1$ in (3.14)-(3.17). Hence for $0<\eta+\zeta$, inequalities (3.14)-(3.17) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.14) reduces to the following inequality as

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\mathrm{d}t \le \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}}\mathrm{d}t, \tag{3.18}$$

where $\zeta<0$, $\eta\ge 0$ and $\eta+\theta\le 0$ and the functions $\overline{G}$ and $\overline{H}$ is defined as

$$\overline{G}(t)=\int_a^t z(s)\mathrm{d}s \quad \text{and} \quad \overline{H}(t)=\int_t^\infty z(s)h(s)\mathrm{d}s. \tag{3.19}$$

Inequality (3.18) is novel and generalizes the continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330] and the continuous inequality obtained by Copson [17, Theorem 3] for $\zeta>1$, $\eta=0$ and $\theta<1$ to the cases $\zeta<0$, $\eta\ge 0$ and $\eta+\theta\le 0$. Moreover inequality (3.18) extends the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (9) in [42, Theorem 3] for $\zeta>1$, $\eta\ge 0$, $\eta+\theta<1$ to the cases $\zeta<0$, $\eta\ge 0$ and $\eta+\theta\le 0$.

Remark 9. If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t-1$ in (3.14)-(3.17).

Using $\overline{G}(t) = \int_a^t z(s)\nabla s = \sum_{k=a+1}^{t} z(k)$, we have $\overline{G}^{\rho}(t) = \overline{G}(t-1) = \sum_{k=a+1}^{t-1} z(k)$. Moreover $\overline{H}(t) = \sum_{k=t+1}^{\infty} z(k)f(k)$. Let us assume that there exist $L_3 > 0$ such that $1 \leq \frac{\overline{G}(t)}{\overline{G}(t-1)} \leq L_3$. For $a = 0$, $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta \leq 0$, in the set of natural numbers, inequalities (3.14)-(3.17) become novel discrete Hardy-Copson type inequalities, two of which obtained from (3.15) and (3.17) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \leq \left[\frac{L_3^{\eta+\theta-1}(\eta+\zeta)}{1-\eta-\theta}\right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}}$$

and for $\eta + \zeta > 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t-1)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t-1)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}}.$$

The discrete Hardy-Copson type inequality obtained by Copson [16, Theorem 2.1] and the discrete Hardy-Copson type inequality obtained by Bennett [10, Corollary 4] or Leindler [35, Proposition 2] for $\zeta > 1$, $\eta = 0$, $0 \leq \theta < 1$ and the ones obtained in [28, Remark 3.18] for $\zeta > 1$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$, are extended to the cases $\zeta < 0$, $\eta \geq 0$, $\eta + \theta \leq 0$ by Theorem 11 and particularly by this remark.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [16, Theorem 2.1], [10, Corollary 4], [35, Proposition 2] and [28, Remark 3.18] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta \leq 0$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this remark.

The next theorem, which is proven for $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta > 1$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$. These previous Hardy-Copson type inequalities are listed as follows:

(a) The discrete inequality obtained by Copson [16, Theorem 2.1] and the discrete inequality obtained by Bennett [10, Corollary 4] or Leindler [35, Proposition 2].

(b) The continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330], the continuous inequality obtained by Copson [17, Theorem 3] and the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (9) in [42, Theorem 3].

(c) The delta inequality (1.6) in Theorem 2 obtained by Saker et al. [51, Theorem 2.1].

(d) The nabla inequality (1.11) in Theorem 6 obtained by Kayar and Kaymakçalan [28, Theorem 3.13].

Theorem 12. *For the functions $z, h, \overline{G}$ and $\overline{H}$ defined as in (1.9), suppose that there exists $L_4 > 0$ such that $1 \geq \dfrac{\overline{G}^{\rho}(t)}{\overline{G}(t)} \geq L_4$ for $t \in (a, \infty)_{\mathbb{T}}$. Let $\zeta < 0,\ \eta \geq 0$ and $0 \leq \eta + \theta < 1$ be real constants.*

(1) *If $\eta + \zeta > 1$, then we have*

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{L_4(\eta+\zeta)}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \nabla t \tag{3.20}$$

and

$$\int_a^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{3.21}$$

(2) *If $0 < \eta + \zeta < 1$, then we have*

$$\int_a^{\infty} \frac{z(t)[\overline{H}^{\rho}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}^{\rho}(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \nabla t \tag{3.22}$$

and

$$\int_a^{\infty} \frac{z(t)[\overline{H}^{\rho}(t)]^{\eta+\zeta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}^{\rho}(t)]^{\eta}}{[\overline{G}^{\rho}(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{3.23}$$

Proof. The combination of the techniques used in the proof of [31, Theorem 3.9] and in the proof of Theorem 9 work for the proof of this theorem. □

Remark 10. The nabla Hardy-Copson type inequalities (3.20)-(3.21) obtained for $\zeta < 0,\ \eta \geq 0$ and $0 \leq \eta + \theta < 1$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.13] for $\zeta > 1,\ \eta \geq 0$ and $0 \leq \eta + \theta < 1$.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Theorem 3.13] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.22)-(3.23), which are obtained for $\zeta < 0,\ \eta \geq 0,\ 0 \leq \eta + \theta < 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 4. *From the inequalities (3.20)-(3.23) obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing $\overline{G}^{\rho}, \overline{G}, \overline{H}^{\rho}, \overline{H}$ presented in (1.9) by $\overline{G}, \overline{G}^{\sigma}, \overline{H}, \overline{H}^{\sigma}$ defined in (1.4), respectively.*

Let the functions $z, h, \overline{G}$ and $\overline{H}$ be defined as in (1.4). Suppose that there exists $M_4 > 0$ such that $1 \geq \dfrac{\overline{G}(t)}{\overline{G}^{\sigma}(t)} \geq M_4$ for $t \in (a, \infty)_{\mathbb{T}}$. In this case for $\zeta < 0,\ \eta \geq 0$ and $0 \leq \eta + \theta < 1$, nabla Hardy-Copson type inequalities (3.20)-(3.23) become novel *delta Hardy-Copson type inequalities, two of which obtained from* (3.20) *and* (3.22) *can be written as follows, respectively. For* $\eta+\zeta>1$, *we have*

$$\int_a^{\infty}\frac{z(t)[\overline{H}^{\sigma}(t)]^{\eta+\zeta}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta}}\Delta t\leq\left[\frac{M_4(\eta+\zeta)}{1-\eta-\theta}\right]^{\zeta}\int_a^{\infty}\frac{z(t)h^{\zeta}(t)[\overline{H}^{\sigma}(t)]^{\eta}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta-\zeta}}\Delta t$$

and for $0<\eta+\zeta<1$, *we have*

$$\int_a^{\infty}\frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta}}\Delta t\leq\left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta}\int_a^{\infty}\frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[\overline{G}^{\sigma}(t)]^{\eta+\theta-\zeta}}\Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.20)-(3.21) *obtained for* $\zeta<0$, $\eta\geq 0$ *and* $0\leq\eta+\theta<1$ *are complements of the delta Hardy-Copson type inequalities given in [51, Theorem 2.1] for* $\zeta>1$, $\eta\geq 0$ *and* $0\leq\eta+\theta<1$.

If $\eta=0$, *then the delta Hardy-Copson type inequalities in [47, Theorem 2.5] obtained for* $\zeta>1$ *and* $0\leq\theta<1$ *are extended to the cases* $\zeta<0$ *and* $0\leq\theta<1$ *by the delta variant of nabla Hardy-Copson type inequalities* (3.20)-(3.21).

Note that the condition $\eta+\zeta>1$ is automatically satisfied in [51, Theorem 2.1] and [47, Theorem 2.5] while the other one, $0<\eta+\zeta<1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.22)-(3.23), which are obtained for $\zeta<0$, $\eta\geq 0$, $0\leq\eta+\theta<1$ and $0<\eta+\zeta<1$ for the first time, offer novelties in the current literature by this corollary.

Remark 11. If the time scale is the set of real numbers, then for all $t\in\mathbb{R}$, the backward jump operator results in $\rho(t)=t$ and $L_4=1$ in (3.20)-(3.23). Hence for $0<\eta+\zeta$, inequalities (3.20)-(3.23) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.20) reduces to the following inequality as

$$\int_a^{\infty}\frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}}\mathrm{d}t\leq\left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta}\int_a^{\infty}\frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}}\mathrm{d}t,\tag{3.24}$$

where $\zeta<0$, $\eta\geq 0$ and $0\leq\eta+\theta<1$ and the functions $\overline{G}$ and $\overline{H}$ is defined as in (3.19).

Inequality (3.24) is novel and generalizes the continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330] and the continuous inequality obtained by Copson [17, Theorem 3] for $\zeta>1$, $\eta=0$ and $\theta<1$ to the cases $\zeta<0$, $\eta\geq 0$ and $0\leq\eta+\theta<1$. Moreover inequality (3.24) extends the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (9) in [42, Theorem 3] for $\zeta>1$, $\eta\geq 0$, $\eta+\theta<1$ to the cases $\zeta<0$, $\eta\geq 0$ and $0\leq\eta+\theta<1$.

Remark 12. If the time scale is the set of natural numbers, then for all $t\in\mathbb{N}$, the backward jump operator results in $\rho(t)=t-1$ in (3.20)-(3.23). Let us assume that there exist $L_4 > 0$ such that $\dfrac{\overline{G}(t)}{\overline{G}(t-1)} \leq \dfrac{1}{L_4}$. For $a = 0$, $\zeta < 0$, $\eta \geq 0$ and $0 \leq \eta + \theta < 1$, in the set of natural numbers, inequalities (3.20)-(3.23) becomes novel discrete Hardy-Copson type inequalities, two of which obtained from (3.20) and (3.22) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \leq \left[\frac{L_4(\eta+\zeta)}{1-\eta-\theta}\right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}}$$

and for $0 < \eta + \zeta < 1$, we have

$$\sum_{t=1}^{\infty} \frac{z(t)[\overline{H}(t-1)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^{\zeta} \sum_{t=1}^{\infty} \frac{z(t)h^{\zeta}(t)[\overline{H}(t-1)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}},$$

where the series $\overline{G}$ and $\overline{H}$ are defined as in Remark 9.

The discrete Hardy-Copson type inequality obtained by Copson [16, Theorem 2.1] and the discrete inequality obtained by Bennett [10, Corollary 4] or Leindler [35, Proposition 2] for $\zeta > 1$, $\eta = 0$, $0 \leq \theta < 1$ and the ones obtained in [28, Remark 3.18] for $\zeta > 1$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$, are extended to the cases $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ by Theorem 12 and particularly by this remark.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [16, Theorem 2.1], [10, Corollary 4], [35, Proposition 2] and [28, Remark 3.18] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta < 0$, $\eta \geq 0$, $0 \leq \eta + \theta < 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this remark.

The next theorem, which is proven for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$. These previous Hardy-Copson type inequalities are listed as follows:

(a) The discrete inequality (1.1) obtained by Hardy as inequality (2) in [24, Theorem B], the discrete inequality obtained by Copson [16, Theorem 1.1] and the discrete inequality obtained by Bennett [10, Corollary 3] or Leindler [35, Proposition 1].

(b) The continuous inequality (1.2) obtained by Hardy as inequality (4) in [24, Theorem B], the continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330], the continuous inequality obtained by Copson [17, Theorem 1] and the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (3) in [42, Theorem 1].

(c) The delta inequality (1.8) in Theorem 4 obtained by Kayar and Kaymakçalan [28, Remark 3.2] and Saker et al. [51].

(d) The nabla inequality (1.13) in Theorem 8 obtained by Kayar and Kaymakçalan [28, Theorem 3.1].

Theorem 13. *Suppose that the functions $z, h, \overline{G}$ and H are defined as in (1.9) and the constant L_4 is defined as in Theorem 12. Let $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ be real numbers.*

(1) *If $\eta + \zeta > 1$, then we have*

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\rho(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \nabla t \tag{3.25}$$

and

$$\int_a^\infty \frac{z(t)[H^\rho(t)]^{\eta+\zeta}}{[\overline{G}^\rho(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{L_4^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\rho(t)]^\eta}{[\overline{G}^\rho(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{3.26}$$

(2) *If $0 < \eta + \zeta < 1$, then we have*

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \nabla t \tag{3.27}$$

and

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}^\rho(t)]^{\eta+\theta}} \nabla t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[\overline{G}^\rho(t)]^{\eta+\theta-\zeta}} \nabla t. \tag{3.28}$$

Proof. The combination of the techniques used in the proof of [31, Theorem 3.12] and in the proof of Theorem 9 work for the proof of this theorem. ☐

Remark 13. The nabla Hardy-Copson type inequalities (3.25)-(3.26) obtained for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.1] for $\zeta > 1$, $\eta \geq 0$ and $\eta + \theta > 1$.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Theorem 3.1] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the nabla inequalities (3.27)-(3.28), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature.

Corollary 5. *From the inequalities (3.25)-(3.28) obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing $\overline{G}^\rho, \overline{G}, H^\rho, H$ presented in (1.9) by $\overline{G}, \overline{G}^\sigma, H, H^\sigma$ defined in (1.4), respectively.*

Let the functions $z, h, \overline{G}$ and H be defined as in (1.4) and the constant M_4 be defined as in Corollary 4. In this case for $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$, nabla Hardy-Copson type inequalities (3.25)-(3.28) become novel delta Hardy-Copson type inequalities, two of which obtained from (3.26) and (3.28) can be written as follows, respectively. For $\eta + \zeta > 1$, we have

$$\int_a^\infty \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{M_4^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H(t)]^\eta}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \Delta t$$

and for $0 < \eta + \zeta < 1$, *we have*

$$\int_a^{\infty} \frac{z(t)[H^{\sigma}(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \Delta t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H^{\sigma}(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.25)-(3.26) *obtained for* $\zeta < 0$, $\eta \geq 0$ *and* $\eta + \theta > 1$ *are complements of the delta Hardy-Copson type inequalities given in [28, Remark 3.2] and [51] for* $\zeta > 1$, $\eta \geq 0$ *and* $\eta + \theta > 1$.

If $\eta = 0$, *then the delta Hardy-Copson type inequalities in [47, Theorem 2.1] obtained for* $\zeta > 1$ *and* $\theta > 1$ *are extended to the cases* $\zeta < 0$ *and* $\theta > 1$ *by the delta variant of nabla Hardy-Copson type inequalities* (3.25)-(3.26).

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [28, Remark 3.2] and [51] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.27)-(3.28), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this corollary.

Remark 14. If the time scale is the set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_4 = 1$ in (3.25)-(3.28). Hence for $0 < \eta + \zeta$, inequalities (3.25)-(3.28) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.25) reduces to the following inequality as

$$\int_a^{\infty} \frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t)]^{\eta+\theta}} \mathrm{d}t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta} \int_a^{\infty} \frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[\overline{G}(t)]^{\eta+\theta-\zeta}} \mathrm{d}t, \tag{3.29}$$

where $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$ and the functions $\overline{G}$ and H is defined as in (3.19) and (3.8), respectively.

Inequality (3.29) is novel and generalizes the continuous inequality (1.2) obtained by Hardy as inequality (4) in [24, Theorem B], the continuous inequality (1.3) obtained by Hardy et al. [26, Theorem 330], the continuous inequality obtained by Copson [17, Theorem 1] for $\zeta > 1$, $\eta = 0$, $\theta > 1$ to the cases $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$. Moreover inequality (3.29) extends the continuous inequality obtained by Pachpatte as inequality (6) in [40, Theorem 1] and Pečarić and Hanjš as inequality (3) in [42, Theorem 1] for $\zeta > 1$, $\eta \geq 0$, $\eta + \theta > 1$ to the case $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$.

Remark 15. If the time scale is the set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t - 1$ in (3.25)-(3.28). Let the constant L_4 be defined as in Remark 12. For $a = 0$, $\zeta < 0$, $\eta \geq 0$ and $\eta + \theta > 1$, in the set of natural numbers, inequalities (3.25)-(3.28) becomes novel discrete Hardy-Copson type inequalities, two of which obtained from (3.26) and (3.28) can be written as follows, respectively. For $\eta+\zeta>1$, we have

$$\sum_{t=1}^{\infty}\frac{z(t)[H(t-1)]^{\eta+\zeta}}{[\overline{G}(t-1)]^{\eta+\theta}}\leq\left[\frac{L_4^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^{\zeta}\sum_{t=1}^{\infty}\frac{z(t)h^{\zeta}(t)[H(t-1)]^{\eta}}{[\overline{G}(t-1)]^{\eta+\theta-\zeta}}$$

and for $0<\eta+\zeta<1$,

$$\sum_{t=1}^{\infty}\frac{z(t)[H(t)]^{\eta+\zeta}}{[\overline{G}(t-1)]^{\eta+\theta}}\leq\left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta}\sum_{t=1}^{\infty}\frac{z(t)h^{\zeta}(t)[H(t)]^{\eta}}{[\overline{G}(t-1)]^{\eta+\theta-\zeta}},$$

where the series $\overline{G}$ and H are defined as in Remark 9 and Remark 3, respectively.

The discrete Hardy-Copson type inequality (1.1) obtained by Hardy as inequality (2) in [24, Theorem B], the discrete inequality obtained by Copson [16, Theorem 1.1] and the discrete inequality obtained by Bennett [10, Corollary 3] or Leindler [35, Proposition 1] for $\zeta>1,\ \eta=0,\ \theta>1$ and the ones obtained in [28, Remark 3.7] for $\zeta>1,\ \eta\geq 0,\ \eta+\theta>1$, are extended to the cases $\zeta<0,\ \eta\geq 0,\ \eta+\theta>1$ by Theorem 13 and particularly by this remark.

Note that the condition $\eta+\zeta>1$ is automatically satisfied in [24, Theorem B], [16, Theorem 1.1], [10, Corollary 3], [35, Proposition 1] and [28, Remark 3.7] while the other one, $0<\eta+\zeta<1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta<0,\ \eta\geq 0$, $\eta+\theta>1$ and $0<\eta+\zeta<1$ for the first time, offer novelties in the current literature by this remark.

The next theorem, which is proven for $\zeta<0,\ \eta\geq 0$ and $\eta+\theta>1$, provides complements of some of the previous Hardy-Copson type inequalities given for $\zeta>1,\ \eta\geq 0$ and $\eta+\theta>1$. These previous Hardy-Copson type inequalities are listed as follows:

(a) The discrete inequality obtained by Bennett [10, Corollary 6] or Leindler [35, Proposition 4].
(b) The continuous inequality obtained by Saker et al. [51, Corollary 2.3] and by Kayar and Kaymakçalan [28, Remark 3.11].
(c) The delta inequality (1.7) in Theorem 3 obtained by Saker et al. [51, Theorem 2.3].
(d) The nabla inequality (1.12) in Theorem 7 obtained by Kayar and Kaymakçalan [28, Theorem 3.8].

Theorem 14. *Suppose that the functions z,h,G and $\overline{H}$ are defined as in (1.9) and the constant L_2 is defined as in Theorem 10. Let $\zeta<0,\ \eta\geq 0$ and $\eta+\theta>1$ be real numbers.*

(1) *If $\eta+\zeta>1$, then we have*

$$\int_a^{\infty}\frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G^{\rho}(t)]^{\eta+\theta}}\nabla t\leq\left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^{\zeta}\int_a^{\infty}\frac{z(t)h^{\zeta}(t)[\overline{H}(t)]^{\eta}}{[G^{\rho}(t)]^{\eta+\theta-\zeta}}\nabla t \tag{3.30}$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t \leq \left[\frac{L_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{3.31}$$

(2) *If* $0<\eta+\zeta<1$, *then we have*

$$\int_a^\infty \frac{z(t)[\overline{H}^\rho(t)]^{\eta+\zeta}}{[G^\rho(t)]^{\eta+\theta}}\nabla t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^\rho(t)]^\eta}{[G^\rho(t)]^{\eta+\theta-\zeta}}\nabla t \tag{3.32}$$

and

$$\int_a^\infty \frac{z(t)[\overline{H}^\rho(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\nabla t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^\rho(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}\nabla t. \tag{3.33}$$

Proof. The combination of the techniques used in the proof of [31, Theorem 3.4] and in the proof of Theorem 9 work for the proof of this theorem. □

Remark 16. The nabla Hardy-Copson type inequalities (3.30)-(3.31) obtained for $\zeta<0,\ \eta\geq 0$ and $\eta+\theta>1$ are complements of the nabla Hardy-Copson type inequalities given in [28, Theorem 3.8] for $\zeta>1,\ \eta\geq 0$ and $\eta+\theta>1$.

Note that the condition $\eta+\zeta>1$ is automatically satisfied in [28, Theorem 3.8] while the other one, $0<\eta+\zeta<1$, has not appeared in the literature before. Therefore the nabla inequalities (3.32)-(3.33), which are obtained for $\zeta<0,\ \eta\geq 0,\ \eta+\theta>1$ and $0<\eta+\zeta<1$ for the first time, offer novelties in the current literature.

Corollary 6. *From the inequalities* (3.30)-(3.33) *obtained by the nabla calculus, we can get the dual inequalities in the delta setting by replacing* $G^\rho, G, \overline{H}^\rho, \overline{H}$ *presented in* (1.9) *by* $G, G^\sigma, \overline{H}, \overline{H}^\sigma$ *defined in* (1.4)*, respectively.*

Let the functions z, h, G *and* $\overline{H}$ *be defined as in* (1.4) *and the constant* M_2 *be defined as in Corollary 2. In this case for* $\zeta<0,\ \eta\geq 0$ *and* $\eta+\theta>1$*, nabla Hardy-Copson type inequalities* (3.30)-(3.33) *become novel delta Hardy-Copson type inequalities, two of which obtained from* (3.31) *and* (3.33) *can be written as follows, respectively. For* $\eta+\zeta>1$*, we have*

$$\int_a^\infty \frac{z(t)[\overline{H}^\sigma(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}}\Delta t \leq \left[\frac{M_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}^\sigma(t)]^\eta}{[G^\sigma(t)]^{\eta+\theta-\zeta}}\Delta t$$

and for $0<\eta+\zeta<1$*, we have*

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}}\Delta t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[G^\sigma(t)]^{\eta+\theta-\zeta}}\Delta t.$$

The delta versions of the nabla Hardy-Copson type inequalities (3.30)-(3.31) *obtained for* $\zeta<0,\ \eta\geq 0$ *and* $\eta+\theta>1$ *are complements of the delta Hardy-Copson type inequalities given in [51, Theorem 2.3] for* $\zeta>1,\ \eta\geq 0$ *and* $\eta+\theta>1$.

If $\eta = 0$, then the delta Hardy-Copson type inequalities in [47, Theorem 2.10] obtained for $\zeta > 1$ and $\theta > 1$ are extended to the cases $\zeta < 0$ and $\theta > 1$ by the delta variant of nabla Hardy-Copson type inequalities (3.30)-(3.31).

Note that the condition $\eta+\zeta > 1$ is automatically satisfied in [51, Theorem 2.3] and [47, Theorem 2.10] while the other one, $0 < \eta+\zeta < 1$, has not appeared in the literature before. Therefore the delta analogues of nabla Hardy-Copson type inequalities (3.32)-(3.33), which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta+\theta > 1$ and $0 < \eta+\zeta < 1$ for the first time, offer novelties in the current literature by this corollary.

Remark 17. If the time scale is set of real numbers, then for all $t \in \mathbb{R}$, the backward jump operator results in $\rho(t) = t$ and $L_2 = 1$ in (3.30)-(3.33). Hence for $0 < \eta+\zeta$, inequalities (3.30)-(3.33) coincide and their delta versions become exactly the same inequalities as them. Therefore together with its coincident inequalities, inequality (3.30) reduces to the following inequality as

$$\int_a^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\mathrm{d}t \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}\mathrm{d}t, \tag{3.34}$$

where $\zeta < 0$, $\eta \geq 0$ and $\eta+\theta > 1$ and the functions G and $\overline{H}$ is defined as in (3.8) and (3.19), respectively.

For the continuous case, when $\zeta > 1$, $\eta \geq 0$ and $\eta+\theta > 1$, the first Hardy-Copson type inequality was established in [51, Corollary 2.3] and [28, Remark 3.11] for the given aforementioned functions G and $\overline{H}$. By this remark, these inequalities are extended to the cases $\zeta < 0$, $\eta \geq 0$ and $\eta+\theta > 1$ by the novel continuous Hardy-Copson type inequality (3.34).

Remark 18. If the time scale is set of natural numbers, then for all $t \in \mathbb{N}$, the backward jump operator results in $\rho(t) = t-1$ in (3.30)-(3.33). Let the constant L_2 be defined as in Remark 6. For $a = 0$, $\zeta < 0$, $\eta \geq 0$ and $\eta+\theta > 1$, in the set of natural numbers, inequalities (3.30)-(3.33) become novel discrete Hardy-Copson type inequalities, two of which obtained from (3.31) and (3.33) can be written as follows, respectively. For $\eta+\zeta > 1$, we have

$$\sum_{t=1}^\infty \frac{z(t)[\overline{H}(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \leq \left[\frac{L_2^{\eta+\theta-1}(\eta+\zeta)}{\eta+\theta-1}\right]^\zeta \sum_{t=1}^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}$$

and for $0 < \eta+\zeta < 1$, we have

$$\sum_{t=1}^\infty \frac{z(t)[\overline{H}(t-1)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}} \leq \left[\frac{\eta+\zeta}{\eta+\theta-1}\right]^\zeta \sum_{t=1}^\infty \frac{z(t)h^\zeta(t)[\overline{H}(t-1)]^\eta}{[G(t)]^{\eta+\theta-\zeta}},$$

where the series $\overline{H}$ and G are defined as in Remark 9 and Remark 3, respectively.

The discrete Hardy-Copson type inequality obtained by Bennett [10, Corollary 6] or Leindler [35, Proposition 4] for $\zeta > 1$, $\eta = 0$, $\theta > 1$ and the ones obtained in

[28, Remark 3.12] for $\zeta > 1$, $\eta \geq 0$, $\eta + \theta > 1$, are extended to the cases $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ by Theorem 14 and particularly by this remark.

Note that the condition $\eta + \zeta > 1$ is automatically satisfied in [10, Corollary 6], [35, Proposition 4] and [28, Remark 3.12] while the other one, $0 < \eta + \zeta < 1$, has not appeared in the literature before. Therefore the discrete Hardy-Copson type inequalities, which are obtained for $\zeta < 0$, $\eta \geq 0$, $\eta + \theta > 1$ and $0 < \eta + \zeta < 1$ for the first time, offer novelties in the current literature by this remark.

4. Applications

This section is devoted to one of the applications of Hardy-Copson inequalities obtained in Section 3. We only employ the delta versions of the nabla inequalities (3.3) and (3.11) presented in Theorem 9 and Theorem 10, respectively, to find nonoscillation criteria for the related half linear dynamic equations. For the other delta inequalities including the term $H^\sigma(t)$, the similar process can be followed and similar theorems can be obtained.

Before proving nonoscillation criteria, let us show Hardy-Copson inequalities that we will use. The nabla inequalities (3.3) and (3.11) presented in Theorem 9 and Theorem 10, respectively, become the following delta Hardy-Copson inequalities.

Corollary 7. *Let the functions z, h, G and H be defined as in (1.4). Let $\zeta < 0$, $\eta \geq 0$ and $0 < \eta + \zeta < 1$.*

(i) *If $\eta + \theta \leq 0$, then we have delta Hardy-Copson inequality*

$$\int_a^\infty \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[G^\sigma(t)]^{\eta+\theta}}\Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[G^\sigma(t)]^{\eta+\theta-\zeta}}\Delta t. \tag{4.1}$$

(ii) *If $0 \leq \eta + \theta < 1$, then we have delta Hardy-Copson inequality*

$$\int_a^\infty \frac{z(t)[H^\sigma(t)]^{\eta+\zeta}}{[G(t)]^{\eta+\theta}}\Delta t \leq \left[\frac{\eta+\zeta}{1-\eta-\theta}\right]^\zeta \int_a^\infty \frac{z(t)h^\zeta(t)[H^\sigma(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}\Delta t. \tag{4.2}$$

We focus on the following second order half linear dynamic equations

$$\Delta(r(t)\Phi(x))^\Delta + c(t)\Phi(x^\sigma) = 0 \tag{4.3}$$

and

$$\Delta(R(t)\Phi(y))^\Delta + C(t)\Phi(y^\sigma) = 0 \tag{4.4}$$

where $r, c, R, C \in C_{rd}(I, \mathbb{R})$, $\Phi(x) = |x|^{p-1} sgn x$ for $p > 1$ and $R(t), r(t) \neq 0$. Moreover, $I := [a, b]_\mathbb{T}$ with $a < \rho(b)$ and $I_a := [a, \infty)$ in case $sup\mathbb{T} = \infty$. For the detailed information about half linear dynamic equations and their oscillation properties, we refer [2, 53, 54]. The oscillation theory of second order half linear dynamic equations has been extensively studied and many techniques have been developed to find conditions for oscillation and nonoscillation of equation (4.3). In order to focus on the two important methods, which are Reid Roundabout Theorem and Sturm Comparison Theorem, we need to give the fundamental definitions of the oscillation theory.

Definition 1 ([53])**.** A solution x of equation (4.3) has a generalized zero at the point t if $x(t)=0$. A solution x of equation (4.3) has a generalized zero in $(t,\sigma(t))$ if $r(t)y(t)y(\sigma(t))<0$. Equation (4.3) is disconjugate on the interval I, if there is no nontrivial solution of equation (4.3) with two (or more) generalized zeros in I.

Definition 2 ([53])**.** If there exists $c\in[a,\infty)$ such that equation (4.3) is disconjugate on $[c,d]$ for every $d>c$, then it is called nonoscillatory (on $[a,\infty)$).

Proposition 1 (Reid Roundabout Theorem, [53], Section 5, Theorem 2)**.** *Let*

$$U=U(a;b)=\{\chi\colon\ \chi\in C^1_{rd}(I,\mathbb{R})\ \text{with}\ \chi(a)=\chi(b)=0\}$$

be the class of admissible functions. The following are equivalent.

(a) *The p-degree functional F on U, which is defined as*

$$F(\chi;a,b)=\int_a^b\left[r(t)|\chi^\Delta(t)|^p-c(t)|\chi^\sigma(t)|^p\Delta t\right],$$

is positive definite on U provided $F(\chi)\geq 0$ for all $\chi\in U$ and $F(\chi)=0$ if and only if $\chi=0$.

(b) *Equation (4.3) is disconjugate on I and so, it is nonoscillatory.*

Proposition 2 (Sturm type Comparison Theorem, [53], Section 6, Theorem 3)**.** *Suppose that $R(t)\geq r(t)$ and $c(t)\geq C(t)$ for $t\in I^\kappa\subseteq\mathbb{T}^\kappa$. If equation (4.3) is disconjugate (nonoscillatory) on I (on $[a,\infty)$), then so is equation (4.4).*

Now we are ready to present necessary and sufficient condition for the nonoscillation of the half linear dynamic equation (4.3).

Theorem 15. *Suppose that the functions z,h,G and H are defined as in (1.9). Let $\zeta<0,\ \eta\geq 0,\ 0<\eta+\zeta<1$ and $p>1$. We define $E=\left[\dfrac{1-\eta-\theta}{\eta+\zeta}\right]^\zeta$.*

(1) *For $\eta+\theta\leq 0$ and for $H(t)\in U$, if $r(t)=\dfrac{z^{1-p}(t)\,h^{\zeta-p}(t)\,[H^\sigma(t)]^\eta}{[G^\sigma(t)]^{\eta+\theta-\zeta}}$ and $c(t)=E\,\dfrac{z(t)\,[H^\sigma(t)]^{\eta+\zeta-p}}{[G^\sigma(t)]^{\eta+\theta}}$, then equation (4.3) is nonoscillatory.*

(2) *For $0\leq\eta+\theta<1$ and for $H(t)\in U$, if $r(t)=\dfrac{z^{1-p}(t)\,h^{\zeta-p}(t)\,[H^\sigma(t)]^\eta}{[G(t)]^{\eta+\theta-\zeta}}$ and $c(t)=E\,\dfrac{z(t)\,[H^\sigma(t)]^{\eta+\zeta-p}}{[G(t)]^{\eta+\theta}}$, then equation (4.3) is nonoscillatory.*

Proof. We follow the technique used in [55, Claim 3.7]. Let us choose $a>0$ for $a\in\mathbb{T}$. For the nonnegative functions h and z, let us define an admissible function $H\in U$ introduced in Proposition 1 as $H(t)=\displaystyle\int_a^t z(s)h(s)\Delta s$. It is obvious that $\Delta H(t)=z(t)h(t)$.

(1) We have

$$\int_a^b \left[r(t)|H^{\Delta}(t)|^p - c(t)|H^{\sigma}(t)|^p\right]\Delta t = \int_a^b [r(t)[z(t)h(t)]^p - c(t)[H^{\sigma}(t)]^p]\,\Delta t$$

$$= \int_a^b \left[\frac{z^{1-p}(t)\,h^{\zeta-p}(t)\,[H^{\sigma}(t)]^{\eta}}{[G^{\sigma}(t)]^{\eta+\theta-\zeta}}[z(t)h(t)]^p \;-E\,\frac{z(t)\,[H^{\sigma}(t)]^{\eta+\zeta-p}}{[G^{\sigma}(t)]^{\eta+\theta}}(H^{\sigma}(t))^p\right]\Delta t$$

$$= \int_a^b \left[\frac{z(t)h^{\zeta}(t)[H^{\sigma}(t)]^{\eta}}{[G^{\sigma}(t)]^{\eta+\theta-\zeta}} - E\frac{z(t)[H^{\sigma}(t)]^{\eta+\zeta}}{[G^{\sigma}(t)]^{\eta+\theta}}\right]\Delta t \geq 0$$

because of the delta Hady-Copson inequality (4.1). Hence, equation (4.3) is nonoscillatory by Proposition 1.

(2) The proof can be obtained from the proof of (1) by replacing $G^{\sigma}(t)$ by $G(t)$ and by using delta Hady-Copson inequality (4.2).

□

Theorem 16. *Let the assumptions of Theorem 15 hold and $R(t) \geq r(t)$ and $C(t) \leq c(t)$ on I^{κ}.*

(i) *If $r(t)$ and $c(t)$ are defined as in (1) of Thereom 15, then equation (4.4) is nonoscillatory.*

(ii) *If $r(t)$ and $c(t)$ are defined as in (2) of Thereom 15, then equation (4.4) is nonoscillatory.*

Proof. After using the fact that if the sequences $r(t)$ and $c(t)$ are defined as in (1) and (2) of Theorem 15, then equation (4.3) is nonoscillatory, if we employ the Sturm type comparison theorem (Proposition 2), then we can prove (i) and (ii), respectively. □

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