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ON n-DERIVATIONS OF GENERALIZED MATRIX ALGEBRAS

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Abstract. In this article, we study n-derivations on generalized matrix algebras under certain restrictions and find that every n-derivation is an extremal n-derivation on generalized matrix algebras.

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1. HISTORICAL DEVELOPMENT

As far as we know numerous algebraists studied n-derivations on variety of rings and algebras which can be seen in [1–4, 6–15, 18] and bibliographic content existing therein. In the year 1993, Brešar et al. [6] proved that 'every biderivation over a noncommutative prime ring can be described as inner biderivation'. Also, in [8] Brešar investigated biderivations on semiprime rings. The readers are encouraged to read the survey paper [10, Section 3] where applications of biderivations to other fields are also described. Benkovič in [4] defined the concept of an extremal biderivation and proved that 'under certain conditions a biderivation of a triangular algebra is a sum of an extremal and an inner biderivation.' Ghosseiri [13] showed that 'every biderivation of upper triangular matrix rings is decomposed into the sum of three biderivations D, ψ and Δ, where $D(E_{11}, E_{11}) = 0$, ψ is an extremal biderivation and Δ is a special kind of biderivation.' Moreover, they proved that ' every biderivation of upper triangular matrices over a noncommutative prime ring is inner which extended some results due to Benkovič [4].' Du and Wang [12] proved that 'under certain conditions a biderivation of a generalized matrix algebra is a sum of an extremal and an inner biderivation.' Also they considered the question 'when a biderivation of a generalized matrix algebra is an inner biderivation?' and showed that 'every biderivation of a full matrix algebra over a unital algebra is inner.' Apart from associative algebras or rings, many authors studied biderivation and related maps on various types of Lie algebras for example see [7, 11] and references therein.

In [18], Wang et al. explored 'n-derivations ($n \geq 3$) on a certain class of triangular algebras.' Also, they put on their major findings on upper triangular matrix algebras and nest algebras. In the light of above literature, we study the n-derivations on generalized matrix algebras and prove that every n-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ is an extremal n-derivation ζ such that $\zeta(x_1,x_2,\ldots,x_n)=[x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$, where $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$ under certain restrictions.

2. Basic Definitions & Preliminaries

Let $\mathcal{A}$ be an algebra over a commutative ring R with unity. For any $x,y\in\mathcal{A}$, $[x,y]=xy-yx$ denotes the commutator and $\mathfrak{Z}(\mathcal{A})$ denote the center of $\mathcal{A}$. An R-linear map $d\colon \mathcal{A}\to\mathcal{A}$ is said to be a derivation if $d(xy)=d(x)y+xd(y)$ for all $x,y\in\mathcal{A}$. If derivation d takes form $d(x)=[x,a]$ for some fixed $a\in\mathcal{A}$, then d is called an inner derivation on $\mathcal{A}$.

An n-linear map $\phi\colon \mathcal{A}\times\mathcal{A}\times\cdots\times\mathcal{A}\to\mathcal{A}$ is said to be an n-derivation, if it is a derivation in each component. In particular, a 2-derivation is a biderivation. A permuting n-derivation $\zeta\colon \mathcal{A}\times\mathcal{A}\times\cdots\times\mathcal{A}\to\mathcal{A}$ is said to be an extremal n-derivation if it is of the form $\zeta(x_1,x_2,\ldots,x_n)=[x_1,[x_2,\ldots,[x_n,a]\ldots]]$ for all $x_1,x_2,\ldots,x_n\in\mathcal{A}$, where $a\in\mathcal{A}$ and $a\notin\mathfrak{Z}(\mathcal{A})$ such that $[[\mathcal{A},\mathcal{A}],a]=0$. An extremal 2-derivation is said to be an extremal biderivation. If $\mathcal{A}$ is a noncommutative algebra, then the map $\phi(x,y)=\lambda[x,y]$ for all $x,y\in\mathcal{A}$, where $\lambda\in\mathfrak{Z}(\mathcal{A})$ is called an inner biderivation.

Let A and B be two unital algebras with unity 1_{A} and 1_{B}, respectively. A *Morita context* consists of two unital R-algebras A and B, two bimodules (A, B)-bimodule M and (B, A)-bimodule N, and two bimodule homomorphisms called the bilinear pairings $\xi_{\mathrm{MN}}\colon \mathrm{M}\underset{\mathrm{B}}{\otimes}\mathrm{N}\longrightarrow\mathrm{A}$ and $\Omega_{\mathrm{NM}}\colon \mathrm{N}\underset{\mathrm{A}}{\otimes}\mathrm{M}\longrightarrow\mathrm{B}$ satisfying the following commutative diagrams:

$$\begin{array}{ccc} \mathrm{M}\underset{\mathrm{B}}{\otimes}\mathrm{N}\underset{\mathrm{A}}{\otimes}\mathrm{M} & \xrightarrow{\xi_{\mathrm{MN}}\otimes I_{\mathrm{M}}} & \mathrm{A}\underset{\mathrm{A}}{\otimes}\mathrm{M} \\ \Big\downarrow{\scriptstyle I_{\mathrm{M}}\otimes\Omega_{\mathrm{NM}}} & & \Big\downarrow{\scriptstyle\cong} \\ \mathrm{M}\underset{\mathrm{B}}{\otimes}\mathrm{B} & \xrightarrow{\cong} & \mathrm{M} \end{array} \quad\text{and}\quad \begin{array}{ccc} \mathrm{N}\underset{\mathrm{A}}{\otimes}\mathrm{M}\underset{\mathrm{B}}{\otimes}\mathrm{N} & \xrightarrow{\Omega_{\mathrm{NM}}\otimes I_{\mathrm{N}}} & \mathrm{B}\underset{\mathrm{B}}{\otimes}\mathrm{N} \\ \Big\downarrow{\scriptstyle I_{\mathrm{N}}\otimes\xi_{\mathrm{MN}}} & & \Big\downarrow{\scriptstyle\cong} \\ \mathrm{N}\underset{\mathrm{A}}{\otimes}\mathrm{A} & \xrightarrow{\cong} & \mathrm{N}\,. \end{array}$$

If $(\mathrm{A},\mathrm{B},\mathrm{M},\mathrm{N},\ \xi_{\mathrm{MN}},\Omega_{\mathrm{NM}})$ is a Morita context (refer [17] basic properties of Morita context), then the set

$$\begin{bmatrix} \mathrm{A} & \mathrm{M} \\ \mathrm{N} & \mathrm{B} \end{bmatrix}=\left\{\begin{bmatrix} a & m \\ n & b \end{bmatrix}\,\middle|\, a\in\mathrm{A}, m\in\mathrm{M}, n\in\mathrm{N}, b\in\mathrm{B}\right\}$$

forms an R-algebra under matrix addition and matrix-like multiplication, where at least one of the two bimodules M and N is distinct from zero. This kind of R-algebra introduced by Morita [17] is generally called as *generalized matrix algebra* of order 2 and represented by

$$\mathfrak{G} = \mathfrak{G}(\mathrm{A},\mathrm{M},\mathrm{N},\mathrm{B}) = \begin{bmatrix} \mathrm{A} & \mathrm{M} \\ \mathrm{N} & \mathrm{B} \end{bmatrix}.$$

All associative algebras with nontrivial idempotents are isomorphic to generalized matrix algebras. The familiar examples of generalized matrix algebras are full matrix algebras and triangular algebras [12]. Also, if $\mathrm{N} = 0$, then $\mathfrak{G}$ is called a triangular algebra.

The center of $\mathfrak{G}$ is $\mathfrak{Z}(\mathfrak{G}) = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \, am = mb, na = bn \text{ for all } m \in \mathrm{M}, n \in \mathrm{N} \right\}$. Also, note that the center $\mathfrak{Z}(\mathfrak{G})$ consists of all diagonal matrices $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a \in \mathfrak{Z}(\mathrm{A}), b \in \mathfrak{Z}(\mathrm{B})$ and $am = mb, na = bn$ for all $m \in \mathrm{M}, n \in \mathrm{N}$. However, if we assume that M is faithful as a left A-module and also as a right B-module, then the conditions $a \in \mathfrak{Z}(\mathrm{A})$ and $b \in \mathfrak{Z}(\mathrm{B})$ become redundant and can be deleted.

Define two natural projections $\pi_{\mathrm{A}} : \mathfrak{G} \to \mathrm{A}$ and $\pi_{\mathrm{B}} : \mathfrak{G} \to \mathrm{B}$ by $\pi_{\mathrm{A}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = a$ and $\pi_{\mathrm{B}}\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = b$. Moreover, $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathrm{A})$ and $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) \subseteq \mathfrak{Z}(\mathrm{B})$ and there exists a unique algebraic isomorphism $\eta\colon \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) \to \pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$ such that $am = m\eta(a)$ and $na = \eta(a)n$ for all $a \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})), m \in \mathrm{M}$ and $n \in \mathrm{N}$.

Let 1_{A} (resp. 1_{B}) be the unit of the algebra A (resp. B) and let I be the unity of generalized matrix algebra $\mathfrak{G}$, $e = \begin{bmatrix} 1_{\mathrm{A}} & 0 \\ 0 & 0 \end{bmatrix}$, $f = I - e = \begin{bmatrix} 0 & 0 \\ 0 & 1_{\mathrm{B}} \end{bmatrix}$ and $\mathfrak{G}_{11} = e\mathfrak{G}e$, $\mathfrak{G}_{12} = e\mathfrak{G}f$, $\mathfrak{G}_{21} = f\mathfrak{G}e$, $\mathfrak{G}_{22} = f\mathfrak{G}f$. Thus $\mathfrak{G} = e\mathfrak{G}e + e\mathfrak{G}f + f\mathfrak{G}e + f\mathfrak{G}f = \mathfrak{G}_{11} + \mathfrak{G}_{12} + \mathfrak{G}_{21} + \mathfrak{G}_{22}$ where $\mathfrak{G}_{11}$ is subalgebra of $\mathfrak{G}$ isomorphic to A, $\mathfrak{G}_{22}$ is subalgebra of $\mathfrak{G}$ isomorphic to B, $\mathfrak{G}_{12}$ is $(\mathfrak{G}_{11}, \mathfrak{G}_{22})$-bimodule isomorphic to M and $\mathfrak{G}_{21}$ is $(\mathfrak{G}_{22}, \mathfrak{G}_{11})$-bimodule isomorphic to N. Also, $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ and $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$ are isomorphic to $e\mathfrak{Z}(\mathfrak{G})e$ and $f\mathfrak{Z}(\mathfrak{G})f$ respectively. Then there is an algebra isomorphisms $\eta\colon e\mathfrak{Z}(\mathfrak{G})e \to f\mathfrak{Z}(\mathfrak{G})f$ such that $am = m\eta(a)$ and $na = \eta(a)n$ for all $m \in e\mathfrak{G}f$ and $n \in f\mathfrak{G}e$. Also, through the rest of paper, it is assume that M is faithful as a left A-module and also as a right B-module.

An (A,B)-bimodule homomorphism $\mathfrak{f}\colon \mathrm{M} \to \mathrm{M}$ is of the standard form if there exist $a_0 \in \mathfrak{Z}(\mathrm{A}),\ b_0 \in \mathfrak{Z}(\mathrm{B})$ such that $\mathfrak{f}(m) = a_0 m + m b_0$ for all $m \in \mathrm{M}$. Similarly, (B,A)-bimodule homomorphism $\mathfrak{g}\colon \mathrm{N} \to \mathrm{N}$ is of the standard form if there exist $a \in \mathfrak{Z}(\mathrm{A}), b \in \mathfrak{Z}(\mathrm{B})$ such that $\mathfrak{g}(n) = na + bn$ for all $n \in \mathrm{N}$. We say that a pair of bimodule homomorphisms $\mathfrak{f}\colon \mathrm{M} \to \mathrm{M}$ and $\mathfrak{g}\colon \mathrm{N} \to \mathrm{N}$ is special if $\mathfrak{f}(m)n + m\mathfrak{g}(n) = 0 = n\mathfrak{f}(m) + \mathfrak{g}(n)m$ for all $m \in \mathrm{M}$ and $n \in \mathrm{N}$.

A special pair of bimodule homomorphisms $\mathfrak{f}\colon \mathrm{M} \to \mathrm{M}$ and $\mathfrak{g}\colon \mathrm{N} \to \mathrm{N}$ is of the standard form if there exist $a_0 \in \mathfrak{Z}(\mathrm{A}), b_0 \in \mathfrak{Z}(\mathrm{B})$ such that $\mathfrak{f}(m) = a_0 m + m b_0$ and $\mathfrak{g}(n) = -na_0 - b_0 n$ for all $m \in \mathrm{M}, n \in \mathrm{N}$.

Now we should mention some important results which are subsequently used in this article.

Lemma 1 ([9, Corollary 2.4]). *Let* $\phi\colon \mathcal{A}\times\mathcal{A}\to\mathcal{A}$ *be a biderivation. Then*

$$\phi(x,y)[u,v]=[x,y]\phi(u,v) \qquad \textit{for all } x,y,u,v\in\mathcal{A}.$$

Lemma 2 ([12, Proposition 3.3]). *Let* $\mathfrak{G}$ *be a generalized matrix algebra such that*

(1) *for each* $n\in\mathrm{N}$ *the condition* $\mathrm{M}n=0=n\mathrm{M}$ *implies* $n=0$,

(2) *every* (A,B)*-bimodule homomorphism of* M *has the standard form.*

Then each special pair of bimodule homomorphisms $\mathfrak{f}\colon \mathrm{M}\to\mathrm{M}$ *and* $\mathfrak{g}\colon \mathrm{N}\to\mathrm{N}$ *has the standard form.*

Lemma 3 ([12, Proposition 3.4]). *Suppose that every derivation of a generalized matrix algebra* $\mathfrak{G}$ *is inner. Then every special pair of bimodule homomorphisms* $\mathfrak{f}\colon \mathrm{M}\to\mathrm{M}$ *and* $\mathfrak{g}\colon \mathrm{N}\to\mathrm{N}$ *has the standard form.*

3. Key Content

In this part, we study significant results of the article and we initiate with the following basic facts:

Lemma 4. *Let* $\mathfrak{G}$ *be a generalized matrix algebra and* $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ *be a 3-derivation. If* $[x,y]=0$*, then* $\phi(x,y,z)\in\mathrm{M}+\mathrm{N}\ \forall\, x,y,z\in\mathfrak{G}$*. Moreover, suppose that*

(1) *for each* $n\in\mathrm{N}$ *the condition* $\mathrm{M}n=0=n\mathrm{M}$ *implies* $n=0$,

(2) *for each* $m\in\mathrm{M}$ *the condition* $m\mathrm{N}=0=\mathrm{N}m$ *implies* $m=0$,

then $\phi(x,y,z)=0$ *for all* $x,y,z\in\mathfrak{G}$.

Proof. Define a map $\phi_z\colon \mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ for some fix $z\in\mathfrak{G}$ by

$$\phi_z(x,y)=\phi(x,y,z) \qquad \text{for all } x,y\in\mathfrak{G}.$$

Thence ϕ_z is a biderivation on $\mathfrak{G}$ and with [9, Corollory 2.4], we find that

$$\begin{aligned}\phi_z(x,y)[u,v]&=[x,y]\phi_z(u,v)\\ \phi(x,y,z)[u,v]&=[x,y]\phi(u,v,z) \qquad \text{for all } x,y,z,u,v\in\mathfrak{G}.\end{aligned}\tag{3.1}$$

Again, we define a map $\psi_y\colon \mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ for any fixed $y\in\mathfrak{G}$ by

$$\psi_y(x,z)=\phi(x,y,z) \qquad \text{for all } x,y,z\in\mathfrak{G}.$$

It follows that

$$\phi(x,y,z)[u,v]=[x,z]\phi(u,y,v) \qquad \text{for all } x,y,z,u,v\in\mathfrak{G}.\tag{3.2}$$

On comparison of (3.1) and (3.2), we find that

$$[x,y]\phi(u,v,z)=[x,z]\phi(u,y,v) \qquad \text{for all } x,y,z,u,v\in\mathfrak{G}.\tag{3.3}$$

Further (3.2) can be rewritten as

$$\phi(x,v,y)[u,z]=[x,y]\phi(u,v,z) \qquad \text{for all } x,y,z,u,v\in\mathfrak{G}.\tag{3.4}$$

From (3.1) and (3.4), we find that

$$\phi(x,y,z)[u,v] = \phi(x,v,y)[u,z] \qquad \text{for all } x,y,z,u,v \in \mathfrak{G}. \tag{3.5}$$

In view of (3.1), we obtain

$$\phi(x,y,z)[e,m] = [x,y]\phi(e,m,z) = 0$$
$$[e,m]\phi(x,y,z) = \phi(e,m,z)[x,y] = 0.$$

Implying to $\phi(x,y,z)\mathrm{M} = 0 = \mathrm{M}\phi(x,y,z)$ for all $x,y,z \in \mathfrak{G}$ and hence we have

$$\begin{aligned} &e\phi(u,v,z)e\mathrm{M} = 0 = \mathrm{M}f\phi(u,v,z)f,\\ &f\phi(u,v,z)e\mathrm{M} = 0 = \mathrm{M}f\phi(u,v,z)e\ \forall\ u,v,z \in \mathfrak{G}. \end{aligned} \tag{3.6}$$

Since M is a faithful left A-module and a faithful right B-module, we find $e\phi(u,v,z)e = 0 = f\phi(u,v,z)f$. Hence $\phi(u,v,z) = e\phi(u,v,z)f + f\phi(u,v,z)$, that is, $\phi(u,v,z) \in \mathrm{M}+\mathrm{N}$ for all $u,v,z \in \mathfrak{G}$.

Now, suppose that the conditions (1) and (2) are true. It follows from (3.6), $f\phi(u,v,z)e = 0$. Similarly, we can show that $e\phi(u,v,z)f = 0$. Therefore, $\phi(u,v,z) = 0$ for all $u,v,z \in \mathfrak{G}$. □

Now it is easy to verify following lemma:

Lemma 5. *Let $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}$ be a 3-derivation. Then*

(1) $\phi(x,y,1) = \phi(1,x,y) = \phi(x,1,y) = 0$ *for all* $x,y \in \mathfrak{G}$,
(2) $\phi(x,y,0) = \phi(0,x,y) = \phi(x,0,y) = 0$ *for all* $x,y \in \mathfrak{G}$,
(3) $\phi(e,e,e) = -\phi(e,e,f) = \phi(e,f,f) = -\phi(f,f,f)$
$= \phi(f,e,f) = -\phi(e,f,e) = -\phi(f,e,e) = \phi(f,f,e)$.

Proposition 1. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring* R *and $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}$ be a 3-derivation on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{B})$,
(2) *If* $\alpha a = 0, \alpha \in \mathfrak{Z}(\mathfrak{G}), 0 \neq a \in \mathfrak{G}$, *then* $\alpha = 0$.
(3) *If* $\mathrm{MN} = 0 = \mathrm{NM}$, *then at least one of the algebras* A *and* B *is noncommutative.*
(4) *Every special pair of bimodule homomorphisms has the standard form.*

Then every 3*-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G} \to \mathfrak{G}$ is an extremal* 3*-derivation ζ such that* $\zeta(x_1,x_2,x_3) = [x_1,[x_2,[x_3,\phi(e,e,e)]]]$, *where* $\phi(e,e,e) \notin \mathfrak{Z}(\mathfrak{G})$.

Proof. Let ϕ be a 3-derivation with $\phi(e,e,e) \neq 0$ and $\phi(e,e,e) \notin \mathfrak{Z}(\mathfrak{G})$. Then with Lemma 4, we see that $\phi(x,y,z) \notin \mathfrak{Z}(\mathfrak{G})$. From (3.1) it follows that

$$\phi(e,e,e)[u,v] = [e,e]\phi(u,v,e) = 0 \qquad \text{for all } u,v \in \mathfrak{G},$$
$$[x,y]\phi(e,e,e) = \phi(x,y,e)[e,e] = 0 \qquad \text{for all } x,y \in \mathfrak{G},$$

this leads to $[\phi(e,e,e),[\mathfrak{G},\mathfrak{G}]] = 0$, then the map defined by

$$\zeta(x,y,z) = [x,[y,[z,\phi(e,e,e)]]]$$

is an extremal 3-derivation of $\mathfrak{G}$. We note that

$$\begin{aligned}\zeta(e,e,e) &= [e,[e,[e,\phi(e,e,e)]]] \\ &= [e,[e,[e,e\phi(e,e,e)f - f\phi(e,e,e)e]]] \\ &= e\phi(e,e,e)f + f\phi(e,e,e)e = \phi(e,e,e).\end{aligned}$$

Let $\psi = \phi - \zeta$. Then ψ is a 3-derivation of $\mathfrak{G}$ satisfying $\psi(e,e,e) = 0$. Now we have to show that every 3-derivations $\psi = 0$ with $\psi(e,e,e) = 0$. We will prove this argument via following sequence of claims:

Claim 1. *For any* $x \in \mathrm{A} \cup \mathrm{B}$, $m \in \mathrm{M}$ *and* $n \in \mathrm{N}$, *we have*

(i) $\psi(x,y,m) = \psi(y,x,m) = \psi(x,m,y) = \psi(y,m,x) = \psi(m,x,y) = \psi(m,y,x) = 0$ *for all* $y \in \mathrm{A} \cup \mathrm{M} \cup \mathrm{B}$,

(ii) $\psi(x,y,n) = \psi(y,x,n) = \psi(x,n,y) = \psi(y,n,x) = \psi(n,x,y) = \psi(n,y,x) = 0$ *for all* $y \in \mathrm{A} \cup \mathrm{N} \cup \mathrm{B}$.

(iii) $\psi(x,y,z) = 0$ *for all* $y,z \in \mathrm{A} \cup \mathrm{B}$.

Since $\psi(e,e,e) = 0$, we find that

$$\begin{aligned}\psi(a_1,a_2,a_3) &= \psi(ea_1e,a_2,a_3) \\ &= e\psi(a_1,a_2,a_3)e + \psi(e,a_2,a_3)a_1 + a_1\psi(e,a_2,a_3) \\ &= e\psi(a_1,a_2,a_3)e + f\psi(e,ea_2e,a_3)a_1 + a_1\psi(e,ea_2e,a_3)f \\ &= e\psi(a_1,a_2,a_3)e + f\psi(e,e,ea_3e)a_2a_1 + a_1a_2\psi(e,e,ea_3e)f \\ &= e\psi(a_1,a_2,a_3)e + f\psi(e,e,e)a_3a_2a_1 + a_1a_2a_3\psi(e,e,e)f \\ &= e\psi(a_1,a_2,a_3)e \in \mathrm{A}. \end{aligned} \tag{3.7}$$

for all $a_1,a_2,a_3 \in \mathrm{A}$. Since $\psi(f,f,f) = -\psi(e,e,e) = 0$ and by similar calculation, we have $\psi(b_1,b_2,b_3) \in \mathrm{B}$ for all $b_1,b_2,b_3 \in \mathrm{B}$. Also note that $\psi(e,e,f) = -\psi(e,e,e) = 0$. Then in view of Lemma 4, we have

$$\begin{aligned}\psi(a_1,a_2,b) &= e\psi(a_1,a_2,b)f + f\psi(a_1,a_2,b)e \\ &= e\psi(ea_1e,a_2,b)f + f\psi(ea_1e,a_2,b)e \\ &= a_1\psi(e,ea_2e,b)f + f\psi(e,ea_2e,b)a_1 \\ &= a_1a_2\psi(e,e,fbf)f + f\psi(e,e,fbf)a_2a_1 \\ &= a_1a_2\psi(e,e,f)b + b\psi(e,e,f)a_2a_1 = 0. \end{aligned} \tag{3.8}$$

for all $a_1,a_2 \in \mathrm{A}$ and $b \in \mathrm{B}$. In view of (3.7), for any $x \in \mathrm{A}, y \in \mathrm{A}, m \in \mathrm{M}$, we have

$$\begin{aligned}\psi(x,y,m) &= \psi(x,y,emf) \\ &= em\psi(x,y,f) + \psi(x,y,em)f \\ &= em\psi(x,y,f)e + e\psi(x,y,m)f + f\psi(x,y,e)mf \\ &= e\psi(x,y,m)f \in \mathrm{M}.\end{aligned}$$

Similarly, we can have $\psi(x,y,m)\in \mathrm{M}$ for all $x\in \mathrm{A}\cup\mathrm{B}, y\in \mathrm{A}\cup\mathrm{M}\cup\mathrm{B}$ and $m\in\mathrm{M}$. With similar reasons, we can obtain $\psi(x,y,n)\in\mathrm{N}$ for all $x\in\mathrm{A}\cup\mathrm{B}, y\in\mathrm{A}\cup\mathrm{N}\cup\mathrm{B}, n\in\mathrm{N}$ and also rest of the cases.

For fix $y\in\mathfrak{G}$, define maps $\mathfrak{f}\colon \mathrm{M}\to\mathrm{M}$ and $\mathfrak{g}\colon\mathrm{N}\to\mathrm{N}$ by $\mathfrak{f}(m)=e\psi(e,y,m)f$ and $\mathfrak{g}(n)=f\psi(e,y,n)e$ for all $m\in\mathrm{M},\ n\in\mathrm{N}$ respectively. Then $\mathfrak{f}$ and $\mathfrak{g}$ are bimodule homomorphisms. Namely for all $a\in\mathrm{A}, b\in\mathrm{B}, m\in\mathrm{M}$ we get

$$\begin{aligned}\mathfrak{f}(amb)&=e\psi(e,y,amb)f\\&=e\psi(e,y,a)mb+a\psi(e,y,m)b+am\psi(e,y,b)f\\&=a\psi(e,y,m)b=a\mathfrak{f}(m)b.\end{aligned}$$

Similarly we obtain that $\mathfrak{g}(bna)=b\mathfrak{g}(n)a$ for all $n\in\mathrm{N}$. Moreover, we find that

$$\begin{aligned}\mathfrak{f}(m)n+m\mathfrak{g}(n)&=e\psi(e,y,m)fn+mf\psi(e,y,n)e=e\psi(e,y,mn)e=0,\\ \mathfrak{g}(n)m+n\mathfrak{f}(m)&=f\psi(e,y,n)em+ne\psi(e,y,m)f=f\psi(e,y,nm)f=0.\end{aligned}$$

By assumption (4) the bimodule homomorphisms $\mathfrak{f}$, $\mathfrak{g}$ have the standard form, then

$$\mathfrak{f}(m)=a_0m+mb_0 \text{ and } \mathfrak{g}(n)=-na_0-b_0n \text{ for } a_0\in\mathfrak{Z}(\mathrm{A}), b_0\in\mathfrak{Z}(\mathrm{B}).$$

With assumption (1), we see that $a_0\in\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ and $b_0\in\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$. We may write

$$\begin{aligned}\mathfrak{f}(m)&=(a_0+\eta^{-1}(b_0))m=\alpha_y m &&\text{for all } m\in\mathrm{M},\\ \mathfrak{g}(n)&=-n(a_0+\eta^{-1}(b_0))=-n\alpha_y &&\text{for all } n\in\mathrm{N},\end{aligned}$$

where $\alpha_y=a_0+\eta^{-1}(b_0)\in\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ (depending on y). Suppose first that $\mathrm{MN}\neq 0$ or $\mathrm{NM}\neq 0$. Then by (3.5) we have

$$\begin{aligned}\psi(e,y,m)[f,n]&=\psi(e,n,y)[f,m]\\ \Longrightarrow\ e\psi(e,y,m)n&=-e\psi(e,n,y)m\\ \alpha_y mne&=-e\psi(e,n,y)me\\ \alpha_y\mathrm{MN}&=0 \text{ for all } y\in\mathfrak{G}.\end{aligned}$$

Further with (3.4), we obtain that

$$\begin{aligned}[f,n]\psi(e,y,m)&=[f,m]\psi(e,n,y)\\ n\psi(e,y,m)f&=-m\psi(e,n,y)f\\ fn\alpha_y m&=-fm\psi(e,n,y)f\\ \eta(\alpha_y)\mathrm{NM}&=0 \qquad \text{for all } y\in\mathfrak{G}.\end{aligned}$$

The assumption (2) imply $\alpha_y=0$ or $\eta(\alpha_y)=0$ and hence $\alpha_y=0$ for all $y\in\mathfrak{G}$.

Suppose next that $\mathrm{MN}=0=\mathrm{NM}$. By assumption (3), one of A and B is noncommutative. Without loss of generality, assume B is a noncommutative algebra and let $b_1,b_2\in\mathrm{B}$ be fixed elements with $[b_1,b_2]\neq 0$. With $e\psi(e,y,m)f=\alpha_y m$, we obtain

$$\psi(e,y,m)[b_1,b_2]=\psi(e,b_2,y)[b_1,m]$$

$$\begin{aligned}\alpha_y m[b_1,b_2] &= -e\psi(e,b_2,y)mb_1\\ m\eta(\alpha_y)[b_1,b_2] &= -e\psi(e,b_2,y)mb_1\\ \mathrm{M}\eta(\alpha_y)[b_1,b_2] &= 0 \qquad \text{for all } y\in\mathfrak{G}.\end{aligned}$$

The faithfulness of M as a right B-module imply to $\eta(\alpha_y)[b_1,b_2]=0$ and from the assumption (2), we get $\eta(\alpha_y)=0$ and hence $\alpha_y=0$ for all $y\in\mathfrak{G}$. It follows that $e\psi(e,y,m)f=0$ for all $y\in\mathfrak{G}$ and $m\in\mathrm{M}$. For any $a\in\mathrm{A}$ and $y\in\mathrm{A}\cup\mathrm{M}\cup\mathrm{B}$, we have

$$\begin{aligned}\psi(a,y,m) &= e\psi(ae,y,m)f\\ &= a\psi(e,y,m)f+e\psi(a,y,m)ef=0.\end{aligned}$$

Likewise, we have $\psi(b,y,m)=0$ for all $b\in\mathrm{B}$. Therefore, $\psi(x,y,m)=0$ for all $x\in\mathrm{A}\cup\mathrm{B}, y\in\mathrm{A}\cup\mathrm{M}\cup\mathrm{B}$ and $m\in\mathrm{M}$. Similarly, we can prove the other relations of part (ii) and part (iii) also.

In view of (3.1) and (ii), for any $m\in\mathrm{M}$

$$\begin{aligned}&\psi(a_1,a_2,a_3)[e,m]=[a_1,a_2]\psi(e,m,a_3)\\ \Longrightarrow\ &\psi(a_1,a_2,a_3)m=[a_1,a_2]\psi(e,m,a_3)\\ \Longrightarrow\ &\psi(a_1,a_2,a_3)\mathrm{M}=0.\end{aligned}$$

By faithfulness of M as a left A-module implies $e\psi(a_1,a_2,a_3)e=0$ and hence $\psi(a_1,a_2,a_3)=0$ for all $a_1,a_2,a_3\in\mathrm{A}$. Taking into account (3.8), similarly we can have the other cases of part (i).

Claim 2. *For any $m\in\mathrm{M}$ and $n\in\mathrm{N}$, we have*

$$\begin{aligned}\psi(m_1,m_2,n)=\psi(m_1,n,m_2)=\psi(n,m_1,m_2)=0 &\qquad \text{for all } m_1,m_2\in\mathrm{M},\\ \psi(n_1,n_2,m)=\psi(n_1,m,n_2)=\psi(m,n_1,n_2)=0 &\qquad \text{for all } n_1,n_2\in\mathrm{N}.\end{aligned}$$

In view of Lemma 4, for any $m_1,m_2\in\mathrm{M}$ and $n\in\mathrm{N}$, we have

$$\begin{aligned}\psi(m_1,m_2,n) &= e\psi(m_1,m_2,ne)f+f\psi(m_1,m_2,ne)e\\ &= fn\psi(m_1,m_2,e)e+f\psi(m_1,m_2,n)e\\ &= f\psi(m_1,em_2,n)e=0.\end{aligned}$$

In similar manner, we can find the other relations.

Claim 3. *For any $x\in\mathrm{A}\cup\mathrm{B}, m\in\mathrm{M}$ and $n\in\mathrm{N}$, we have*

$$\begin{aligned}\psi(x,n,m)=\psi(n,x,m)=\psi(x,m,n)=\psi(n,m,x)=\psi(m,x,n)=\psi(m,n,x)=0,\\ \psi(x,m,n)=\psi(m,x,n)=\psi(x,n,m)=\psi(m,n,x)=\psi(n,x,m)=\psi(n,m,x)=0.\end{aligned}$$

For any $m\in\mathrm{M},\ n\in\mathrm{N}$, we find that

$$\begin{aligned}\psi(x,n,m) &= \psi(x,n,em)\\ &= e\psi(x,n,m)+\psi(x,n,e)m\\ &= e\psi(x,fn,m)=0.\end{aligned}$$

Similarly, we can prove rest of the cases.

Claim 4. $\psi(m,m_1,m_2)=0\ \forall\ m,m_1,m_2\in \mathrm{M}$, *and* $\psi(n,n_1,n_2)=0\ \forall\ n,n_1,n_2\in \mathrm{N}$.

In view of Lemma 4, it is easy to see that

$$\begin{aligned}\psi(m,m_1,m_2)&=e\psi(m,em_1,m_2)f+f\psi(m,em_1,m_2)e\\&=e\psi(m,m_1,m_2)f\in \mathrm{M}\qquad \text{for all } m,m_1,m_2\in \mathrm{M}.\end{aligned}$$

For fix $m',m''\in \mathrm{M}$, the map $\mathfrak{l}\colon \mathrm{M}\to\mathrm{M}$ defined by $\mathfrak{l}(m)=\psi(m,m',m'')$ for all $m\in\mathrm{M}$ is a bimodule homomorphism.

$$\begin{aligned}\mathfrak{l}(amb)&=\psi(amb,m',m'')\\&=\psi(am,m',m'')b+am\psi(b,m',m'')\\&=a\psi(m,m',m'')b+\psi(a,m',m'')mb+am\psi(b,m',m'')\\&=a\psi(m,m',m'')b=a\mathfrak{l}(m)b.\end{aligned}$$

Now we have to show that $\mathfrak{l}(m)n=0=n\mathfrak{l}(m)$ for all $m\in\mathrm{M},\ n\in\mathrm{N}$.

$$\begin{aligned}\psi(mn,m',m'')&=m\psi(n,m',m'')e+\psi(m,m',m'')n\\0&=\psi(m,m',m'')n=\mathfrak{l}(m)n.\end{aligned}$$

Similarly, we can obtain that $n\mathfrak{l}(m)=0$ for all $m\in\mathrm{M}$ and $n\in\mathrm{N}$.

Likewise, we have $\psi(n,n_1,n_2)\in\mathrm{N}$. Fix $n',n''\in\mathrm{N}$, the map $\mathfrak{h}\colon\mathrm{N}\to\mathrm{N}$ defined by $\mathfrak{h}(n)=\psi(n,n',n'')$ is a bimodule homomorphism and $\mathfrak{h}(bna)=b\mathfrak{h}(n)a$ for all $a\in\mathrm{A},b\in\mathrm{B},n\in\mathrm{N}$. Also, we can see that $\mathfrak{h}(n)m=0=m\mathfrak{h}(n)$ for all $m\in\mathrm{M},n\in\mathrm{N}$. Particularly, we see $\mathfrak{k}$ and $\mathfrak{h}$ are a special pair of bimodule homomorphisms. By assumptions (1) and (4), we get $\mathfrak{k}(m)=\gamma_{m',m''}m$ and $\mathfrak{h}(n)=-n\gamma_{m',m''}$, where $\gamma_{m',m''}\in\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$. Suppose first that $\mathrm{MN}\neq 0$ or $\mathrm{NM}\neq 0$. Then

$$\gamma_{m',m''}mn=\mathfrak{k}(m)n=0=n\mathfrak{k}(m)=n\gamma_{m',m''}m.$$

That is, $\gamma_{m',m''}\mathrm{MN}=0=\eta(\gamma_{m',m''})\mathrm{NM}$. The assumption (2) implies that $\gamma_{m',m''}=0$ or $\eta(\gamma_{m',m''})=0$. So $\gamma_{m',m''}=0$. Hence $\mathfrak{k}(m)=0=\mathfrak{h}(n)$ for all $m\in\mathrm{M},n\in\mathrm{N}$.

Suppose next that $\mathrm{MN}=0=\mathrm{NM}$. The assumption (3) implies that one of A and B is noncommutative. Without loss of generality, we assume that B is a noncommutative algebra and let $b_1,b_2\in\mathrm{B}$ be fixed elements with $[b_1,b_2]\neq 0$. By (3.1) and $\psi(m,m',m'')=\gamma_{m',m''}m$, we obtain that

$$\begin{aligned}\psi(m,m',m'')[b_1,b_2]&=[m,m']\psi(b_1,b_2,m'')\\\gamma_{m',m''}m[b_1,b_2]&=0\\\mathrm{M}\eta(\gamma_{m',m''})[b_1,b_2]&=0.\end{aligned}$$

By faithfulness of the right B-module M, $\eta(\gamma_{m',m''})[b_1,b_2]=0$ and by assumption (2), we get $\eta(\gamma_{m',m''})=0$ and hence $\gamma_{m',m''}=0$. Therefore, $\mathfrak{k}(m)=0=\mathfrak{h}(n)$ for all $m\in\mathrm{M},n\in\mathrm{N}$. This proves our claim.

Thence, we see that $\psi(x,y,z)=0$ for all $x,y,z\in\mathfrak{G}$. Since ψ is linear in each argument, we obtain that $\psi=0$. This completes the proof. $\square$

Proposition 2. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring* R *and $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ be a 3-derivation on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(\mathrm{B})$,
(2) *For each $m\in\mathrm{M}$, the condition $m\mathrm{N}=0=\mathrm{N}m$ implies $m=0$.*
(3) *For each $n\in\mathrm{N}$, the condition $\mathrm{M}n=0=n\mathrm{M}$ implies $n=0$.*
(4) *Every special pair of bimodule homomorphisms has the standard form.*

Then every 3-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\mathfrak{G}\to\mathfrak{G}$ is an extremal 3-derivation ζ such that $\zeta(x_1,x_2,x_3)=[x_1,[x_2,[x_3,\phi(e,e,e)]]]$, where $\phi(e,e,e)\notin\mathfrak{Z}(\mathfrak{G})$.

Proof. The whole proof is similar to proof of Proposition 1 except some modifications in Claim 1 & Claim 4 according to the assumptions of present proposition. Now we attempt to rewrite these proofs as follows:

Let ϕ be a 3-derivation with $\phi(e,e,e)\neq 0$ and $\phi(e,e,e)\notin\mathfrak{Z}(\mathfrak{G})$. It is easy to see $[\phi(e,e,e),[\mathfrak{G},\mathfrak{G}]]=0$ and the map $\zeta(x,y,z)=[x,[y,[z,\phi(e,e,e)]]]$ is an extremal 3-derivation of $\mathfrak{G}$. Also note that $\zeta(e,e,e)=\phi(e,e,e)$. Let $\psi=\phi-\zeta$. Then ψ is a 3-derivation of $\mathfrak{G}$ satisfying $\psi(e,e,e)=0$.

Now we have to show that every 3-derivations $\psi=0$ with $\psi(e,e,e)=0$. We will verify this argument via upcoming sequence of claims:

Claim 5. *For any $x\in\mathrm{A}\cup\mathrm{B}$, $m\in\mathrm{M}$ and $n\in\mathrm{N}$, we have*

(i) $\psi(x,y,m)=\psi(y,x,m)=\psi(x,m,y)=\psi(y,m,x)=\psi(m,x,y)=\psi(m,y,x)=0$ *for all $y\in\mathrm{A}\cup\mathrm{M}\cup\mathrm{B}$,*
(ii) $\psi(x,y,n)=\psi(y,x,n)=\psi(x,n,y)=\psi(y,n,x)=\psi(n,x,y)=\psi(n,y,x)=0$ *for all $y\in\mathrm{A}\cup\mathrm{N}\cup\mathrm{B}$.*
(iii) $\psi(x,y,z)=0$ *for all $y,z\in\mathrm{A}\cup\mathrm{B}$.*

It is easy to verify that $\psi(x,y,m)\in\mathrm{M}$ for all $y\in\mathrm{A}\cup\mathrm{M}\cup\mathrm{B},x\in\mathrm{A}\cup\mathrm{B}$ and $\psi(x,y,n)\in\mathrm{N}$ for all $y\in\mathrm{A}\cup\mathrm{N}\cup\mathrm{B},x\in\mathrm{A}\cup\mathrm{B}$ and rest of the cases follow similarly. For fix $y\in\mathfrak{G}$, define maps $\mathfrak{f}\colon \mathrm{M}\to\mathrm{M}$ and $\mathfrak{g}\colon \mathrm{N}\to\mathrm{N}$ by $\mathfrak{f}(m)=e\psi(e,y,m)f$ and $\mathfrak{g}(n)=f\psi(e,y,n)e$ for all $m\in\mathrm{M},n\in\mathrm{N}$ respectively. Then $\mathfrak{f}$ and $\mathfrak{g}$ are bimodule homomorphisms. For all $a\in\mathrm{A},b\in\mathrm{B},m\in\mathrm{M},n\in\mathrm{N}$, we have $\mathfrak{f}(amb)=a\mathfrak{f}(m)b$ and $\mathfrak{g}(bna)=b\mathfrak{g}(n)a$. Moreover, $\mathfrak{f}(m)n+m\mathfrak{g}(n)=0=\mathfrak{g}(n)m+n\mathfrak{f}(m)$ for all $m\in\mathrm{M},n\in\mathrm{N}$. By assumption (4) the bimodule homomorphism $\mathfrak{f}$ and $\mathfrak{g}$ have the standard form, then

$$\mathfrak{f}(m)=a_0m+mb_0 \text{ and } \mathfrak{g}(n)=-na_0-b_0n \qquad \textit{for } a_0\in\mathfrak{Z}(\mathrm{A}),b_0\in\mathfrak{Z}(\mathrm{B}).$$

Now we use the assumption (1) to see that $a_0\in\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ and $b_0\in\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G}))$. We may write

$$\mathfrak{f}(m)=(a_0+\eta^{-1}(b_0))m=\alpha_y m \qquad \text{for all } m\in\mathrm{M},$$
$$\mathfrak{g}(n)=-n(a_0+\eta^{-1}(b_0))=-n\alpha_y \qquad \text{for all } n\in\mathrm{N},$$

where $\alpha_y = a_0 + \eta^{-1}(b_0) \in \pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G}))$ (depending on y). By (3.5) we have

$$\psi(e,y,m)[f,n] = \psi(e,n,y)[f,m]$$
$$e\psi(e,y,m)ne = -e\psi(e,n,y)me$$
$$e\psi(e,y,m)f\mathrm{N} = 0 \qquad \text{for all } y \in \mathfrak{G}.$$

Further with (3.4), we obtain that

$$[f,n]\psi(e,y,m) = [f,m]\psi(e,n,y)$$
$$fn\psi(e,y,m)f = -fm\psi(e,n,y)f$$
$$\mathrm{N}e\psi(x,y,m)f = 0 \qquad \text{for all } y \in \mathfrak{G}.$$

The above two expression with assumption (2) imply $\mathfrak{f}(m) = e\psi(e,y,m)f = 0$ for all $y \in \mathfrak{G}$. Now for any $a \in \mathrm{A}$ and $y \in \mathrm{A} \cup \mathrm{M} \cup \mathrm{B}$, we may write

$$\begin{aligned}\psi(a,y,m) &= e\psi(ae,y,m)f\\ &= a\psi(e,y,m)f + e\psi(a,y,m)ef = 0.\end{aligned}$$

Likewise, we have $\psi(b,y,m) = 0$ for all $b \in \mathrm{B}$. Therefore, $\psi(x,y,m) = 0$ for all $x \in \mathrm{A} \cup \mathrm{B}, y \in \mathrm{A} \cup \mathrm{M} \cup \mathrm{B}$ and $m \in \mathrm{M}$. Similarly, we can prove the other relations of part (ii) and part (iii) also.

Claim 6. *For any $m \in \mathrm{M}$ and $n \in \mathrm{N}$, we have*

$$\psi(m_1,m_2,n) = \psi(m_1,n,m_2) = \psi(n,m_1,m_2) = 0 \qquad \textit{for all } m_1,m_2 \in \mathrm{M},$$
$$\psi(n_1,n_2,m) = \psi(n_1,m,n_2) = \psi(m,n_1,n_2) = 0 \qquad \textit{for all } n_1,n_2 \in \mathrm{N}.$$

Claim 7. *For any $x \in \mathrm{A} \cup \mathrm{B}, m \in \mathrm{M}$ and $n \in \mathrm{N}$, we have*

$$\psi(x,n,m) = \psi(n,x,m) = \psi(x,m,n) = \psi(n,m,x) = \psi(m,x,n) = \psi(m,n,x) = 0,$$
$$\psi(x,m,n) = \psi(m,x,n) = \psi(x,n,m) = \psi(m,n,x) = \psi(n,x,m) = \psi(n,m,x) = 0.$$

Claim 8. $\psi(m,m_1,m_2) = 0 \ \forall\, m,m_1,m_2 \in \mathrm{M}$, *and* $\psi(n,n_1,n_2) = 0 \ \forall\, n,n_1,n_2 \in \mathrm{N}$.

In view of Lemma 4 and assumptions (2), (3), we can have $\psi(m,m_1,m_2) = 0$ for all $m,m_1,m_2 \in \mathrm{M}$, and $\psi(n,n_1,n_2) = 0$ for all $n,n_1,n_2 \in \mathrm{N}$.

Thence, we see that $\psi(x,y,z) = 0$ for all $x,y,z \in \mathfrak{G}$. Since ψ is linear in each argument, we obtain that $\psi = 0$. Therefore, ϕ is an extremal 3-derivation ζ. □

At this moment, we are equipped to demonstrate a significant result of this article for $n \geq 3$ as below:

Theorem 1. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring R and $\phi\colon \mathfrak{G} \times \mathfrak{G} \times \cdots \times \mathfrak{G} \to \mathfrak{G}$ be a n-derivation (for $n \geq 3$) on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{B})$,

(2) *If* $\alpha a = 0, \alpha \in \mathfrak{Z}(\mathfrak{G}), 0 \neq a \in \mathfrak{G}$, *then* $\alpha = 0$.

(3) *If* $\mathrm{MN} = 0 = \mathrm{NM}$, *then at least one of the algebras* A *and* B *is noncommutative.*
(4) *Every special pair of bimodule homomorphisms has the standard form.*

Then every n-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ *is an extremal n-derivation* ζ *such that* $\zeta(x_1,x_2,\ldots,x_n) = [x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$, *where* $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$.

Proof. For $n=3$ result follows from the Proposition 1. For $n\geq 4$ we apply induction method. Now fix $x_4,\ldots,x_n\in\mathfrak{G}$. Set

$$\phi_{x_4,\ldots,x_n}(x_1,x_2,x_3) = \phi(x_1,x_2,x_3,x_4,\ldots,x_n) \qquad \text{for all } x_1,x_2,x_3\in\mathfrak{G}.$$

Then $\phi_{x_4,\ldots,x_n}(x_1,x_2,x_3)$ is a 3-derivation. By Proposition 1, it follows that

$$\phi_{x_4,\ldots,x_n}(x_1,x_2,x_3) = [x_1,[x_2,[x_3,\phi(e,e,e)]]] \qquad \text{for all } x_1,x_2,x_3\in\mathfrak{G},$$

where $\phi(e,e,e)\notin\mathfrak{Z}(\mathfrak{G})$ (depending on $x_4,\ldots,x_n$) with the property $[\phi(e,e,e),[\mathfrak{G},\mathfrak{G}]] = 0$. Particularly, we have that $\phi_{x_4,\ldots,x_n}(e,e,e) = y$ and so $\phi(e,e,e,x_4,\ldots,x_n) = y$ for all $y\notin\mathfrak{Z}(\mathfrak{G})$. Hence

$$\phi(x_1,x_2,\ldots,x_n) = [x_1,[x_2,[x_3,\phi(e,e,e,x_4,\ldots,x_n)]]] \text{ for all } x_1,x_2,\ldots,x_n\in\mathfrak{G}. \tag{3.9}$$

Clearly, $\phi(e,x_2,x_3,\ldots,x_n)$ is a $(n-1)$-derivation on $\mathfrak{G}$. By induction, we get

$$\phi(e,x_2,x_3,\ldots,x_n) = [x_2,[x_3,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$$

for all $x_2,\ldots,x_n\in\mathfrak{G}$, where $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$ and $[\phi(e,e,\ldots,e),[\mathfrak{G},\mathfrak{G}]] = 0$. Particularly,

$$\phi(e,e,e,x_4,\ldots,x_n) = [x_4,[x_5,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$$

for all $x_4,\ldots,x_n\in\mathfrak{G}$, where we used that $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$. From (3.9) we have

$$\phi(x_1,x_2,x_3,\ldots,x_n) = [x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$$

for all $x_1,x_2,\ldots,x_n\in\mathfrak{G}$. Hence we obtain the expected result. □

In view of [12, Proposition 3.4], we come with the following consequence:

Corollary 1. *Let* $\mathfrak{G}$ *be a generalized matrix algebra over a commutative ring* R *and* $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ *be a n-derivation (for* $n\geq 3$*) on* $\mathfrak{G}$. *Suppose that* ϕ *satisfies*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{G})) = \mathfrak{Z}(\mathrm{B})$,
(2) *If* $\alpha a = 0, \alpha\in\mathfrak{Z}(\mathfrak{G}), 0\neq a\in\mathfrak{G}$, *then* $\alpha = 0$.
(3) *If* $\mathrm{MN} = 0 = \mathrm{NM}$, *then at least one of the algebras* A *and* B *is noncommutative.*
(4) *Every derivation* $\mathfrak{G}$ *is inner.*

Then every n-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ *is an extremal n-derivation* ζ *such that* $\zeta(x_1,x_2,\ldots,x_n) = [x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$, *where* $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$.

Forthwith, we present another significant result of this article as follows:

Theorem 2. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring* R *and $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ be a n-derivation (for $n\geq 3$) on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{A}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(A)$ *and* $\pi_{B}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(B)$,
(2) *For each $m\in M$, the condition $mN=0=Nm$ implies $m=0$.*
(3) *For each $n\in N$, the condition $Mn=0=nM$ implies $n=0$.*
(4) *Every special pair of bimodule homomorphisms has the standard form.*

Then every n-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ is an extremal n-derivation ζ such that $\zeta(x_1,x_2,\ldots,x_n)=[x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$, where $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$.

Proof. In view of Proposition 2, proof is similar to the proof of Theorem 1. □

On account of [12, Proposition 3.3, Proposition 3.4], we come with the following results respectively.

Corollary 2. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring* R *and $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ be a n-derivation (for $n\geq 3$) on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{A}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(A)$ *and* $\pi_{B}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(B)$,
(2) *For each $m\in M$, the condition $mN=0=Nm$ implies $m=0$.*
(3) *For each $n\in N$, the condition $Mn=0=nM$ implies $n=0$.*
(4) *Every (A,B)-bimodule homomorphism of* M *is of the standard form.*

Then every n-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ is an extremal n-derivation ζ such that $\zeta(x_1,x_2,\ldots,x_n)=[x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$, where $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$.

Corollary 3. *Let $\mathfrak{G}$ be a generalized matrix algebra over a commutative ring* R *and $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ be a n-derivation (for $n\geq 3$) on $\mathfrak{G}$. Suppose that ϕ satisfies*

(1) $\pi_{A}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(A)$ *and* $\pi_{B}(\mathfrak{Z}(\mathfrak{G}))=\mathfrak{Z}(B)$,
(2) *For each $m\in M$, the condition $mN=0=Nm$ implies $m=0$.*
(3) *For each $n\in N$, the condition $Mn=0=nM$ implies $n=0$.*
(4) *Every derivation $\mathfrak{G}$ is inner.*

Then every n-derivation $\phi\colon \mathfrak{G}\times\mathfrak{G}\times\cdots\times\mathfrak{G}\to\mathfrak{G}$ is an extremal n-derivation ζ such that $\zeta(x_1,x_2,\ldots,x_n)=[x_1,[x_2,\ldots,[x_n,\phi(e,e,\ldots,e)]\ldots]]$, where $\phi(e,e,\ldots,e)\notin\mathfrak{Z}(\mathfrak{G})$.

4. Applications

On application of our significant results to some classical examples of generalized matrix algebras, we prevail the following consequences:

Corollary 4. *Let $\mathfrak{M}_s(R)$ be the algebra of all $s\times s$ matrices over a commutative ring* R*, where $s\geq 2$ is an integer. Then every n-derivation (for $n\geq 3$) is an extremal n-derivation on $\mathfrak{M}_s(R)$.*

Corollary 5 ([18, Theorem 2]). *Let $\mathfrak{T} = Tri(\mathrm{A},\mathrm{M},\mathrm{B})$ be a triangular algebra. If the following conditions hold:*

(1) $\pi_{\mathrm{A}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{A})$ *and* $\pi_{\mathrm{B}}(\mathfrak{Z}(\mathfrak{T})) = \mathfrak{Z}(\mathrm{B})$,
(2) *either* A *or* B *does not contain nonzero central ideals,*
(3) *each derivation of* A *is inner,*

then every n-derivation $(n \geq 3)$ $\phi\colon \mathfrak{T}\times\mathfrak{T}\times\cdots\times\mathfrak{T}\to\mathfrak{T}$ *is an extremal n-derivation.*

5. For Future Discussions

In this part, we make an effort to collect a few specific queries related to the literature of the article. But before that, we should bring up some basic notions of related subject matter. In view of [5, Propostion 2.1, 2.2], we can write the structure of automorphisms on generalized matrix algebras respectively as follows:

Lemma 6. *Let $\mathfrak{G} = \mathfrak{G}(\mathrm{A},\mathrm{M},\mathrm{N},\mathrm{B})$ be a generalized matrix algebra and $(\gamma,\delta,\mu,\nu,m_0,n_0)$ be a 6-tuple such that $\gamma\colon \mathrm{A}\to\mathrm{A}$ & $\delta\colon \mathrm{B}\to\mathrm{B}$ are algebraic automorphisms, $\mu\colon \mathrm{M}\to\mathrm{M}$ is a $\gamma-\delta-$bimodule automorphism, $\nu\colon \mathrm{N}\to\mathrm{N}$ is a $\delta-\gamma-$bimodule automorphism and $m_0\in\mathrm{M}$ & $n_0\in\mathrm{N}$ are fixed elements such that following conditions are satisfied:*

(i) $[m_0,\mathrm{N}] = 0$ *and* $(\mathrm{N},m_0) = 0$,
(ii) $[\mathrm{M},n_0] = 0$ *and* $(n_0,\mathrm{M}) = 0$,
(iii) $[\mu(m),\nu(n)] = \gamma([m,n])$ *and* $(\nu(n),\mu(m)) = \delta((n,m))$.

Then the map $\alpha_1\colon \mathfrak{G}\to\mathfrak{G}$ *defined by*

$$\alpha_1\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \gamma(a) & \gamma(a)m_0 - m_0\delta(b) + \mu(m) \\ n_0\gamma(a) - \delta(b)n_0 + \nu(n) & \delta(b) \end{bmatrix}$$

is an algebraic automorphism.

Lemma 7. *Let $\mathfrak{G} = \mathfrak{G}(\mathrm{A},\mathrm{M},\mathrm{N},\mathrm{B})$ be a generalized matrix algebra and $(\rho,\sigma,\mu,\nu,m_*,n_*)$ be a 6-tuple such that $\rho\colon \mathrm{A}\to\mathrm{B}$ & $\sigma\colon \mathrm{B}\to\mathrm{A}$ are algebraic automorphisms, $\mu\colon (\mathrm{M},+)\to(\mathrm{N},+)$ & $\nu\colon (\mathrm{N},+)\to(\mathrm{M},+)$ are group automorphisms such that $\mu(amb) = \rho(a)\mu(m)\sigma(b)$ & $\nu(bna) = \sigma(b)\nu(n)\rho(a)$ for all $a\in\mathrm{A}, b\in\mathrm{B}, m\in\mathrm{M}, n\in\mathrm{N}$ and $m_*\in\mathrm{M}$ & $n_*\in\mathrm{N}$ are fixed elements such that following conditions are satisfied:*

(i) $[m_*,\mathrm{N}] = 0$ *and* $(\mathrm{N},m_*) = 0$,
(ii) $[\mathrm{M},n_*] = 0$ *and* $(n_*,\mathrm{M}) = 0$,
(iii) $(\mu(m),\nu(n)) = \rho([m,n])$ *and* $[\nu(n),\mu(m)] = \sigma((n,m))$.

Then the map $\alpha_2\colon \mathfrak{G}\to\mathfrak{G}$ *defined by*

$$\alpha_2\left(\begin{bmatrix} a & m \\ n & b \end{bmatrix}\right) = \begin{bmatrix} \sigma(a) & m_*\rho(a) - \sigma(b)m_* + \nu(n) \\ \rho(a)n_* - n_*\sigma(b) + \mu(m) & \rho(b) \end{bmatrix}$$

is an algebraic automorphism.

Let α be an automorphism on R-algebra $\mathcal{A}$. An R-linear map $d\colon \mathcal{A} \to \mathcal{A}$ is said to be an α-derivation if $d(xy) = d(x)y + \alpha(x)d(y)\ \forall\, x,y \in \mathcal{A}$. An R-linear map $g\colon \mathcal{A} \to \mathcal{A}$ is said to be a generalized α-derivation associated with an α-derivation d if $g(xy) = g(x)y + \alpha(x)d(y)$ for all $x,y \in \mathcal{A}$. An n-linear map $\Phi\colon \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A} \to \mathcal{A}$ is said to be a generalized $\alpha - n-$derivation, if it is a generalized α-derivation in each component. In particular, a generalized $\alpha - 2-$derivation is a generalized α-biderivation. Also, if $\alpha = I_{\mathcal{A}}$, then a generalized $I_{\mathcal{A}} - n-$derivation is a generalized $n-$derivation. Now in view of [9, 16], it is reasonable to raise the following questions as:

Question 1. *What is the most general form of generalized n-derivations on triangular algebras and which constraints are needed to apply on triangular algebras?*

Question 2. *What is the most general form of generalized α-biderivations on generalized matrix algebras and which constraints are needed to apply on generalized matrix algebras?*

In general, one can also explore the following query:

Question 3. *What is the most general form of generalized $\alpha - n-$derivations on generalized matrix algebras and which constraints are needed to apply on generalized matrix algebras?*

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