Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 921–942

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4219

DYNAMIC ANALYSIS AND CHAOS CONTROL IN A DISCRETE PREDATOR-PREY MODEL WITH HOLLING TYPE IV AND NONLINEAR HARVESTING

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Received 23 April, 2022

Abstract. In this paper, we investigate the dynamical behavior of a two dimensional discretized prey predator system. The model is formulated in terms of difference equations and derived by using the higher-order implicit Runge Kutta method with a very small step size to attain a discrete time version of its continuous counterpart. The existence of fixed points as well as their local asymptotic stability are proved. Further, it is shown that the model experiences Neimark-Sacker bifurcation (NSB for short) and the periodic doubling bifurcation (PDB) in a small neighborhood of the coexistence fixed point under certain parametric conditions. This analysis utilizes bifurcation theory and the center manifold theorem. The chaos influenced by NSB is stabilized. Finally, we use numerical simulations and computer analysis to check our theories and show more complex behaviors.

2010 *Mathematics Subject Classification:* 65Q10; 34H10; 34G20; 39A28; 39A30; 39A33; 92D25

Keywords: discrete model, asymptotic stability, bifurcation analysis, chaos control

1. INTRODUCTION

When a predator and its prey interact, the population dynamics may be represented using a prey-predator model that is either discrete or continuous (in time) [10, 14]. Discrete time prey-predator models based on difference equations are best suited for predicting and describing the dynamics of populations with non-overlapping generations [6]. The discrete case is especially important because it can exhibit more interesting dynamic behaviors, including bifurcations and chaos. It may also offer a more efficient computational model for numerical simulations. Controlling chaos in nonlinear dynamical systems is a challenging area of research. Chaos occurs suddenly and might lead to unpredictable behavior.

In [8], a discrete time predator prey model with six parameters is investigated, and the hybrid control strategy is applied to control Neimark-Sacker bifurcation. The Neimark-Sacker bifurcation in a discrete time glycolytic oscillator model have been studied in [11]. Prime period and periodic points of the discrete model are investigated. A further hybrid control strategy is applied to control chaos influenced by Neimark-Sacker bifurcation. In [1], nonlinear dynamics and chaotic behavior of a multi-strain tuberculosis model under the fractal fractional operator in Atangana-Baleanu sense are established. In [20], the local dynamics of a discrete nonlinear prey–predator model are obtained on asymptotic properties of fixed points and Neimark–Sacker bifurcations. Some results on the global stability are studied. In [17], a new approach related to the global stability are established for a special class of discrete time evolutionary models for both single species and multi-species dynamics, that are derived according to the evolutionary game theory. The reader interested to other works on the asymptotic stability, bifurcation theory and chaos control is referred to, among many others, to [3, 4, 7, 9, 11, 12, 14, 15, 15, 16, 19, 21].

Motivated by the previous cited works, we consider the following class of prey-predator system incorporating Holling functional response type IV and nonlinear harvesting in the prey [22]

$$\frac{dx}{dt} = x(1-x) - \underbrace{\frac{xy}{\frac{1}{\alpha}x^2 + x + \beta}}_{\text{Functional response Holling IV}} - \underbrace{\frac{ax}{x+b}}_{\text{Nonlinear harvesting}}, \tag{1.1}$$

$$\frac{dy}{dt} = cy\left(1 - \frac{dy}{x}\right), \tag{1.2}$$

here x and y model the prey and predator densities, respectively. All the model's parameters $a, b, c, d, \alpha, \beta$ are nonnegative. The coefficient α is the tolerance of the prey, β represent half saturation constant. The parameters b stands for the effort of harvesting, and c is intrinsic growth rate of the predator. Finally, the parameter d models the amount of prey required to support one predator at equilibrium. The qualitative behavior of (1.1)-(1.2) is investigated in [22]. In [2], Euler scheme is implemented to discretize the system (1.1)-(1.2). In this work, arguing as in [18], and considering that the variables and functions switch only at regular intervals, the following formula is built through the conversion form (1.1)-(1.2) by implementing the largest integer function $[t]$:

$$\frac{1}{x(t)}\frac{dx}{dt} = (1-x) - \frac{y[t]}{\frac{1}{\alpha}x^2[t] + x[t] + \beta} - \frac{a}{x[t]+b}, \tag{1.3}$$

$$\frac{1}{y(t)}\frac{dy}{dt} = c\left(1 - \frac{dy[t]}{x[t]})\right), \tag{1.4}$$

applying the piecewise constant arguments method for differential equations $[t]$, thus integrating the system (1.3)-(1.4) on the interval $[n, n+1]$ we have

$$\ln\frac{x_{n+1}}{x_n} = (1-x_n) - \frac{y_n}{\frac{1}{\alpha}x_n^2 + x_n + \beta} - \frac{a}{x_n+b}, \tag{1.5}$$

$$\ln\frac{y_{n+1}}{y_n} = c\left(1-\frac{dy_n}{x_n})\right). \tag{1.6}$$

For $n = 0,1,2...$, we obtain the following discrete time model.

$$x_{n+1} = x_n \exp\left((1-x_n) - \frac{y_n}{\frac{1}{\alpha}x_n^2 + x_n + \beta} - \frac{a}{x_n+b}\right), \tag{1.7a}$$

$$y_{n+1} = y_n \exp\left(c\left(1-\frac{dy_n}{x_n}\right)\right). \tag{1.7b}$$

The aim of this research is to find the system's fixed points (1.7a)-(1.7b) and analyze the asymptotic stability conditions of these fixed points. Furthermore, the interesting aspect of this study, is to prove rigorously, by using center manifold theory, that the system possesses NSB and PDB near the positive fixed point. Moreover, we control the chaos influenced by NSB. The paper is organized as follows: In Section 2, the existence and asymptotic stability of the fixed points are investigated. In Section 3, we prove that the discretized system admits NSB near the coexistence fixed point. The existence of PDB is also proved analytically by using center manifold theory in Section 4. A state feedback method is implemented to control chaos in Section 5. Finally, some numerical simulations are given in 6, followed by a conclusion in 7.

2. Existence of the fixed points and their stability

The fixed points for the system (1.7) are the solutions of the two isoclines

$$(1-x) - \frac{y}{\frac{1}{\alpha}x^2 + x + \beta} - \frac{a}{x+b} = 0, \quad 1 - \frac{dy}{x} = 0. \tag{2.1}$$

For $y = 0$, it follows that the system (1.7) has two boundary fixed points noted $(x_1, 0)$ and $(x_2, 0)$, where x_1 and x_2 are the roots of the quadratic equation $x^2 + (b-1)x + a - b = 0$, where $x_{1,2} = \frac{1-b\pm\sqrt{(1+b)^2-4a}}{2}$ *and* $(1+b)^2 > 4a$. Moreover, we assume that $0 < b < 1$ and $b < a < \frac{1}{4}(b+1)^2$, then both x_1 and x_2 are positive. Let $x_{1,2} = \bar{x}$ and we write the boundary fixed point $E_1(\bar{x}, 0)$.

For the coexistence fixed point $E_2(x^*, y^*)$ the components are given as follows $y^* = \frac{1}{d}x^*$, where x^* is a positive real root for the quadratic equation

$$\alpha_1 x^4 + \alpha_2 x^3 + \alpha_3 x^2 + \alpha_4 x + \alpha_5 = 0.$$

where $\alpha_1 = d$, $\alpha_2 = d(\alpha + b - 1)$, $\alpha_3 = d(\alpha(\beta - 1) + a + b(\alpha_1)) + \alpha$, and $\alpha_4 = \alpha(ad + (b-1)\beta b - bd + b)$, $\alpha_5 = \alpha\beta d(a-b)$. Arguing as [22], there exist a unique coexistence fixed point. We rewrite (2.1) as $y = \frac{x}{d}$ and $y = (1 - x - \frac{a}{b+x})(\beta + \frac{x^2}{a} + x)$. The Jacobian matrix of the system (1.7a)-(1.7b) at any fixed point (x, y) is given by

$$J(x,y) = \begin{pmatrix} j_{11} & j_{12} \\ j_{21} & j_{22} \end{pmatrix}, \tag{2.2}$$

where

$$
\begin{aligned}
j_{11} &= \exp\left(1 - x - \frac{y}{(\frac{1}{\alpha}x^2 + x + \beta)} - \frac{a}{x+b}\right) \\
&\quad \times \left[1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha} + 1)}{(\frac{1}{\alpha}x^{*2} + x^* + \beta)^2} + \frac{a}{(x^* + b)^2}\right)\right)\right], \\
j_{12} &= -\frac{x}{\frac{1}{\alpha}x^2 + x + \beta} \exp\left(1 - x - \frac{y}{(\frac{1}{\alpha}x^2 + x + \beta)} - \frac{a}{x+b}\right), \\
j_{21} &= \frac{cdy^2}{x^2} \exp\left(c(1 - \frac{dy}{x})\right), \\
j_{22} &= \exp\left(c(1 - d\frac{y}{x})\right)\left(1 - \frac{ycd}{x}\right).
\end{aligned}
$$

The fixed point $E_1(K,0)$ has two eigenvalues $\lambda_1 = 1 - x\left(1 - \frac{a}{(x+b)^2}\right)$ and $\lambda_2 = \exp(c)$ with $\lambda_2 > 1$. Thus both the boundary fixed point $E_1(\bar{x},0)$ of the system (1.7) are unstable.

For the fixed point $E_2(x^*, y^* = \frac{x^*}{d})$, we have

$$
J(E_2) = \begin{pmatrix} 1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha} + 1)}{(\frac{1}{\alpha}x^{*2} + x^* + \beta)^2} + \frac{a}{(x^* + b)^2}\right)\right) & -\frac{x^*}{\frac{1}{\alpha}x^{*2} + x^* + \beta} \\ \frac{c}{d} & 1 - c \end{pmatrix}. \tag{2.3}
$$

The characteristic equation associated to (2.3) is [6]

$$
\eta^2 - \mathrm{tr}J(E_2)\eta + \det J(E_2) = 0, \tag{2.4}
$$

where

$$
\mathrm{tr}J(E_2) = 2 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha} + 1)}{(\frac{1}{\alpha}x^{*2} + x^* + \beta)^2} + \frac{a}{(x^* + b)^2}\right)\right) - c, \tag{2.5}
$$

and

$$
\begin{aligned}
\det J(E_2) &= \left(1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha} + 1)}{(\frac{1}{\alpha}x^{*2} + x^* + \beta)^2} + \frac{a}{(x^* + b)^2}\right)\right)\right)(1 - c) \\
&\quad + \frac{cx^*}{\frac{d}{\alpha}x^{*2} + dx^* + d\beta}.
\end{aligned} \tag{2.6}
$$

The discriminant of (2.4) is

$$
\Delta = \mathrm{tr}\Big(J(E_2)\Big)^2 - 4\det\Big(J(E_2)\Big). \tag{2.7}
$$

Theorem 1. *We assume* $x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)>2$, *and we set*

$$\Lambda_1=\frac{2d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-2\right)}{d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-2\right)+\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}},$$

$$\Lambda_2=\frac{dx^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)}{d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-1\right)+\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}}.$$

The coexistent fixed point E_2 *is locally asymptotically stable if* $\Lambda_1<c<\Lambda_2$ *and unstable otherwise.* E_2 *is nonhyperbolic if either* $c=\Lambda_1$ *or* $c=\Lambda_2$.

Theorem 2. *We assume* $x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)>2$, *and we set*

$$\Lambda_1=\frac{2d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-2\right)}{d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-2\right)+\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}},$$

$$\Lambda_2=\frac{dx^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)}{d\left(x^*\left(1-\left(y^*\frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2}+\frac{a}{(x^*+b)^2}\right)\right)-1\right)+\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}}.$$

The coexistent fixed point E_2 *is locally asymptotically stable if* $\Lambda_1<c<\Lambda_2$ *and unstable otherwise.* E_2 *is nonhyperbolic if either* $c=\Lambda_1$ *or* $c=\Lambda_2$.

3. Neimark Sacker bifurcation

If the discriminant Δ defined in (2.7) is negative and $c=\Lambda_2$ hold, then the eigenvalues of (2.7) are pair of conjugate complex numbers with modulus 1. Thus, this conditions can be written as

$$N_s=\left\{(\alpha,\beta,a,b,c,d)>0,\ \Delta<0,\ c=\Lambda_2\right\}. \tag{3.1}$$

If we vary c in the neighborhood of $c=\overline{c}$ keeping other parameters in (3.1) constant, then the coexistence fixed point E_2 undergoes Neimark-Sacker bifurcation. Taking a perturbation c^* where ($c^*\ll 1$) of the parameter c in the neighborhood of $c=\overline{c}$ in the system (1.7a)-(1.7b), we have

$$x_{n+1}=x_n\exp\left((1-x_n)-\frac{y_n}{\frac{1}{\alpha}x_n^2+x_n+\beta}-\frac{a}{x_n+b}\right)=f(x_n,y_n,c^*), \tag{3.2a}$$

$$y_{n+1} = y_n \exp\left((c+c^*)\left(1-\frac{dy_n}{x_n})\right)\right) = g(x_n, y_n, c^*). \tag{3.2b}$$

We transform the (3.2) into the origin ($v_n = x_n - x^*$ and $w_n = y_n - y^*$), and expanding the resulted system up to second order near the origin, one obtain

$$v_{n+1} = \gamma_1 v_n + \gamma_2 w_n + \gamma_{12} v_n w_n + \gamma_{11} v_n^2 + \gamma_{22} w_n^2 + O\left((\mid v_n \mid + \mid w_n \mid)^2\right), \tag{3.3a}$$

$$w_{n+1} = \delta_1 v_n + \delta_2 w_n + \delta_{12} v_n w_n + \delta_{11} v_n^2 + \delta_{22} w_n^2 + O\left((\mid v_n \mid + \mid w_n \mid)^2\right), \tag{3.3b}$$

where

$$\begin{aligned}
\gamma_1 &= f_x(x^*, y^*, 0) = 1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right),\\
\gamma_2 &= f_y(x^*, y^*, 0) = -\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta},\\
\gamma_{12} &= f_{xy}(x^*, y^*, 0) = \frac{-1+x^*}{\frac{x^{*2}}{\alpha}+x^*+\beta} + \frac{2x^{*2}+\alpha x^*}{\alpha(\frac{x^{*2}}{\alpha}+x^*+\beta)^2} - \frac{2x^{*2}y^*+\alpha x^* y^*}{\alpha(\frac{x^{*2}}{\alpha}+x^*+\beta)^3}\\
&\quad - \frac{ax^*}{(x+b)^2(\frac{x^{*2}}{\alpha}+x^*+\beta)},\\
\gamma_{11} &= f_{xx}(x^*, y^*, 0) = -1 + \frac{2x^*y^*+\alpha y^*+y^*x^*}{\alpha(\frac{x^{*2}}{\alpha}+x^*+\beta)^2} - \frac{4x^{*2}y^*+\alpha^2x^*y^*+4\alpha x^{*2}y^*}{\alpha^2(\frac{x^{*2}}{\alpha}+x^*+\beta)^3}\\
&\quad + \frac{2x^{*3}y^{*2}+2\alpha x^{*2}y^*}{\alpha^2(\frac{x^{*2}}{\alpha}+x^*+\beta)^4} + \frac{y^{*2}x^*}{2(\frac{x^{*2}}{\alpha}+x^*+\beta)^4} + \frac{2x^{*2}y^*a+\alpha a x^* y^*}{\alpha(x+b)^2(\frac{x^{*2}}{\alpha}+x^*+\beta)^2}\\
&\quad + \frac{a}{(x+b)^2} - \frac{ax}{(x+b)^3} + \frac{a^2x^*}{2(x^*+b)^4} - \frac{2x^{*2}y^*+\alpha x^* y^*}{\alpha(\frac{x^{*2}}{\alpha}+x^*+\beta)^2} - \frac{ax^*}{(x^*+b)^3} + \frac{x^*}{2},\\
\gamma_{22} &= f_{yy}(x^*, y^*, 0) = \frac{x^*}{2((\frac{x^{*2}}{\alpha}+x^*+\beta)^2)},\\
\delta_1 &= \frac{c}{d},\\
\delta_2 &= 1-c,\\
\delta_{12} &= g_{xy}(x^*, y^*, 0) = \frac{c}{x^*}(2-\frac{c}{x^*}),\\
\delta_{11} &= g_{xx}(x^*, y^*, 0) = \frac{cx^*}{2d}(-1+\frac{c}{x^{*2}}),\\
\delta_{22} &= g_{yy}(x^*, y^*, 0) = \frac{cd}{x^*}(-1+\frac{c}{2}).
\end{aligned}$$

The roots of the characteristic equation associated with the linearized map (3.3) at $(v_n, w_n) = (0,0)$ are given by

$$\eta_{1,2}(c^*) = \frac{\mathrm{tr}J(c^*) \pm i\sqrt{4\det J(c^*) - (\mathrm{tr}(c^*))^2}}{2}, \quad |\eta_{1,2}(c^*)| = \sqrt{\det J(c^*)}.$$

When $c^* = 0$, we have

$$\det(J(0)) = 1 \text{ ,and } \frac{d|\eta_{1,2}|}{dc^*}\,|_{c^*=0} \neq 0. \tag{3.4}$$

Additionally, we required that when $c^* = 0$, $\eta_{1,2}^m \neq 1, m = 1,2,3,4$. This is equivalent to $trJ(0) \neq -2,-1,1,2$.

Let $\zeta = Re(\eta_{1,2})$, and $\xi = Im(\eta_{1,2})$. The model (3.3) is written as

$$\begin{pmatrix} v_{n+1} \\ w_{n+1} \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{pmatrix} \begin{pmatrix} v_n \\ w_n \end{pmatrix} + \begin{pmatrix} \gamma_{12} v_n w_n + \gamma_{11} v_n^2 + \gamma_{22} w_n^2 \\ \delta_{12} v_n w_n + \delta_{11} v_n^2 + \delta_{22} w_n^2 \end{pmatrix}. \tag{3.5}$$

Let us consider the invertible matric P associated to the eigenvalue $\eta_{1,2} = \zeta \pm i\xi$,

$$P = \begin{pmatrix} \gamma_2 & 0 \\ \zeta - \gamma_1 & -\xi \end{pmatrix}.$$

Using the following translation

$$\begin{pmatrix} v_n \\ w_n \end{pmatrix} = \begin{pmatrix} \gamma_2 & 0 \\ \zeta - \gamma_1 & -\xi \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix}.$$

The system (3.5) can be written as

$$P\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \gamma_1 & \gamma_2 \\ \delta_1 & \delta_2 \end{pmatrix} P \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} \widetilde{f}(X_n, Y_n) \\ \widetilde{g}(X_n, Y_n) \end{pmatrix},$$

where

$$\begin{aligned} \widetilde{f}(X_n,Y_n) &= \Big(\gamma_{12}\gamma_2(\zeta-\gamma_1) + \gamma_{11}\gamma_2^2 + \gamma_{22}(\zeta-\gamma_1)^2\Big)X_n^2 - \Big(\gamma_{12}\gamma_2\xi \\ &\quad + 2(\zeta-\gamma_1)\gamma_{22}\xi\Big)X_nY_n + \gamma_{22}\xi^2Y_n^2, \\ \widetilde{g}(X_n,Y_n) &= \Big(\delta_{12}\gamma_2(\zeta-\gamma_1) + \delta_{11}\gamma_2^2 + \delta_{22}(\zeta-\gamma_1)^2\Big)X_n^2 - \Big(\delta_{12}\gamma_2\xi \\ &\quad + 2(\zeta-\gamma_1)\delta_{22}\xi\Big)X_nY_n + \delta_{22}\xi^2Y_n^2. \end{aligned}$$

Thus

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \zeta & -\xi \\ -\xi & \zeta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + P^{-1} \begin{pmatrix} \widetilde{f}(X_n, Y_n) \\ \widetilde{g}(X_n, Y_n) \end{pmatrix},$$

where

$$P^{-1} = \begin{pmatrix} \frac{1}{\gamma_2} & 0 \\ \frac{\zeta-\gamma_1}{\xi\gamma_2} & -\frac{1}{\xi} \end{pmatrix}.$$

$$\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} \zeta & -\xi \\ -\xi & \zeta \end{pmatrix} \begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} F(X_n,Y_n) \\ G(X_n,Y_n) \end{pmatrix}, \tag{3.6}$$

with

$$F(X_n,Y_n) = \frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big)X_n^2 - \frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big)X_nY_n + \frac{1}{\gamma_2}\gamma_{22}\xi^2Y_n^2,$$

and

$$\begin{aligned} G(X_n,Y_n) = {} & \Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big) \\ & - \frac{1}{\xi}\Big(\delta_{12}\gamma_2(\zeta-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\zeta-\gamma_1)^2\Big)\Bigg)X_n^2 \\ & - \Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big) - \frac{1}{\xi}\Big(\delta_{12}\gamma_2\xi \\ & +2(\zeta-\gamma_1)\delta_{22}\xi\Big)\Bigg)X_nY_n + \left(\frac{(\zeta-\gamma_1)\gamma_{22}\xi^2}{\xi\gamma_2} - \frac{\delta_{22}\xi^2}{\xi}\right)Y_n^2. \end{aligned}$$

In order for (3.6) to undergo a Neimark Sacker bifurcation, it is required that the following quantity is non zero [13]

$$N = -\Re\left[\frac{(1-2\overline{\eta})\overline{\eta}^2}{1-\eta}\tau_{11}\tau_{20}\right] - \frac{1}{2}\mid\tau_{11}\mid^2 - \mid\tau_{02}\mid^2 + \Re(\overline{\eta}\tau_{21}), \tag{3.7}$$

where

$$\begin{aligned} \tau_{02} = \frac{1}{4}\Bigg[& \Bigg(\frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big) - \frac{1}{\gamma_2}\gamma_{22}\xi^2 \\ & + \Bigg(\frac{\zeta-\gamma_1}{\xi\alpha_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big) - \frac{1}{\xi}\Big(\delta_{12}\gamma_2\xi+2(\zeta-\gamma_1)\delta_{22}\xi\Big)\Bigg)\Bigg) \\ & + i\Bigg(\Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big) \\ & - \frac{1}{\xi}\Big(\delta_{12}\gamma_2(\zeta-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\zeta-\gamma_1)^2\Big)\Bigg) - \left(\frac{(\zeta-\gamma_1)\gamma_{22}\xi^2}{\xi\gamma_2} - \frac{\delta_{22}\xi^2}{\xi}\right) \\ & - \frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\Big)\Bigg)\Bigg], \end{aligned}$$

$$\begin{aligned} \tau_{11} = \frac{1}{2}\Bigg[& \Bigg(\frac{1}{\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big) + \frac{1}{\gamma_2}\gamma_{22}\xi^2\Bigg) \\ & + i\Bigg(\Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_2}\Big(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\Big) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{\xi}\left(\delta_{12}\gamma_2(\zeta-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\zeta-\gamma_1)^2\right)\Bigg)\\
&+\left(\frac{(\zeta-\gamma_1)\gamma_{22}\xi^2}{\xi\gamma_2}-\frac{\delta_{22}\xi^2}{\xi}\right)\Bigg)\Bigg],\\
\tau_{20}=\frac{1}{4}\Bigg[&\Bigg(\frac{1}{\gamma_2}\left(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\right)\\
&-\frac{1}{\gamma_2}\gamma_{22}\xi^2-\Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_2}\left(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\right)\\
&-\frac{1}{\xi}\left(\delta_{12}\gamma_2\xi+2(\zeta-\gamma_1)\delta_{22}\xi\right)\Bigg)\Bigg)\\
&+i\Bigg(\Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_1 2}\left(\gamma_{12}\gamma_2(\zeta-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\zeta-\gamma_1)^2\right)\\
&-\frac{1}{\xi}\left(\delta_{12}\gamma_2(\zeta-\gamma_1)+\delta_{11}\gamma_2^2+\zeta_{22}(\zeta-\gamma_1)^2\right)\Bigg)\\
&+\Bigg(\frac{\zeta-\gamma_1}{\xi\gamma_2}\left(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\right)\\
&-\frac{1}{\xi}\left(\delta_{12}\gamma_2\xi+2(\zeta-\gamma_1)\delta_{22}\xi\right)\Bigg)+\frac{1}{\gamma_2}\left(\gamma_{12}\gamma_2\xi+2(\zeta-\gamma_1)\gamma_{22}\xi\right)\Bigg)\Bigg],\\
\tau_{21}=0.&
\end{aligned}
$$

Based on the above analysis, we state the following result on NSB.

Theorem 3. *If the condition* (3.4) *holds and N defined in* (3.7) *is nonzero then the model* (1.7a)-(1.7b) *undergoes NSB about the coexistent fixed point $E_2(x^*, y^*)$ when c^* varies near the origin. and $(\alpha, \beta, a, b, c, d) \in N_b$. Moreover, if $N < 0$ ($N > 0$) then an attracting (respectively repelling) invariant closed curve bifurcates from the fixed point $E_2(x^*, y^*)$ for $c > \bar{c}$ (respectively, $c < \bar{c}$).*

4. Period Doubling Bifurcation

For the fixed point $E_2(x^*, y^*)$ associated to the system (1.7a)-(1.7b). We define the space P_d as

$$
P_d = \left\{ c = \Lambda_1, \quad \mathrm{tr}^2 J(E_2) > 4\det J(E_2), \quad (\alpha, \beta, a, b, c, d) > 0 \right\}, \tag{4.1}
$$

one of the eigenvalues of $J(x^*, y^*)$ is -1 and the other is neither 1 nor -1. Therefore the system (1.7a)-(1.7b) undergoes PDB at the fixed point $E(x^*, y^*)$ if c varies in the small neighborhood of $c = \widehat{c}$ and $(\alpha, \beta, a, b, c, d) \in P_d$. Giving a perturbation c^*

(where $c^* \ll 1$) of the parameter c in the neighborhood of $c = \widehat{c}$ to the system (1.7a)-(1.7b) which is rewritten as follows

$$x_{n+1} = x_n \exp\left((1-x_n) - \frac{y}{\frac{1}{\alpha}x_n^2 + x_n + \beta} - \frac{a}{x_n + b}\right) = f(x_n, y_n, c^*), \tag{4.2a}$$

$$y_{n+1} = y_n \exp\left((c+c^*)\left(1 - \frac{dy_n}{x_n})\right)\right) = g(x_n, y_n, c^*). \tag{4.2b}$$

Let $v_n = x_n - x^*$, $w_n = y_n - y^*$, Then from (4.2) we set

$$v_{n+1} = (v_n + x^*)\exp\left((1-(v_n+x^*)) - \frac{(w_n+y^*)}{\frac{1}{\alpha}(v_n+x^*)^2 + (v_n+x^*) + \beta} - \frac{a}{(v_n+x^*)+b}\right) - x^*, \tag{4.3a}$$

$$w_{n+1} = (w_n + y^*)\exp\left((c+c^*)(1 - \frac{d(w_n+y^*)}{(v_n+x^*)})\right) - y^*. \tag{4.3b}$$

Expanding (4.3) in Taylor series about $(v_n, w_n, c^*) = (0,0,0)$, and considering the terms up to second order, we have

$$v_{n+1} = \gamma_1 v_n + \gamma_2 w_n + \gamma_{12} v_n w_n + \gamma_{11} v_n^2 + \gamma_{22} w_n^2, \tag{4.4}$$

$$\begin{aligned} w_{n+1} =& \delta_1 v_n + \delta_2 w_n + \delta_{12} v_n w_n + \delta_{11} v_n^2 + \delta_{22} w_n^2 + \delta_{13} c^* v_n + \delta_{13} c^* v_n \\ &+ \delta_{23} c^* w_n + \delta_{123} c^* v_n w_n + \delta_{113} c^* v_n^2 + \delta_{223} c^* w_n^2. \end{aligned} \tag{4.5}$$

where

$$\gamma_1 = f_x(x^*, y^*, 0) = 1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2} + x^* + \beta)^2} + \frac{a}{(x^*+b)^2}\right)\right),$$

$$\gamma_2 = f_y(x^*, y^*, 0) = -\frac{x^*}{\frac{1}{\alpha}x^{*2} + x^* + \beta},$$

$$\begin{aligned} \gamma_{12} = f_{xy}(x^*, y^*, 0) =& \frac{-1+x^*}{\frac{x^{*2}}{\alpha} + x^* + \beta} + \frac{2x^{*2} + \alpha x^*}{\alpha(\frac{x^{*2}}{\alpha} + x^* + \beta)^2} - \frac{2x^{*2}y^* + \alpha x^* y^*}{\alpha(\frac{x^{*2}}{\alpha} + x^* + \beta)^3} \\ &- \frac{ax^*}{(x+b)^2(\frac{x^{*2}}{\alpha} + x^* + \beta)} \end{aligned}$$

$$\begin{aligned} \gamma_{11} = f_{xx}(x^*, y^*, 0) =& -1 + \frac{2x^*y^* + \alpha y^* + y^* x^*}{\alpha(\frac{x^{*2}}{\alpha} + x^* + \beta)^2} - \frac{4x^{*2}y^* + \alpha^2 x^* y^* + 4\alpha x^{*2} y^*}{\alpha^2(\frac{x^{*2}}{\alpha} + x^* + \beta)^3} \\ &+ \frac{a}{(x+b)^2} + \frac{2x^{*3}y^{*2} + 2\alpha x^{*2} y^*}{\alpha^2(\frac{x^{*2}}{\alpha} + x^* + \beta)^4} + \frac{y^{*2}x^*}{2(\frac{x^{*2}}{\alpha} + x^* + \beta)^4} \\ &+ \frac{2x^{*2}y^* a + \alpha a x^* y^*}{\alpha(x+b)^2(\frac{x^{*2}}{\alpha} + x^* + \beta)^2} - \frac{ax}{(x+b)^3} + \frac{a^2 x^*}{2(x^*+b)^4} \end{aligned}$$

$$-\frac{2x^{*2}y^*+\alpha x^*y^*}{\alpha(\frac{x^{*2}}{\alpha}+x^*+\beta)^2}-\frac{ax^*}{(x^*+b)^3}+\frac{x^*}{2},$$

$$\gamma_{22}=f_{yy}(x^*,y^*,0)=\frac{x^*}{2(\frac{x^{*2}}{\alpha}+x^*+\beta)^2)},$$

$$\beta_1=g_x(x^*,y^*,0)=\frac{c}{d},$$

$$\beta_2=g_y(x^*,y^*,0)=1-c,$$

$$\beta_{12}=g_{xy}(x^*,y^*,0)=\frac{c}{x^*}(2-\frac{c}{x^*}),$$

$$\beta_{13}=g_{xc^*}(x^*,y^*,0)=\frac{1}{d},$$

$$\beta_{11}=g_{xx}(x^*,y^*,0)=\frac{cx^*}{2h}(-1+\frac{s}{x^{*2}}),$$

$$\beta_{22}=g_{yy}(x^*,y^*,0)=\frac{cd}{x^*}(-1+\frac{s}{2}),$$

$$\beta_{23}=g_{yc^*}(x^*,y^*,0)=-1,$$

$$\beta_{123}=g_{xyc^*}(x^*,y^*,0)=\frac{2}{x^*}(1-\frac{c}{x^*}),$$

$$\beta_{113}=g_{xxc^*}(x^*,y^*,0)=\frac{-x^*}{2d}+\frac{c}{dx^*},$$

$$\beta_{223}=g_{yyc^*}(x^*,y^*,0)=\frac{d}{x^*}(-1+\frac{c}{2})+\frac{cd}{2x^*}.$$

Now we define an invertible matrix $T=\begin{pmatrix}\gamma_2 & \gamma_2\\ -1-\gamma_1 & \eta_2-\gamma_1\end{pmatrix}$, and use the transformation $\begin{pmatrix}v_n\\ w_n\end{pmatrix}=T\begin{pmatrix}X_n\\ Y_n\end{pmatrix}$. Writing

$$v_n=\gamma_2(X_n+Y_n), w_n=-(1+\gamma_1)X_n+(\eta_2-\gamma_1)Y_n.$$

Then the system (4.4) becomes

$$T\begin{pmatrix}X_{n+1}\\ Y_{n+1}\end{pmatrix}=\begin{pmatrix}\gamma_1 & \gamma_2\\ \delta_1 & \delta_2\end{pmatrix}T\begin{pmatrix}X_n\\ Y_n\end{pmatrix}+T\begin{pmatrix}f_1(X_n,Y_n,c^*)\\ g_1(X_n,Y_n,c^*)\end{pmatrix},$$

where

$$f_1(X_n,Y_n,c^*)=\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)X_n^2 \tag{4.6}$$

$$+\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)-\gamma_{12}\gamma_2(1+\gamma_1)+2\gamma_{11}\gamma_2^2-\gamma_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)X_nY_n$$

$$+\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\eta_2-\gamma_1)^2\Big)Y_n^2,$$

and

$$
\begin{aligned}
g_1(X_n,Y_n,c^*) = & \Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big)X_n^2 \\
&+\Big(\delta_{113}\gamma_2^2+\delta_{223}(1+\gamma_1)^2-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_n^2c^* \\
&+\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\eta_2-\gamma_1)^2\Big)Y_n^2 \\
&+\Big(\delta_{223}\gamma_2^2+\delta_{223}(\eta_2-\gamma_1)^2+\delta_{123}\gamma_2(\eta_2-\gamma_1)\Big)Y_n^2c^* \\
&+\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)-\delta_{12}\gamma_2(1+\gamma_1)+2\delta_{11}\gamma_2^2-2\delta_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)X_nY_n \\
&+\Big(2\delta_{113}\gamma_2^2+2\delta_{223}(1+\gamma_1)(\eta_2-\gamma_1)+\delta_{123}\gamma_2(\eta_2-\gamma_1) \\
&-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_nY_nc^*+\Big(\delta_{13}\gamma_2-\delta_{23}(1+\gamma_1)\Big)X_nc^* \\
&+\Big(\delta_{13}\gamma_2+\delta_{23}(\eta_2-\gamma_1)\Big)Y_nc^*.
\end{aligned} \tag{4.7}
$$

Thus

$$
\begin{pmatrix} X_{n+1} \\ Y_{n+1} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & \eta_2 \end{pmatrix}\begin{pmatrix} X_n \\ Y_n \end{pmatrix} + \begin{pmatrix} F_1(X_n,Y_n,c^*) \\ G_1(X_n,Y_n,c^*) \end{pmatrix}, \tag{4.8}
$$

where

$$
\begin{aligned}
F_1(X_n,Y_n,c^*) = & \frac{\eta_2-\gamma_1}{1+\eta_2}\Bigg(\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)X_n^2 \\
&+\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)-\gamma_{12}\gamma_2(1+\gamma_1)+2\gamma_{11}\gamma_2^2-2\gamma_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)X_nY_n \\
&+\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(\eta_2-\gamma_1)^2\Big)Y_n^2 \\
&-\frac{1}{\eta_2+1}\Bigg(\Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big)X_n^2 \\
&+\Big(\delta_{113}\gamma_2^2+\delta_{223}(1+\gamma_1)^2-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_n^2c^* \\
&+\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\eta_2-\gamma_1)^2\Big)Y_n^2 \\
&+\Big(\delta_{223}\gamma_2^2+\delta_{223}(\eta_2-\gamma_1)^2+\delta_{123}\gamma_2(\eta_2-\gamma_1)\Big)Y_n^2c^*
\end{aligned}
$$

$$+\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)-\delta_{12}\gamma_2(1+\gamma_1)+2\delta_{11}\gamma_2^2-2\delta_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)X_nY_n$$
$$+\Big(2\delta_{113}\gamma_2^2+2\delta_{223}(1+\gamma_1)(\eta_2-\gamma_1)+\delta_{123}\gamma_2(\eta_2-\gamma_1)$$
$$-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_nY_nc^*+\Big(\delta_{13}\gamma_2-\delta_{23}(1+\gamma_1)\Big)X_nc^*$$
$$+\Big(\delta_{13}\gamma_2+\delta_{23}(\eta_2-\gamma_1)\Big)Y_nc^*\Big)\Big).$$

and

$$G_1(X_n,Y_n,c^*)=\frac{1+\gamma_1}{1+\eta_2}\Big(\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2\Big)X_n^2$$
$$+\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)-\gamma_{12}\gamma_2(1+\gamma_1)+2\gamma_{11}\gamma_2^2\Big)X_nY_n$$
$$+\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)+\gamma_{11}\gamma_2^2\Big)Y_n^2\Big)$$
$$+\frac{1}{\eta_2+1}\Big(\Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big)X_n^2$$
$$+\Big(\delta_{113}\gamma_2^2+\delta_{223}(1+\gamma_1)^2-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_n^2c^*$$
$$+\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(\eta_2-\gamma_1)^2\Big)Y_n^2$$
$$+\Big(\delta_{223}\gamma_2^2+\delta_{223}(\eta_2-\gamma_1)^2+\delta_{123}\gamma_2(\eta_2-\gamma_1)\Big)Y_n^2c^*$$
$$+\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)-\delta_{12}\gamma_2(1+\gamma_1)+2\delta_{11}\gamma_2^2-2\delta_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)X_nY_n$$
$$+\Big(2\delta_{113}\gamma_2^2+2\delta_{223}(1+\gamma_1)(\eta_2-\gamma_1)+\delta_{123}\gamma_2(\eta_2-\gamma_1)$$
$$-\delta_{123}\gamma_2(1+\gamma_1)\Big)X_nY_nc^*+\Big(\delta_{13}\gamma_2-\delta_{23}(1+\gamma_1)\Big)X_nc^*$$
$$+\Big(\delta_{13}\gamma_2+\delta_{23}(\eta_2-\gamma_1)\Big)Y_nc^*\Big)\Big).$$

Hereafter we determine the center manifold $\mathcal{W}_c(0,0)$ of (4.8) about $(0,0)$ in a small neighborhood of c^*. By center manifold theorem, there exists a center manifold

$\mathcal{W}_c(0,0)$ that can be represented as follows:

$$\mathcal{W}_c(0,0)=\{(X_n,Y_n):Y_n=\phi(X_n,c^*)=a_1X_n^2+a_2X_nc^*+a_3c^{*2}+O((\mid X_n\mid+\mid c^*\mid)^2)\},\tag{4.9}$$

where $O((\mid X_n\mid+\mid c^*\mid)^2)$ is a function with order at least three in their variables (X_t,s^*), Moreover, the center manifold must satisfy

$$\phi\Big(-X_n+F_1(X_n,\phi(X_n,c^*)),c^*),c^*\Big)-\eta_2\phi(X_n,c^*)-G_1(X_n,\phi(X_n,c^*),c^*)=0.\tag{4.10}$$

By equating (4.10), we obtain

$$\begin{aligned}a_1&=\frac{1+\gamma_1}{1-\eta_2^2}\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)\\&\quad+\frac{1}{1-\eta_2^2}\Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big),\\a_2&=\frac{-1}{1+\eta_2}\Big(\delta_{13}\gamma_2-\beta_{23}(1+\alpha_1)\Big),\quad a_3=0.\end{aligned}$$

Therefore, we consider the map which is the map (4.8) restricted to the center manifold $\mathcal{W}_c(0,0)$

$$f=X_{n+1}=-X_n+c_1X_nc^*+c_2X_n^2+c_3X_n^2c^*+c_4X_n^3,\tag{4.11}$$

where

$$\begin{aligned}c_1&=-\frac{1}{1+\eta_2}\Big(\delta_{13}\gamma_2-\delta_{23}(1+\gamma_1)\Big),\\c_2&=\frac{\eta_2-\gamma_1}{1+\eta_2}\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)\\&\quad-\frac{1}{\eta_2+1}\Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big),\\c_3&=\frac{-1}{1+\eta_2}\Big(\delta_{13}\gamma_2-\delta_{23}(1+\gamma_1)\Big)\Big[\frac{\eta_2-\gamma_1}{1+\eta_2}\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)-\gamma_{12}\gamma_2(1+\gamma_1)+2\gamma_{11}\gamma_2^2\\&\quad-2\gamma_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)-\frac{1}{1+\eta_2}\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)-\delta_{12}\gamma_2(1+\gamma_1)+2\delta_{11}\gamma_2^2\\&\quad-2\gamma_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)\Big]-\frac{1}{1+\eta_2}\Big(\delta_{113}\gamma_2^2+\delta_{223}(1+\gamma_1)^2-\delta_{123}\gamma_2(1+\gamma_1)\Big)\\&\quad\times\Big(\frac{1+\gamma_1}{(1-\eta_2^2)(1+\eta_2)}\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)\\&\quad+\frac{1}{(1-\eta_2^2)(1+\eta_2)}\Big(-\delta_{12}\gamma_2(1+\gamma_1)+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big)\Big)\end{aligned}$$

$$
\begin{aligned}
&\times\Big(\delta_{13}\gamma_2+\delta_{23}(\eta_2-\gamma_1)\Big),\\
c_4 =& \left(\frac{1+\gamma_1}{1-\eta_2^2}\Big(-\gamma_{12}\gamma_2(1+\gamma_1)+\gamma_{11}\gamma_2^2+\gamma_{22}(1+\gamma_1)^2\Big)+\frac{1}{1-\eta_2^2}\Big(-\delta_{12}\gamma_2(1+\gamma_1)\right.\\
&\left.+\delta_{11}\gamma_2^2+\delta_{22}(1+\gamma_1)^2\Big)\right)\left[\frac{\eta_2-\gamma_1}{1+\eta_2}\Big(\gamma_{12}\gamma_2(\eta_2-\gamma_1)-\gamma_{12}\gamma_2(1+\gamma_1)+2\gamma_{11}\gamma_2^2\Big)\right.\\
&-\frac{1}{1+\eta_2}\Big(\delta_{12}\gamma_2(\eta_2-\gamma_1)-\delta_{12}\gamma_2(1+\gamma_1)+2\delta_{11}\gamma_2^2\\
&\left.-2\delta_{22}(1+\gamma_1)(\eta_2-\gamma_1)\Big)\right].
\end{aligned}
$$

In order for the map (4.11) to undergo a period-doubling bifurcation, we require that the following discriminatory quantities are non-zero [13]:

$$
\sigma_1=\left(\frac{\partial^2 f}{\partial X_n \partial c^*}+\frac{1}{2}\frac{\partial f}{\partial c^*}\frac{\partial^2 f}{\partial^2 X_n}\right)|_{(0,0)}\neq 0,
$$

and

$$
\sigma_2=\left(\frac{1}{6}\frac{\partial^3 f}{\partial X_n^3}+(\frac{1}{2}\frac{\partial^2 f}{\partial X_n^2})^2\right)|_{(0,0)}\neq 0.
$$

After calculating we get

$$
\sigma_1=c_1+c_3,\ \ \sigma_2=c_4+c_2^2.
$$

From the above analysis we have the following theorem.

Theorem 4. *If $\sigma_2\neq 0$, and $\sigma_1\neq 0$, then the system (1.7a)-(1.7b) experiences a period-doubling bifurcation about the unique positive fixed point $E_2(x^*,y^*)$ when c^* varies in a small neighborhood of $O(0,0)$. Moreover, if $\sigma_2>0$ (resp $\sigma_2<0$), then the period 2 points that bifurcate from $E_2(x^*,y^*)$ are stable (unstable).*

5. Chaos Control

In this paper, we apply the state feedback method [3] to stabilize the chaotic orbits at an unstable fixed point of system. Towards this we introduce a feedback control force P_n such that

$$
x_{n+1}=x_n\exp\left((1-x_n)-\frac{y_n}{\frac{1}{\alpha}x_n^2+x_n+\beta}-\frac{a}{x_n+b}\right), \tag{5.1}
$$

$$
y_{n+1}=y_n\exp\left(\widehat{c}\Big(1-\frac{dy_n}{x_n})\Big)\right)-\underbrace{\mu_1(x_n-x^*)-\mu_2(y_n-y^*)}_{\text{The control force } P_n}. \tag{5.2}
$$

where μ_1, μ_2 are feedback gains and $\widehat{c}$ is the nominal value for c which belongs to some chaotic regions. The Jacobian matrix of (5.1)-(5.2) at E_2 is

$$J(E_2) = \begin{pmatrix} 1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right) & -\frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta} \\ \frac{\widehat{c}}{d} - \mu_1 & 1 - \widehat{c} - \mu_2 \end{pmatrix}. \tag{5.3}$$

The corresponding characteristic equation of (5.3) is

$$\lambda^2 - \left(2 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right) - \widehat{c} - \mu_2\right)\right)\lambda \tag{5.4}$$
$$+ \left(1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right)\left(1 - \widehat{c} - \mu_2\right)$$
$$+ \frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}\left(\frac{\widehat{c}}{d} - \mu_1\right) = 0.$$

Let λ_1 , λ_2 are the eigenvalues of the characteristic Eq.(5.4) then sum and the product of their roots are given by

$$\lambda_1 + \lambda_2 = 2 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right) - \widehat{c} - \mu_2\right), \tag{5.5}$$

$$\lambda_1\lambda_2 = \left(1 - x^*\left(1 - \left(y^* \frac{(\frac{2x^*}{\alpha}+1)}{(\frac{1}{\alpha}x^{*2}+x^*+\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right)\left(1 - \widehat{c} - \mu_2\right)$$
$$+ \frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}\left(\frac{\widehat{c}}{d} - \mu_1\right). \tag{5.6}$$

Lemma 1. *The system* (5.1)-(5.2) *is asymptotically stable if all the eigenvalues of the characteristic Eq.* (5.4) *have modulus less than 1.*

Proof. The marginal stability lines can be obtained from the conditions $\lambda_1 = \pm 1$, $\lambda_1\lambda_2 = 1$. For the conditions $\lambda_1\lambda_2 = 1$, Eq.(5.6) gives

$$L_1 : \frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}\mu_1 + \left(1 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right)\mu_2$$
$$= \left[1 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right]\left(1 - \widehat{c}\right)$$
$$+ \frac{x^*\widehat{c}}{d(\frac{1}{\alpha}x^{*2}+x^*+\beta)}, \tag{5.7}$$

The Eq. (5.7) expresses the first condition for marginal stability. For $\lambda_1 = 1$, the Eq. (5.5) yields

$$L_2 : \frac{x^*}{\frac{1}{\alpha}x^{*2}+x^*+\beta}\mu_1 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\mu_2\right.$$

$$= \widehat{c} + \frac{x^*\widehat{c}}{d(\frac{1}{\alpha}x^{*2} + x^* + \beta)} - \widehat{c}\left[1 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^* + \alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right],$$

similarly for $\lambda_1 = -1$, it gives

$$L_3 : \frac{x^*}{\frac{1}{\alpha}x^{*2} + x^* + \beta}\mu_1 + \left(2 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right)$$
$$= \left[1 - x^*\left(1 - \left(\frac{x^*}{d}\frac{2x+\alpha}{(x^{*2}+\alpha x^*+\alpha\beta)^2} + \frac{a}{(x^*+b)^2}\right)\right)\right]\left(2 - \widehat{c}\right) + 2 - \widehat{c}$$
$$+ \frac{x^*\widehat{c}}{d(\frac{1}{\alpha}x^{*2} + x^* + \beta)}.$$

The lines L_1, L_2, L_3 give the conditions for the eigenvalues to have absolute value less than 1. The triangular region bounded by these lines accommodates stable eigenvalues. □

6. Numerical simulations

Let $(\alpha, \beta, a, b, d) = (0.19, 1, 0.0001, 0.75, 0.3)$ [2] and initial conditions $(x_0, y_0) = (0.5, 0.5)$. The coexistent fixed point $E_2(0.4, 1.35)$ is locally asymptotically stable if c takes higher values than 3, which means that all the orbits attract towards the positive fixed point $E_2(0.4, 1.35)$. By decreasing the value of c from $c = 0.3$ to $c = 0.25$, the system (1.7a)-(1.7b) loses its stability. In particular, we observe the appearance of a closed invariant curve which indicate that the discrete-time model (1.7a)-(1.7b) experiences Neimark-Sacker bifurcation about $E_2 = (0.4, 1.35)$. To see this, we compute the value of N defined in (3.7), that is $N = -0.0081364312 < 0$. Hence, the model (1.7a)-(1.7b) admits supercritical Neimark-Sacker bifurcation if $c < 0.3$ and meanwhile, stable curve appears, see Fig. 2.

Additional simulations are plotted to show the emergence of periodic doubling bifurcation for $(\alpha, \beta, a, b, d) = (0.19, 1, 0.0001, 1, 0.3)$ and initial conditions $(0.5, 0.5)$. For these values E_2 undergo periodic doubling bifurcation, see Fig. 3.

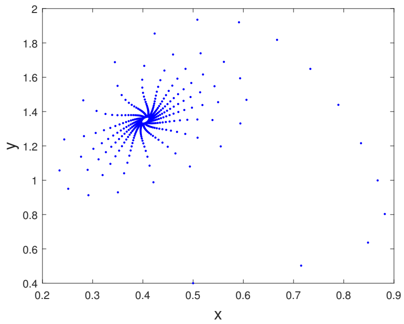

FIGURE 1. *Phase portrait of the system* (1.7) *for* $c = 0.3$.

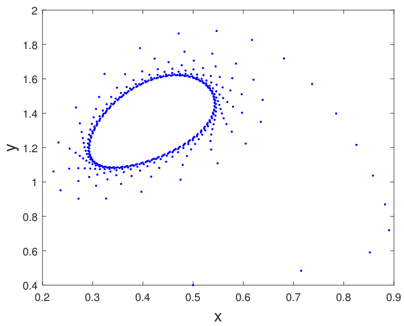

FIGURE 2. *Supercritical Neimark Sacker bifurcation of the system* (1.7) *for* $c = 0.25$.

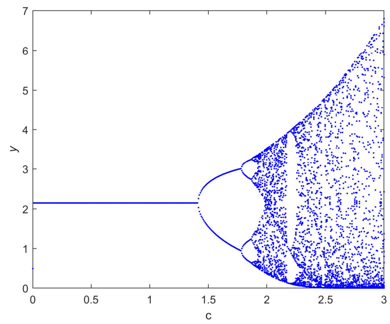

FIGURE 3. Period doubling bifurcation with respect to c.

Now, we use the state feedback method described in Section. 5 to control the chaos influenced by N-S bifurcation, the following set of parameter is taken : $(\alpha, \beta, a, b, d) = (0.19, 1, 0.0001, 0.75, 0.3)$ and $\widehat{c} = 0.25$. Using lemma 1, the domain of stability of the controlled system (5.1)-(5.2) is drawn in Fig.4.

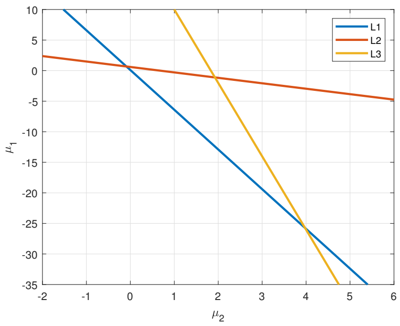

FIGURE 4. Stability region of the controlled system (5.1)-(5.2).

Now, in order to stabilize the chaos in the system (1.7), we consider the feedback controlling force $P_n = \mu_1(x_n - 0.4) + \mu_2(y_n - 1.35)$ with feedback gains $\mu_1 = -0.2$, $\mu_2 = 0.35$chosen from the triangular region from Fig.4. Accordingly to these values, the system (1.3)-(1.4) converges to the fixed point $(0.4, 1.35)$ as shown in Fig.5. The periodic solution is stabilized at time $t = 150$ and reproduce the efficiency of the implemented method.

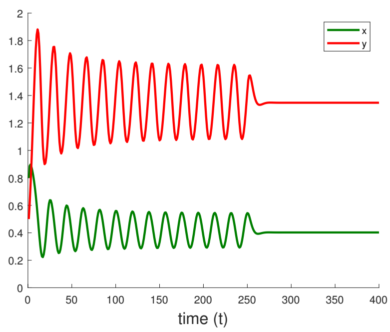

FIGURE 5. The chaos is controlled after time t=250.

7. CONCLUDING REMARKS

In this paper, we explored the dynamical proprieties of a discrete-time two-dimensional prey predator system. The model is formulated in terms of difference equations, by a discretization of a differential prey predator system involving a nonlinear harvesting effect on the prey and Holling functional response type IV. The discretization is based on the higher-order implicit Runge Kutta method with a very small step size to attain a discrete time version of its continuous counterpart. The existence and local asymptotic stability of the fixed points are investigated. In order to support the complexity of (1.7a)-(1.7b) the presence of Neimark-Sacker bifurcation and periodic doubling bifurcation for the coexistence fixed point $E_2(x^*, y^*)$ is proved analytically by using bifurcation theory and center manifold theory. In addition, the paper provides the method of state feedback and parameter perturbation for the bifurcation control. Numerical simulations carried out to verify our theoretical analysis. In particular, we showed that the discrete model (1.7a)-(1.7b) loses its asymptotic stability when the intrinsic growth rate of the predator $c$varies in the neighborhood of 3.1 and 4.1, and when the fixed point $E_2 = (0.4, 1.35)$ loses stability (all orbits are not attracted to this fixed point), a feedback chaos control strategy is used to stabilize the chaos. Numerical simulations give evidence of the successful implementation of the method. Using Evolutionary Game theory to explain how phenotypic traits change over time will be a fascinating topic for our future work [17]. We want to do an asymptotic study of the global stability of the present discrete model. As a result, new methods for determining the global stability of fixed points, such as the method of Lyapunov functions, are required [5].

ACKNOWLEDGMENT

We would like to thank anonymous referees very much for their valuable comments and suggestions. Professor Saber Elaydi is thanked by K.M for his excellent guidance during his Ph.D.

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