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FIXED POINT THEOREMS OF ASYMPTOTICALLY REGULAR MAPS

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Abstract. We prove some fixed point theorems for asymptotically regular self-mappings, not necessarily orbitally continuous or k-continuous, satisfying weaker Proinov contraction. Our results extend several recent results in the literature and provide more answers to an open question raised by Rhoades. Some nontrivial examples are also given to illustrate our results.

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1. INTRODUCTION AND PRELIMINARIES

In 2006, Proinov [25] proved an interesting fixed point theorem which includes a wide class of well-known fixed point theorems in the existing literature (see, [11, 14–16, 18, 24]). Before stating the result by Proinov, we recall here some related definitions and notations. We also refer the reader to [1] for various related terminologies in fixed point iterative methods in metric spaces.

Definition 1. [13] Let (X,d) be a metric space and $T : X \to X$ be a mapping. A Picard iterate of T is a sequence $\{x_n\}$ in X defined by $x_n = T^n x$ for some $x \in X$.

Definition 2. [13, 25] Let T be a self-mapping on a metric space (X,d). A fixed point z of T is said to be contractive if all of the Picard iterates of T converge to this fixed point.

Note that if z is a contractive fixed point of $T : X \to X$, then z is the unique fixed point of T [13].

Definition 3. [6] A self-mapping T on a metric space (X,d) is called asymptotically regular at $x \in X$, if $\lim_{n\to\infty} d(T^n x, T^{n+1}x) = 0$. If T is asymptotically regular at all $x \in X$, then T is said to be asymptotically regular.

For some fixed point results concerning asymptotically regular mappings, we refer the reader to, e.g., [8, 9, 12, 17, 25] and references therein.

Let Φ [25] denote the class of all functions $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying: for any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < t < \delta$ implies $\varphi(t) \leq \varepsilon$. Proinov's fixed point theorem is stated as follows.

Theorem 1. *[25] Let (X,d) be a complete metric space. Let T be a continuous and asymptotically regular self-mapping on X such that:*

(i) *There exists $\varphi \in \Phi$ such that $d(Tx,Ty) \leq \varphi(D(x,y))$ for all $x,y \in X$, where $D(x,y) = d(x,y) + \gamma[d(x,Tx) + d(y,Ty)]$, $\gamma \geq 0$;*

(ii) *$d(Tx,Ty) < D(x,y)$ for all $x,y \in X$ with $x \neq y$.*

Then T has a contractive fixed point.

Moreover, if $D(x,y) = d(x,y) + d(x,Tx) + d(y,Ty)$ and φ is continuous and satisfies $\varphi(t) < t$ for all $t > 0$, then the continuity of T can be dropped.

The first conclusion of Theorem 1 still holds true if the condition (i) is replaced by another condition. More precisely, as shown in [25], the first part of Theorem 1 is equivalent to the following.

Theorem 2. *[25] Let (X,d) be a complete metric space. Let T be a continuous and asymptotically regular self-mapping on X such that:*

(i) *For any $\varepsilon > 0$ there exists $\delta > \varepsilon$ such that $\varepsilon < D(x,y) < \delta$ implies $d(Tx,Ty) \leq \varepsilon$, where $D(x,y) = d(x,y) + \gamma[d(x,Tx) + d(y,Ty)]$, $\gamma \geq 0$;*

(ii) *$d(Tx,Ty) < D(x,y)$ for all $x,y \in X$ with $x \neq y$.*

Then T has a contractive fixed point.

We see that, except the cases $0 \leq \gamma < 1$, the above result demands the continuity of the considered mapping T.

Recently, the study of contractive conditions for mappings which are not continuous at fixed points is an active research direction. The main idea is to give new notions which are weaker than continuity of the mapping such that the existence of fixed point for mappings satisfying a contractive type condition is ensured. Some results in this direction can be found in ([2–5, 18–20, 22, 23] and references therein). These results also provide answers to an open question raised by Rhoades in [26].

Let (X,d) be a metric space and $T : X \to X$ be a mapping. The orbit of T at a point $x \in X$ is the set $O(T,x) = \{x, Tx, T^2x, \cdots, T^nx, \cdots\}$. We now recall some weaker notions of continuity.

Definition 4. [7] Let (X,d) be a metric space. A mapping $T : X \to X$ is called orbitally continuous at a point $z \in X$ if for any sequence $\{x_n\} \subset O(T,x)$ for some $x \in X$, $x_n \to z$ implies $Tx_n \to Tz$ as $n \to \infty$.

Definition 5. [18] Let (X,d) be a metric space. A mapping $T : X \to X$ is called k-continuous, $k = 1, 2, \cdots$, at a point $z \in X$ if $T^kx_n \to Tz$ whenever $\{x_n\}$ is a sequence in X such that $T^{k-1}x_n \to z$ as $n \to \infty$.

Definition 6. [19] Let (X,d) be a metric space. A mapping $T : X \to X$ is called weakly orbitally continuous if the set $\{y \in X : \lim_{i\to\infty} T^{m_i}y = u \Longrightarrow \lim_{i\to\infty} TT^{m_i}y = Tu\}$ is non-empty whenever the set $\{x \in X : \lim_{i\to\infty} T^{m_i}x = u\}$ is non-empty.

Remark 1. The following observations are evident (see, e.g., [19, 20]).

(i) Continuity of T^k and k-continuity of T are independent conditions when $k > 1$.
(ii) 1- continuity of T is equivalent to the continuity of T and

$$\text{continuity} \implies 2-\text{continuity} \implies 3-\text{continuity} \implies \ldots,$$

but not conversely.
(iii) Continuity of T implies orbital continuity of T, but not conversely.
(iv) Orbital continuity of T implies weak orbital continuity of T, but the converse need not be true.
(v) k-continuous mapping of T is orbitally continuous but the converse need not be true.

In [2, 4], replacing the continuity of the mapping T in Theorem 1 by k-continuity of T and orbital continuity of T, respectively, the authors proved the following improvement of Theorem 1.

Theorem 3. *[2, 4] Let (X,d) be a complete metric space. Let T be an asymptotically regular self-mapping on X satisfying (i) and (ii) of Theorem 1. Suppose T is k-continuous for some $k \in \mathbb{N}$ or T is orbitally continuous. Then T has a contractive fixed point.*

Hicks and Rhoades [10] introduced the following notion of orbital lower semi-continuity.

Definition 7. [10] Let (X,d) be a metric space and $T : X \to X$. A mapping $G : X \to \mathbb{R}$ is said to be T-orbitally lower semi-continuous at a point $z \in X$ if $\{x_n\}$ is a sequence in $O(T,x)$ for some $x \in X$, $\lim_{n\to\infty} x_n = z$ implies $G(z) \le \liminf_{n\to\infty} G(x_n)$.

In [17], Nguyen showed that the T-orbital lower semi-continuity of $x \to d(x,Tx)$ is weaker than both orbital continuity and k-continuity of T. The following example illustrates that fact.

Example 1. Let $X = \{0\} \cup \{1/n : n = 1,2,\ldots\} \cup \{1+1/n : n = 1,2,\ldots\}$ with the metric $d(x,y) = |x-y|$ for all $x,y \in X$. We define a mapping $T : X \to X$ as follows:

$$Tx = \begin{cases} 0 & \text{if } \ x = 0 \\ 2 & \text{if } \ x = 1 \\ 1+\frac{1}{n+1} & \text{if } \ x = \frac{1}{n}, \qquad n = 2,\ldots \\ \frac{1}{n} & \text{if } \ x = 1+\frac{1}{n}, \quad n = 1,2,\ldots \end{cases}$$

We can check that T is neither orbitally continuous nor k-continuous for any $k \geq 1$. Indeed, one has

$$O(T,1) = \left\{1, 1+\frac{1}{2}, \frac{1}{2}, 1+\frac{1}{3}, \frac{1}{3}, \ldots, 1+\frac{1}{n}, \frac{1}{n}, \ldots\right\}.$$

Let $\{x_n\}$ be a sequence in $O(T,1)$ with $x_n = \frac{1}{n}$ for $n \geq 1$. Then, $x_n \to 0$ as $n \to \infty$. However, $Tx_n = 1 + \frac{1}{n+1} \to 1 \neq T0$. Hence, T is not orbitally continuous. Let $\{u_n\}$ be in X with $u_n = \frac{1}{n}$ for $n \geq 1$. For $i = 1, 2, \cdots$, we have $T^{2i-1}u_n = 1 + \frac{1}{n+i}$, $T^{2i}u_n = \frac{1}{n+i}$ and $T^{2i+1}u_n = 1 + \frac{1}{n+i+1}$. Since $\lim_{n\to\infty} T^{2i-1}u_n = 1$ and $\lim_{n\to\infty} T^{2i}u_n = 0 \neq 2 = T1$, T is not $2i$-continuous. Since $\lim_{n\to\infty} T^{2i+1}u_n = 1 \neq T0$, T is not $(2i+1)$-continuous. Therefore, T is not k-continuous for any $k \geq 1$.

Moreover,

$$d(x,Tx) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ 1 - \frac{1}{n(n+1)} & \text{if } x = \frac{1}{n}, \quad n = 1, 2, \ldots \\ 1 & \text{if } x = 1 + \frac{1}{n}, \quad n = 1, 2, \ldots \end{cases}$$

and one can check that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous in X.

The following examples show that the weak orbital continuity of a mapping T on a metric space (X,d) and the T-orbital lower semi-continuity of the mapping $x \to d(x,Tx)$ are independent.

Example 2. Let (X,d) as in Example 1. We define a mapping $T : X \to X$ as follows.

$$Tx = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ 1 + \frac{1}{n+1} & \text{if } x = \frac{1}{n}, \quad n = 1, 2, \cdots \\ \frac{1}{n} & \text{if } x = 1 + \frac{1}{n}, \quad n = 1, 2, \cdots \end{cases}$$

Let $y \in X$. If $\{x_n\} \subset O(T,y)$ converges, then it converges either to 0 or 1. If $\{x_n\}$ converges to 0, then $x_n = \frac{1}{n}$ for large n. In this case, $\lim_{n\to\infty} Tx_n = 1 \neq \frac{1}{2} = T1$. If $\{x_n\}$ converges to 1, then $x_n = 1 + \frac{1}{n}$ for n large. In this case, $\lim_{n\to\infty} Tx_n = 0 \neq \frac{3}{2} = T1$. Thus, T is not weakly orbitally continuous

Moreover,

$$d(x,Tx) = \begin{cases} \frac{1}{2} & \text{if } x = 0 \\ 1 - \frac{1}{n(n+1)} & \text{if } x = \frac{1}{n}, \quad n = 1, 2, \ldots \\ 1 & \text{if } x = 1 + \frac{1}{n}, \quad n = 1, 2, \ldots \end{cases}$$

and we can check that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous in X.

Example 3. Let $X = [0,1]$ with the usual metric. Let $T : X \to X$ be defined by

$$Tx = \begin{cases} 1 & \text{if } x = 0, \\ \frac{x}{2} & \text{if } 0 < x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

It is easy to see that T is weakly orbitally continuous on X. We have

$$d(x,Tx) = \begin{cases} 1 & \text{if} \quad x = 0, \\ \frac{x}{2} & \text{if} \quad 0 < x < 1, \\ 0 & \text{if} \quad x = 1. \end{cases}$$

Let $\{x_n\}$ in $O(T,x)$ with $x \in X$ be such that $\{x_n\}$ converges to 0. Then $x \in (0,1)$ and in this case we may assume that $x_n = \frac{x}{2^n}$ for all n. We have

$$\liminf_{n\to\infty} f(x_n) = \lim_{n\to\infty} \frac{x}{2^{n+1}} = 0 < 1 = f(0).$$

Thus, mapping $x \to f(x) = d(x,Tx)$ is not T-orbitally lower semi-continuous at $z = 0$.

In this paper, we show that contractive condition given by Proinov still guarantees the existence of fixed points under weaker forms of continuity. We not only relax the continuity requirement in the results proved by Proinov and others, but also relax the condition (i) in Theorems 1 and 2. Our results also provide more answers to an open question posed in [26]. Some examples are also given to support our results.

2. Fixed point results

Our first result is the following theorem:

Theorem 4. *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotically regular mapping such that:*

(i) *There exists $\varphi \in \Phi$ such that $d(T^{i+1}x, T^{j+1}x) \le \varphi(D(T^i x, T^j x))$ for all $x \in X$, $i, j \in \mathbb{N} \cup \{0\}$, where $\gamma \ge 0$ and $D(x,y) = d(x,y) + \gamma[d(x,Tx) + d(y,Ty)]$;*

(ii) *$d(Tx,Ty) < D(x,y)$ for all $x, y \in X$ with $x \ne y$.*

Then T has a contractive fixed point $z \in X$ provided that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous or T is weakly orbitally continuous on X.

Proof. Let $x \in X$ and define the sequence $\{x_n\}$ as follows:

$$x_n = T^n x, \quad n = 0,1,2\ldots$$

Since T is asymptotically regular, we have

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = \lim_{n\to\infty} d(T^n x, T^{n+1} x) = 0. \tag{2.1}$$

We will show that $\{x_n\}$ is a Cauchy sequence. Let $\varepsilon > 0$ and $\varepsilon < \delta < 2\varepsilon$. By (2.1), there exists $N > 0$ such that

$$d(T^n x, T^{n+1} x) = d(x_n, x_{n+1}) < \frac{\delta - \varepsilon}{1 + 2\gamma}, \quad \text{for all } n \ge N. \tag{2.2}$$

We will prove that

$$d(T^n x, T^m x) < \frac{\delta + 2\gamma\varepsilon}{1 + 2\gamma}, \quad \text{for all } m \ge n \ge N. \tag{2.3}$$

Fix $n \geq N$. We prove (2.3) by induction on m. It is obvious that (2.2) holds for $m = n$. Since

$$\frac{\delta - \varepsilon}{1+2\gamma} < \frac{\delta + 2\gamma\varepsilon}{1+2\gamma},$$

(2.3) also holds for $m = n+1$. Assume now that (2.3) holds for an integer $m \geq n+1$, we shall show that it holds for $m+1$.

It is enough to show that

$$d(T^{n+1}x, T^{m+1}x) \leq \varepsilon. \tag{2.4}$$

Indeed, if (2.4) holds, then by the triangle inequality and (2.2),

$$\begin{aligned} d(T^n x, T^{m+1}x) &\leq d(T^n x, T^{n+1}x) + d(T^{n+1}x, T^{m+1}x) \\ &< \frac{\delta - \varepsilon}{1+2\gamma} + \varepsilon = \frac{\delta + 2\gamma\varepsilon}{1+2\gamma}, \end{aligned}$$

i.e., (2.3) holds for $m+1$.

We consider two cases.

Case 1. $D(T^n x, T^m x) \leq \varepsilon$. By (ii), it follows that

$$d(T^{n+1}x, T^{m+1}x) \leq D(T^n x, T^m x) \leq \varepsilon,$$

i.e., (2.4) holds.

Case 2. $D(T^n x, T^m x) > \varepsilon$. Using (2.2) and (2.3), we have

$$\begin{aligned} D(T^n x, T^m x) &= d(T^n x, T^m x) + \gamma\left[d(T^n x, T^{n+1}x) + d(T^m x, T^{m+1}x)\right] \\ &< \frac{\delta + 2\gamma\varepsilon}{1+2\gamma} + 2\gamma.\frac{\delta - \varepsilon}{1+2\gamma} = \delta. \end{aligned}$$

Hence, $\varepsilon < D(T^n x, T^m x) < \delta$. By the property of φ, it holds $\varphi(D(T^n x, T^m x)) \leq \varepsilon$. Thus, by (i), one has

$$d(T^{n+1}x, T^{m+1}x) \leq \varphi(D(T^n x, T^m x)) \leq \varepsilon,$$

i.e., (2.4) holds.

We have proved (2.3). Therefore, $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} x_n = \lim_{n\to\infty} T^n x = z.$$

Assume that $x \to d(x, Tx)$ is T-orbitally lower semi-continuous. Since $\{x_n\} \subset O(T, x)$ and $x_n \to z$ satisfying $d(x_n, Tx_n) = d(T^n x, T^{n+1}x) \to 0$ as $n \to \infty$, by the T-orbital lower semi-continuity of $x \to d(x, Tx)$, we have $d(z, Tz) \leq \liminf_{n\to\infty} d(x_n, Tx_n) = 0$. Thus, z is a fixed point of T.

Suppose now that T is weakly orbitally continuous. Since $T^n x_0 \to z$ for each x_0, by virtue of weak orbital continuity of T we get, $T^n y_0 \to z$ and $T^{n+1} y_0 \to Tz$ for some $y_0 \in X$. This implies that $z = Tz$ since $T^{n+1} y_0 \to z$. Therefore, z is a fixed point of T. Uniqueness of the fixed point z follows from (ii). □

Remark 2. Theorem 4 improves Theorem 1 in the following ways. Firstly, the continuity condition on the considered mapping T is replaced by a weaker condition: the weakly orbital continuity of T or T-orbitally lower semi-continuity of $x \to d(x,Tx)$. Secondly, in Theorem 1, it is required that the inequality $d(Tx,Ty) \leq \varphi(D(x,y))$ holds for all $x,y \in X$ whereas, in Theorem 4, we only ask the inequality holds for x,y in the same orbit of T. Later, we will provide examples (see, Example1 and Example 3) which can apply to our result but cannot apply to the result by Proinov.

Equivalent version of Theorem 4 is the following result:

Theorem 5. *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotically regular mapping such that:*

(i) *For any $\varepsilon > 0$, there exists $\delta > \varepsilon$ such that $\varepsilon < D(T^i x, T^j x) < \delta$ implies $d(T^{i+1}x, T^{j+1}x) \leq \varepsilon$ for all $x \in X$, $i, j \in \mathbb{N} \cup \{0\}$;*
(ii) *$d(Tx,Ty) < D(x,y)$ for all $x,y \in X$ with $x \neq y$.*

Then T has a contractive fixed point $z \in X$ provided that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous or T is weakly orbitally continuous on X.

Inspired by [27], we have the following result:

Proposition 1. *Let $T : X \to X$ be a mapping. Consider the following statements:*

(a) *T has a contractive fixed point;*
(b) *$d(Tx,Ty) < D(x,y)$ for all $x,y \in X$, $x \neq y$;*
(c) *For any $\varepsilon > 0$, there exists $\delta > \varepsilon$ such that $\varepsilon < D(T^i x, T^j x) < \delta$ implies $d(T^{i+1}x, T^{j+1}x) \leq \varepsilon$ for all $x \in X, i, j \in \mathbb{N} \cup \{0\}$.*

If (a) and (b) hold, then (c) holds.

Proof. Let $z \in X$ be a contractive fixed point of T. Then, $\{T^n x\}$ converges to z for each $x \in X$. Assume that (a) and (b) hold but (c) does not hold. Then, there exist $x \in X$, $\varepsilon > 0$ and sequences $\{u(n)\}$ and $\{w(n)\}$ in $\mathbb{N} \cup \{0\}$ such that $u(n) < w(n)$ and

$$\varepsilon < D\left(x_{u(n)}, x_{w(n)}\right) < \varepsilon + \frac{1}{n} \quad \text{and} \quad \varepsilon < d\left(x_{u(n)+1}, x_{w(n)+1}\right) \tag{2.5}$$

for any $n \in \mathbb{N}$, where $x_k = T^k x$ for each $k \in \mathbb{N} \cup \{0\}$.

Without loss of generality, we may assume that either $\lim_{n\to\infty} u(n) = \infty$ holds or there exists $p \in \mathbb{N} \cup \{0\}$ such that $u(n) = p$ for any $n \in \mathbb{N}$. Similarly, we can also assume that either $\lim_{n\to\infty} w(n) = \infty$ or there exists $q \in \mathbb{N} \cup \{0\}$ such that $w(n) = q$ for any $n \in \mathbb{N}$. From (b) and (2.5), we have

$$\varepsilon < d\left(x_{u(n)+1}, x_{w(n)+1}\right) < D\left(x_{u(n)}, x_{w(n)}\right) < \varepsilon + \frac{1}{n}.$$

Thus,

$$0 < \varepsilon = \lim_{n\to\infty} d\left(x_{u(n)+1}, x_{w(n)+1}\right) = \lim_{n\to\infty} D\left(x_{u(n)}, x_{w(n)}\right). \tag{2.6}$$

If $\lim_{n\to\infty} u(n) = \infty$, then $\lim_{n\to\infty} w(n) = \infty$. By (2.6) and the fact that $\{T^n x\}$ converges to z, we have $\varepsilon = d(z,z) = 0$ which is a contradiction. We have two possible cases:

Case 1: $u(n) = p$ and $w(n) = q$ for all n. In this case, using (2.6) we have

$$d(Tx_p, Tx_q) = d(x_{p+1}, x_{q+1}) = \varepsilon = D(x_p, x_q)$$

which contradicts (b).

Case 2: $u(n) = p$ for all n and $\lim\limits_{n\to\infty} w(n) = \infty$. Using (2.6), we obtain

$$d(Tx_p, Tz) = d(x_{p+1}, z) = D(x_p, z) = \varepsilon$$

which contradicts (b). Therefore, (c) holds. □

We now provide some examples to illustrate our results.

Example 4. Consider the set ℓ^∞ of all bounded real sequences with the metric

$$d(x,y) = \sup\{|x_n - y_n| : n = 1,2,3,\cdots\} \quad \text{for all } x = (x_n), y = (y_n) \in \ell^\infty.$$

For each $n \in \mathbb{N}$, denote by $e(n)$ the sequence with all elements equal to 0 except the nth element equal to 1, i.e.,

$$e(n) = (0,\cdots,0,1,0,\cdots).$$

Let $a > 1$ and κ be a positive integer. Consider the set

$$X = \left\{a^{1-m/n} e(n) : m,n = 1,2,,3,\cdots\right\} \cup \{\mathbf{0}\},$$

where $\mathbf{0}$ is the zero sequence in ℓ^∞. We define a mapping $T : X \to X$ by

$$T(\mathbf{0}) = \mathbf{0}, \quad \text{and} \quad T\left(a^{1-m/n} e(n)\right) = a^{1-(m+\kappa)/n} e(n) \text{ for all } m,n = 1,2,3,\cdots.$$

Then,

(a) (X,d) is a complete metric space.
(b) Condition (i) of Theorem 1 does not hold and Theorems 1, 2 and 3 are not applicable to T.
(c) Conditions (a) and (b) of Proposition 1 hold and therefore condition (i) of Theorem 5 holds.
(d) Theorems 5 and 4 are applicable to T.

Proof. (a) It is evident that (ℓ^∞, d) is a complete metric space. One can easily check that X is a closed subset of ℓ^∞ and thus (X,d) is a complete metric space.

(b) Assume to the contrary that condition (i) of Theorem 1 holds. That is, there exist $\varphi \in \Phi$ and $\gamma > 0$ such that $d(Tx,Ty) \le \varphi(D(x,y))$ for all $x,y \in X$. Let $x = \mathbf{0}$ and $y = a^{1-(n-\kappa-1)/n}$ with $n > \kappa + 1$. We have

$$D_n := D(x,y) = a^{(\kappa+1)/n} + \gamma(a^{(\kappa+1)/n} - a^{1/n}) \quad \text{and} \quad d(Tx,Ty) = a^{1/n}.$$

Let $\varepsilon = 1$. Since $D_n > 1$ for all $n > \kappa + 1$ and $\lim_{n\to\infty} D_n = 1$, there exists $\delta > 1 = \varepsilon$ such that

$$\varepsilon < D_n = D(x,y) < \delta$$

for all n sufficiently large. By the property of Φ, one has

$$a^{1/n} = d(Tx,Ty) \le \varphi(D(x,y)) \le \varepsilon = 1.$$

This is a contradiction. Thus, condition (i) of Theorem 1 does not hold. Therefore, we cannot apply Theorems 1, 2 and 3 to the mapping T.

(c) It is evident that T has a unique fixed point $z = \mathbf{0}$. Moreover, for $x \in X$ and $i \in \mathbb{N} \cup \{0\}$, we have

$$T^i x = \begin{cases} \mathbf{0} & \text{if} \quad x = \mathbf{0} \\ a^{1-\frac{m+i\kappa}{n}} e(n) & \text{if} \quad x = a^{1-\frac{m}{n}} e(n). \end{cases}$$

It is obvious that $\{T^i x\}$ converges to $\mathbf{0}$ for each $x \in X$. Hence, condition (a) of Proposition 1 holds. One can also check, for $\gamma = 2$, that $d(Tx,Ty) < D(x,y)$ for all $x,y \in X$ with $x \neq y$, i.e., condition (b) of Proposition 1 holds. By Proposition 1, condition (i) of Theorem 5 holds.

(d) It is easy to check that T is asymptotically regular and continuous on X. As in (c), conditions (i) and (ii) of Theorem 5 hold. Thus, all the conditions of Theorem 5 hold and we can apply this theorem to T. □

Example 5. Let (X,d) and T be as in Example 3 and let $\gamma = 2$. One can easily check that

(a) T is asymptotically regular on X,
(b) $d(T^{i+1}x, T^{j+1}x) \le 1/2D(T^i x, T^j x)$ for all $x \in X$ and $i, j \in \mathbb{N} \cup \{0\}$,
(c) $d(Tx,Ty) < D(x,y)$ for all $x,y \in X$, $x \neq y$.

Since (X,d) is complete and T is weakly orbitally continuous on X, we can apply Theorem 4 to T. In fact, T has a unique fixed point $z = 1$. Since T is neither $k-$continuous nor orbitally continuous, hence we cannot apply Theorems 1 and 3 to this example. Note that T is not continuous at the fixed point $z = 1$.

The following corollaries are easy consequences of Theorem 4.

Corollary 1. *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotically regular mapping such that:*

(i) *There exists $\varphi \in \Phi$ such that $d(T^{i+1}x, T^{j+1}x) \le \varphi(m(T^i x, T^j x))$ for all $x \in X$, $i, j \in \mathbb{N} \cup \{0\}$, where*

$$m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty) + d(y,Tx)]\};$$

(ii) *$d(Tx,Ty) < m(x,y)$ for all $x,y \in X$ with $x \neq y$.*

Then T has a contractive fixed point $z \in X$ provided that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous or T is weakly orbitally continuous on X.

Corollary 2. *Let (X,d) be a complete metric space and $T : X \to X$ be an asymptotically regular mapping such that:*

(i) *For any $\varepsilon > 0$, there exists $\delta > \varepsilon$ such that $\varepsilon < m(T^i x, T^j x) < \delta$ implies $d(T^{i+1}x, T^{j+1}x) \leq \varepsilon$ for all $x \in X$, $i, j \in \mathbb{N} \cup \{0\}$;*

(ii) *$d(Tx,Ty) < m(x,y)$ for all $x, y \in X$ with $x \neq y$.*

Then T has a contractive fixed point $z \in X$ provided that $x \to d(x,Tx)$ is T-orbitally lower semi-continuous or T is weakly orbitally continuous on X.

Existence of a fixed point of T is still guaranteed if condition (ii) in Theorem 4 holds for all $x, y \in O(T, x_0)$ and $x \neq y$ for some $x_0 \in X$. In this case, we only require T to be asymptotically regular at x_0 and condition (i) of Theorem 4 can also be relaxed. However, in this case, T may have more than one fixed point. More precisely, we have the following result whose proof follows from the proof of Theorem 4.

Theorem 6. *Let (X,d) be a complete metric space and $x_0 \in X$. Assume that $T : X \to X$ is asymptotically regular at $x_0 \in X$ such that:*

(i) *There exists $\varphi \in \Phi$ such that $d(T^{i+1}x, T^{j+1}x) \leq \varphi(D(T^i x, T^j x))$ for all $x \in O(T, x_0)$, $i, j \in \mathbb{N} \cup \{0\}$;*

(ii) *$d(Tx,Ty) < D(x,y)$ for all $x, y \in O(T, x_0)$, with $x \neq y$.*

Then T has a fixed point $z \in X$ and $T^n x$ converges to z for each $x \in O(T, x_0)$.

Example 6. Let (X,d) and T be as in Example 1. Then, T is asymptotically regular at $x_0 = 0$. It is obvious that all conditions of Theorem 6 are satisfied and T has a fixed point $z = 0$.

Remark 3. Our results generalize and subsume various results given in [2–6, 11, 13, 15, 16, 19–25]. In addition to it, we provide more answers to the open problem regarding the existence of contractive mapping which has fixed point but is discontinuous at the fixed point [26].

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REFERENCES

[1] V. Berinde, A. Petrusel, and I. A. Rus, " Remarks on the terminology of the mappings in fixed point iterative methods in metric spaces," *Fixed Point Theory*, vol. 24, no. 2, pp. 525–540, 2023, doi: 10.24193/fpt-ro.2023.2.05.

[2] R. K. Bisht, "A remark on convergence theory for iterative processes of Proinov contraction," *Commun. Korean Math. Soc.*, vol. 34, no. 4, pp. 1157–1162, 2019, doi: 10.4134/CKMS.c180382.

[3] R. K. Bisht and R. P. Pant, "A remark on discontinuity at fixed point," *J. Math. Anal. Appl.*, vol. 445, no. 2, pp. 1239–1242, 2017, doi: 10.1016/j.jmaa.2016.02.053.

[4] R. K. Bisht, R. P. Pant, and V. Rakočević, "Proinov contractions and discontinuity at fixed point," *Miskolc Math. Notes*, vol. 20, no. 1, pp. 131–137, 2019, doi: 10.18514/MMN.2019.2277.

[5] R. K. Bisht and V. Rakočević, "Generalized Meir-Keeler type contractions and discontinuity at fixed point," *Fixed Point Theory*, vol. 19, no. 1, pp. 57–64, 2018, doi: 10.24193/fpt-ro.2018.1.06.

[6] F. E. Browder and W. V. Petryshyn, "The solution by iteration of nonlinear functional equations in Banach spaces," *Bull. Am. Math. Soc.*, vol. 72, pp. 571–575, 1966, doi: 10.1090/S0002-9904-1966-11544-6.

[7] L. B. Ćirić, "On contraction type mappings," *Math. Balk.*, vol. 1, pp. 52–57, 1971.

[8] P. Debnath and M. de La Sen, "Contractive inequalities for some asymptotically regular set-valued mappings and their fixed points," *Symmetry*, vol. 12, no. 3, 2020, doi: 10.3390/sym12030411.

[9] P. Debnath and M. de La Sen, "On some new contractive conditions for asymptotically regular set-valued mappings," *International Journal of Analysis and Application*, vol. 18, no. 4, pp. 586–593, 2020.

[10] T. L. Hicks and B. E. Rhoades, "A Banach type fixed point theorem," *Math. Japon.*, vol. 24, pp. 327–330, 1979.

[11] J. Jachymski, "Equivalent conditions and the Meir-Keeler type theorems," *J. Math. Anal. Appl.*, vol. 194, no. 1, pp. 293–303, 1995, doi: 10.1006/jmaa.1995.1299.

[12] A. Jeribi, N. Kaddachi, and Z. Laouar, "Fixed point theorems for weakly asymptotically regular mappings in Banach spaces with an application," *Numer. Funct. Anal. Optim.*, vol. 43, no. 1, pp. 68–87, 2022, doi: 10.1080/01630563.2021.2001820.

[13] S. Leader, "Fixed points for operators on metric spaces with conditional uniform equivalence of orbits," *J. Math. Anal. Appl.*, vol. 61, pp. 466–474, 1977, doi: 10.1016/0022-247X(77)90131-7.

[14] T.-C. Lim, "On characterizations of Meir-Keeler contractive maps," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 46, no. 1, pp. 113–120, 2001, doi: 10.1016/S0362-546X(99)00448-4.

[15] J. Matkowski, "Integrable solutions of functional equations," *Diss. Math.*, vol. 127, 1975.

[16] A. Meir and E. Keeler, "A theorem on contraction mappings," *J. Math. Anal. Appl.*, vol. 28, pp. 326–329, 1969, doi: 10.1016/0022-247X(69)90031-6.

[17] L. V. Nguyen, "On fixed points of asymptotically regular mappings," *Rend. Circ. Mat. Palermo (2)*, vol. 70, no. 2, pp. 709–719, 2021, doi: 10.1007/s12215-020-00527-0.

[18] A. Pant and R. P. Pant, "Fixed points and continuity of contractive maps," *Filomat*, vol. 31, no. 11, pp. 3501–3506, 2017, doi: 10.2298/FIL1711501P.

[19] A. Pant, R. P. Pant, and M. C. Joshi, "Caristi type and Meir-Keeler type fixed point theorems," *Filomat*, vol. 33, no. 12, pp. 3711–3721, 2019, doi: 10.2298/FIL1912711P.

[20] A. Pant, R. P. Pant, V. Rakočević, and R. K. Bisht, "Generalized Meir-Keeler type contractions and discontinuity at fixed point. II," *Math. Slovaca*, vol. 69, no. 6, pp. 1501–1507, 2019, doi: 10.1515/ms-2017-0325.

[21] A. Pant, R. P. Pant, and W. Sintunavarat, "Analytical Meir-Keeler type contraction mappings and equivalent characterizations," *Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM*, vol. 115, no. 1, p. 15, 2021, id/No 37, doi: 10.1007/s13398-020-00939-8.

[22] R. P. Pant, "Discontinuity and fixed points," *J. Math. Anal. Appl.*, vol. 240, no. 1, pp. 284–289, 1999, doi: 10.1006/jmaa.1999.6560.

[23] R. P. Pant, N. Y. Özgür, and N. Taş, "On discontinuity problem at fixed point," *Bull. Malays. Math. Sci. Soc. (2)*, vol. 43, no. 1, pp. 499–517, 2020, doi: 10.1007/s40840-018-0698-6.

[24] S. park and J. S. Bae, "Extension of a fixed point theorem of Meir-Keeler," *Ark. Mat.*, vol. 19, no. 0, pp. 223–228, 1991.

[25] P. D. Proinov, "Fixed point theorems in metric spaces," *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, vol. 64, no. 3, pp. 546–557, 2006, doi: 10.1016/j.na.2005.04.044.

[26] B. E. Rhoades, "Contractive definitions and continuity," Fixed point theory and its applications, Proc. Conf., Berkeley/Calif. 1986, Contemp. Math. 72, 233-245 (1988)., 1988.

[27] T. Suzuki, "The weakest contractive conditions for Edelstein's mappings to have a fixed point in complete metric spaces," *J. Fixed Point Theory Appl.*, vol. 19, no. 4, pp. 2361–2368, 2017, doi: 10.1007/s11784-017-0430-4.

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