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A NUMERICAL COMPARATIVE STUDY FOR THE SINGULARLY PERTURBED NONLINEAR VOLTERRA-FREDHOLM INTEGRO-DIFFERENTIAL EQUATIONS ON LAYER-ADAPTED MESHES

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Abstract. This article deals with the singularly perturbed nonlinear Volterra-Fredholm integro-differential equations. Firstly, some priori bounds are presented. Then, the finite difference scheme is constructed on non-uniform mesh by using interpolating quadrature rules [5] and composite numerical integration formulas. The error estimates are derived in the discrete maximum norm. Finally, theoretical results are performed on two examples and they are compared for both Bakhvalov (B-type) and Shishkin (S-type) meshes.

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1. Introduction

Nonlinear integro-differential equations (NIDEs) are effective tools for modelling of many real life situations. Their applications appear in physical processes [14], biological events [27], population dynamics [20], financial problems [1] and other areas [31].

In the literature, many papers have been written about NIDEs. In general, solving of such problems is hard and even sometimes impossible with classical analytical techniques. Thus, reliable numerical methods have been developed such as Bernstein polynomial method [8], collocation techniques [32], Nyström discretization [24], Legendre wavelets [22], finite difference methods [11, 29] and Galerkin finite element approach [18].

The above-mentioned studies have only dealt with the regular cases (absent the layer behavior). This article concerns with the following initial-value problem of the singularly perturbed nonlinear Volterra-Fredholm integro-differential equation:

$$Lu := \varepsilon u'(t) + f(t,u(t)) + \int_0^t K_1(t,s,u(s))ds + \lambda \int_0^T K_2(t,s,u(s))ds = 0, \quad (1.1)$$

$$u(0) = A, \quad t \in I = (0,T], \quad (1.2)$$

where $0 < \varepsilon \ll 1$ is the perturbation parameter, λ is an arbitrary parameter, $\bar{I} = [0,T]$ and A is a given constant. $f(t,u(t))$ $((t,u) \in \bar{I} \times \mathbb{R})$ and $K_1(t,s,u(s))$, $K_2(t,s,u(s))$ $((t,s,u) \in \bar{I} \times \bar{I} \times \mathbb{R})$ are sufficiently smooth functions and $0 < \alpha \leq \left|\frac{\partial f}{\partial u}\right| \leq p^* < \infty$. In [16], A. A. Hamoud and K. P. Ghadle have discussed the existence and uniqueness results of the problem (1.1)-(1.2) without singular perturbation. In [31], different variational techniques have been proposed for Volterra-Fredholm integro-differential equations. Furthermore, the linear form of the problem (1.1)-(1.2) has been considered by using finite difference schemes in [9, 13].

Singular perturbation phenomena is classified by a small parameter ε multiplying highest order derivative terms in the differential equation. The class of such problems generally involves the boundary layers. The solution of the problem changes rapidly within layer region whereas it behaves slowly and regularly outside of the layer region. To overcome this complexity, robust discretizations are needed [12, 21, 23, 26]. Some existence and uniqueness results about singularly perturbed problems have been given in [15, 23]. In recent times, notable techniques and various numerical schemes have been presented for singularly perturbed integro-differential equations (see [2, 6, 9, 10, 13, 17, 19, 25, 29, 33–35]). Our aim in this paper is to present a uniform numerical method for solving singularly perturbed nonlinear integro-differential equations and compare the obtained results on Bakhvalov and Shishkin type meshes. In addition to the subject, Bakhvalov mesh was developed by N. S. Bakhvalov in 1969 [7] and G. I. Shishkin designed the piecewise equidistant mesh [30].

The rest of the paper is as follows: The properties of the analytical solution of the problem (1.1)-(1.2) are given in Section 2. In Section 3, the difference scheme is established on non-uniform mesh. The stability and convergence of the method are investigated in Section 4. In Section 5, some numerical experiments are presented.

2. Preliminaries

This section is devoted to the asymptotic bounds of the problem (1.1)-(1.2).

Lemma 1 (see [19, Lemma 2.1]). *Take into account the following initial-value problem:*

$$\varepsilon v'(t) + a(t)v(t) = F(t), \quad t \in I, \quad (2.1)$$

$$v(0) = A. \quad (2.2)$$

Let $a(t) \geq \alpha > 0$, $F(t) \in C(\bar{I})$, $|F(t)| \leq \mathcal{F}(t)$ and $\mathcal{F}(t)$ is a nondecreasing function. Then, the solution of the problem (2.1)-(2.2) holds that

$$|v(t)| \leq |A| + \alpha^{-1}\mathcal{F}(t), \quad t \in I.$$

Lemma 2. *Suppose that*

$$p, q \in C^1[0,T], \quad \frac{\partial}{\partial t}G_1(t,s) \in C^1[0,T]^2, \quad \frac{\partial}{\partial t}G_2(t,s) \in C^1[0,T]^2 \tag{2.3}$$

and

$$\gamma = e^{\alpha^{-1}\bar{G}_1 T}\alpha^{-1}|\lambda| \max_{0\leq t\leq T} \int\limits_0^t |G_2(t,s)|\,ds < 1.$$

Then, the solution $u(t)$ of the problem (1.1)-(1.2) satisfies that

$$\|u\|_\infty \leq C_0 \tag{2.4}$$

and

$$|u'(t)| \leq C\left\{1 + \frac{1}{\varepsilon}e^{\frac{-\alpha t}{\varepsilon}}\right\}, \quad t \in [0,T], \tag{2.5}$$

where

$$C_0 = (1-\gamma)^{-1}\left(|A| + \alpha^{-1}\|q\|_\infty\right)e^{\alpha^{-1}\bar{G}_1 T}$$

and

$$\bar{G}_1 = \max_{\bar{I}\times\bar{I}} |G_1(t,s)|.$$

Here, for any continuous function $v(t)$ defined on the corresponding interval, we use the maximum norm $\|v\|_\infty = \max\limits_{[0,T]}|v(t)|$ and C (in some cases subscripted) is a generic positive constant [2, 29].

Proof. Firstly, we prove the relation (2.4). Applying the mean value theorem to the functions in the equation (1.1), we find

$$\varepsilon u'(t) + p(t)u(t) + \int_0^t G_1(t,s)u(s)ds + \lambda\int_0^T G_2(t,s)u(s)ds = q(t), \quad t \in I \tag{2.6}$$

where

$$p(t) = \frac{\partial}{\partial u}f(t,\tilde{u}), \quad \tilde{u} = \rho u, \quad 0 < \rho < 1,$$

$$G_1(t,s) = \frac{\partial}{\partial u}K_1(t,s,\bar{u}), \quad \bar{u} = \theta u, \quad 0 < \theta < 1,$$

$$G_2(t,s) = \frac{\partial}{\partial u}K_2(t,s,\breve{u}), \quad \breve{u} = \zeta u, \quad 0 < \zeta < 1,$$

and

$$q(t) = -f(t,0) - \int_0^t K_1(t,s,0)ds - \lambda\int_0^T K_2(t,s,0)ds.$$

From here, we have

$$\varepsilon u'(t) + p(t)u(t) = F(t), \quad t \in I, \tag{2.7}$$

where

$$F(t) = q(t) - \int_0^t G_1(t,s)u(s)ds - \lambda \int_0^T G_2(t,s)u(s)ds. \tag{2.8}$$

Then, we estimate (2.8) as

$$|F(t)| \le \|q\|_\infty + \bar{G}_1 \int_0^t |u(s)|ds + |\lambda| \int_0^T |G_2(t,s)||u(s)|ds.$$

Considering Lemma 1 for the equation (2.7), we get

$$|u(t)| \le \delta + \alpha^{-1}\bar{G}_1 \int_0^t |u(s)|ds, \tag{2.9}$$

where

$$\delta = |A| + \alpha^{-1}\|q\|_\infty + \alpha^{-1}|\lambda| \int_0^T |G_2(t,s)||u(s)|ds.$$

Applying the Gronwall's inequality to the relation (2.9), we obtain

$$\|u\|_\infty \le \frac{\left(|A| + \alpha^{-1}\|q\|_\infty\right)\exp(\alpha^{-1}\bar{G}_1 T)}{\left(1 - \alpha^{-1}|\lambda|\exp(\alpha^{-1}\bar{G}_1 T)\max\limits_{t\in[0,T]} \int_0^t |G_2(t,s)|ds\right)},$$

which points to the proof of (2.4). Now, we show the proof of the relation (2.5) (see [9, 11, 13, 29]). From (2.6), we write

$$|u'(0)| \le \frac{1}{\varepsilon}\left(|q(0)| + |p(0)||A| + |\lambda| \int_0^T |G_2(0,s)||u(s)|ds\right) \le \frac{C}{\varepsilon}. \tag{2.10}$$

Differentiating the equation (2.7), we find

$$\varepsilon v' + p(t)v = H(t), \quad v(t) = u'(t), \tag{2.11}$$

where

$$H(t) = q'(t) - p'(t)u(t) - \int_0^t \frac{\partial}{\partial t}G_1(t,s)u(s)ds - G_1(t,t)u(t) - \lambda \int_0^T \frac{\partial}{\partial t}G_2(t,s)u(s)ds.$$

Because of (2.3) and (2.4), we have

$$|H(t)| \le C. \tag{2.12}$$

From (2.11), it can be seen that

$$u'(t) = u'(0)e^{-\frac{1}{\varepsilon}\int_0^t p(\eta)d\eta} + \frac{1}{\varepsilon}\int_0^t H(\xi)\, e^{-\frac{1}{\varepsilon}\int_\xi^t p(\eta)d\eta} d\xi.$$

By taking into account (2.10) and (2.12), it is obtained:

$$|u'(t)| \le \frac{C}{\varepsilon}e^{\frac{-\alpha t}{\varepsilon}} + \alpha^{-1}\|H\|_\infty\left(1 - e^{\frac{-\alpha t}{\varepsilon}}\right),$$

which shows the validity of (2.5). □

3. Discrete Scheme

In this section, we construct the difference scheme on the non-uniform mesh.

Notation 1 ([28]). Let ω_N be any non-uniform mesh on $[0,T]$:

$$\omega_N = \{0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T,\ \tau_i = t_i - t_{i-1},\ i = 1,2,\ldots,N\}$$

and

$$\overline{\omega}_N = \omega_N \cup \{t_0 = 0\}.$$

For any mesh function on $\overline{\omega}_N$, we use [28]

$$v_i = v(t_i), \quad v_{\bar{t},i} = \frac{v_i - v_{i-1}}{\tau_i}, \quad \|v\|_\infty \equiv \|v\|_{\infty,\overline{\omega}_N} := \max_{0\le i\le N}|v_i|.$$

Definition 1 ([3, 21, 23, 26, 30, Shishkin-Type Mesh])**.** For an even number N, we divide each of the subintervals $[0,\sigma]$ and $[\sigma,T]$ into $\frac{N}{2}$ equidistant subintervals. The transition parameter σ is stated as

$$\sigma = \min\left\{\frac{T}{2}, \alpha^{-1}\varepsilon \ln N\right\}.$$

We use the notation $\tau^{(1)}$ for the mesh width in $[0,\sigma]$ and the notation $\tau^{(2)}$ for the width in $[\sigma,T]$. Hence, the mesh stepsizes hold

$$\tau^{(1)} = \frac{2\sigma}{N}, \quad \tau^{(2)} = \frac{2(T-\sigma)}{N},$$

$$\tau^{(1)} \le TN^{-1}, \quad TN^{-1} \le \tau^{(2)} \le 2TN^{-1}, \quad \tau^{(1)} + \tau^{(2)} = 2TN^{-1}.$$

t_i mesh points are shown by

$$\overline{\omega}_N = \begin{cases} t_i = i\tau^{(1)}, & i = 0,1,\ldots,\frac{N}{2},\ t_i \in [0,\sigma]; \\ t_i = \sigma + \left(i - \frac{N}{2}\right)\tau^{(2)}, & i = \frac{N}{2}+1,\ldots,N,\ t_i \in [\sigma,T]. \end{cases}$$

Definition 2 ([4, 7, 21, 26, Bakhvalov-Type Mesh])**.** We divide each of the subintervals $[0,\sigma]$ and $[\sigma,T]$ into $\frac{N}{2}$ equidistant subintervals. The transition point σ is expressed by

$$\sigma = \min\left\{\frac{T}{2}, \alpha^{-1}\varepsilon\left|\ln\varepsilon\right|\right\}.$$

t_i node points are described as follows:

If $\sigma < \frac{T}{2}$,

$$t_i = \begin{cases} -\alpha^{-1}\varepsilon\ln\left[1-(1-\varepsilon)\,2i/N\right], & i = 0,1,\ldots,\frac{N}{2}, \\ \sigma + \left(i-\frac{N}{2}\right)\tau, & i = \frac{N}{2}+1,\ldots,N, \end{cases}$$

and if $\sigma = \frac{T}{2}$,

$$t_i = \begin{cases} -\alpha^{-1}\varepsilon\ln\left[1-\left(1-\exp(\frac{-\alpha T}{2\varepsilon}\right)2i/N\right], & i = 0,1,\ldots,\frac{N}{2}, \\ \sigma + \left(i-\frac{N}{2}\right)\tau, & i = \frac{N}{2}+1,\ldots,N, \end{cases}$$

where $\tau = 2\left(T-\sigma\right)/N$.

Now, we establish the finite difference scheme. To design the difference method, we utilize that

$$\tau_i^{-1}\int_{t_{i-1}}^{t_i}\varepsilon u'(t)dt + \tau_i^{-1}\int_{t_{i-1}}^{t_i} f(t,u)dt + \tau_i^{-1}\int_{t_{i-1}}^{t_i}\int_0^t K_1(t,s,u(s))dsdt + \tau_i^{-1}\lambda\int_{t_{i-1}}^{t_i}\int_0^T K_2(t,s,u(s))dsdt = 0. \quad (3.1)$$

Applying interpolating quadrature rules [5] and some manipulations in [11, 29] for the first term and the second term of the relation (3.1), we find

$$\tau_i^{-1}\int_{t_{i-1}}^{t_i}\left[\varepsilon u'(t)+f(t,u)\right]dt = \varepsilon u_{\bar{t},i} + f(t_i,u_i) + R_i^{(1)} \quad (3.2)$$

where

$$R_i^{(1)} = -\tau_i^{-1}\int_{t_{i-1}}^{t_i}\left(\xi - t_{i-1}\right)\frac{\partial}{\partial\xi}f(\xi,u(\xi))d\xi. \quad (3.3)$$

For the third and fourth terms of (3.1), using interpolating quadrature rules again [5, 11, 29] and right side rectangle formula, we get

$$\tau_i^{-1}\int_{t_{i-1}}^{t_i}\int_0^t K_1(t,s,u(s))dsdt + \tau_i^{-1}\lambda\int_{t_{i-1}}^{t_i}\int_0^T K_2(t,s,u(s))dsdt$$
$$= \sum_{j=1}^{i}\tau_j K_{1,ij}(t_i,t_j,u_j) + \lambda\sum_{j=1}^{N}\tau_j K_{2,ij}(t_i,t_j,u_j) + R_i^{(2)} + R_i^{(3)} + R_i^{(4)} + R_i^{(5)} \quad (3.4)$$

where remainder terms

$$R_i^{(2)} = -\tau^{-1}\int_{t_{i-1}}^{t_i}\left(\xi - t_{i-1}\right)\left(\int_0^t\frac{\partial}{\partial\xi}K_1(\xi,s,u(s))ds\right)d\xi, \quad (3.5)$$

$$R_i^{(3)} = \sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} (\xi - t_{j-1}) \frac{\partial}{\partial \xi} K_1(t_i, \xi, u(\xi)) d\xi, \tag{3.6}$$

$$R_i^{(4)} = -\lambda \tau_i^{-1} \int_{t_{i-1}}^{t_i} (\xi - t_{i-1}) \left(\int_0^T \frac{\partial}{\partial \xi} K_2(\xi, s, u(s)) ds \right) d\xi \tag{3.7}$$

and

$$R_i^{(5)} = \lambda \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} (\xi - t_{j-1}) \frac{\partial}{\partial \xi} K_2(t_i, \xi, u(\xi)) d\xi. \tag{3.8}$$

From the relations (3.2) and (3.4), the following difference problem is written:

$$\varepsilon u_{\bar{t},i} + f(t_i, u_i) + \sum_{j=1}^{i} \tau_j K_{1,i,j}(t_i, t_j, u_j) + \lambda \sum_{j=1}^{N} \tau_j K_{2,i,j}(t_i, t_j, u_j) + R_i = 0, \tag{3.9}$$

$$u_0 = A, \quad i = 1, 2, \ldots, N,$$

where the error term is expressed by

$$R_i = \sum_{k=1}^{5} R_i^{(k)}. \tag{3.10}$$

Omitting the error term R_i in (3.9), we can present that

$$\varepsilon y_{\bar{t},i} + f(t_i, y_i) + \sum_{j=1}^{i} \tau_j K_{1,ij}(t_i, t_j, y_j) + \lambda \sum_{j=1}^{N} \tau_j K_{2,ij}(t_i, t_j, y_j) = 0, \tag{3.11}$$

$$y_0 = A, \quad i = 1, 2, \ldots, N.$$

4. Convergence Analysis

Let the error function $z_i = y_i - u_i$, $i = 0, 1, 2, \ldots, N$ be the solution of the following discrete problem:

$$lz_i = \varepsilon z_{\bar{t},i} + [f(t_i, y_i) - f(t_i, u_i)] + \sum_{j=1}^{i} \tau_j [K_{1,ij}(t_i, t_j, y_j) - K_{1,ij}(t_i, t_j, u_j)]$$

$$+ \lambda \sum_{j=1}^{N} \tau_j [K_{2,ij}(t_i, t_j, y_j) - K_{2,ij}(t_i, t_j, u_j)] = R_i, \tag{4.1}$$

$$i = 1, 2, \ldots, N, \quad z_0 = 0, \tag{4.2}$$

where R_i is given by (3.10).

Lemma 3 (see [19, Lemma 4.1]). *Consider the following difference problem:*

$$\varepsilon v_{\bar{t},i} + p_i v_i = F_i, \quad i = 0, 1, 2, \ldots, N, \quad v_0 = A. \tag{4.3}$$

Let $|F_i| \leq \mathcal{F}_i$ and $\mathcal{F}_i$ be nondecreasing function. Then, the solution of (4.3) satisfies

$$|v_i| \leq |A| + \alpha^{-1} \mathcal{F}_i, \quad i = 0, 1, 2, \ldots, N.$$

Lemma 4. *Let z_i be the solution of the problem (4.1)-(4.2). If*

$$\bar{\gamma} = \alpha^{-1} |\lambda| e^{\alpha^{-1}\bar{G}_1 t_i} \max_{1\leq i\leq N} \sum_{j=1}^{N} \tau_j \left|G_{2,ij}\right| < 1,$$

it is held that

$$\|z\|_{\infty,\overline{\omega}_N} \leq \alpha^{-1} (1-\bar{\gamma})^{-1} e^{\alpha^{-1}\bar{G}_1 t_i} \|R\|_{\infty}.$$

Proof. From (4.1), applying the mean value theorem, we have

$$\varepsilon z_{\bar{t},i} + p_i z_i + \sum_{j=1}^{i} \tau_j G_{1,ij} z_j + \lambda \sum_{j=1}^{N} \tau_j G_{2,ij} z_j = R_i, \quad i = 1,2,\ldots,N, \tag{4.4}$$

where

$$p_i = \frac{\partial}{\partial u} f(t_i, u_i + \mu_1 z_i), \qquad 0 < \mu_1 < 1,$$
$$G_{1,ij} = \frac{\partial}{\partial u} K_1(t_i, s_j, u_i + \mu_2 z_i), \qquad 0 < \mu_2 < 1$$

and

$$G_{2,ij} = \frac{\partial}{\partial u} K_2(t_i, s_j, u_i + \mu_3 z_i), \qquad 0 < \mu_3 < 1.$$

Then, the difference equation (4.4) can be written in the form

$$\varepsilon z_{\bar{t},i} + p_i z_i = F_i$$

where

$$F_i = R_i - \sum_{j=1}^{i} \tau_j G_{1,ij} z_j - \lambda \sum_{j=1}^{N} \tau_j G_{2,ij} z_j. \tag{4.5}$$

From (4.5), we get

$$|F_i| \leq \|R\|_{\infty} + \sum_{j=1}^{i} \tau_j \left|G_{1,ij}\right| \left|z_j\right| + |\lambda| \sum_{j=1}^{N} \tau_j \left|G_{2,ij}\right| \left|z_j\right|. \tag{4.6}$$

Furthermore, considering Lemma 3 and the inequality (4.6), we have

$$|z_i| \leq \bar{\delta} + \alpha^{-1} \bar{G}_1 \sum_{j=1}^{i} \tau_j \left|z_j\right|, \tag{4.7}$$

where

$$\bar{\delta} = \alpha^{-1} \|R\|_{\infty} + \alpha^{-1} |\lambda| \sum_{j=1}^{N} \tau_j \left|G_{2,ij}\right| \left|z_j\right|.$$

Applying the difference analogue of Gronwall's inequality to the relation (4.7), it can be written obviously that

$$\|z\|_\infty \leq \frac{e^{\alpha^{-1}\bar{G}_1 t_i}\alpha^{-1}\|R\|_\infty}{1-\alpha^{-1}|\lambda|\, e^{\alpha^{-1}\bar{G}_1 t_i}\max\limits_{1\leq i\leq N}\sum\limits_{j=1}^{N}\tau_j\left|G_{2,ij}\right|},$$

which shows the proof of the lemma (see [9, 11, 13, 29]). □

Lemma 5. *Under the conditions of Lemma 2, the following estimate is satisfied on Shishkin-type mesh*

$$\|R\|_{\infty,\omega_N} \leq CN^{-1}\ln N$$

and for the Bakhvalov-type mesh, it is written that

$$\|R\|_{\infty,\omega_N} \leq CN^{-1}.$$

Proof.

a) Here, we present a parallel approach to [11, 29]. Firstly, we consider the term (3.3). From the relation (3.3), we obtain

$$\begin{aligned}\left|R_i^{(1)}\right| &\leq \tau_i^{-1}\int_{t_{i-1}}^{t_i}(\xi-t_{i-1})\left|\frac{\partial}{\partial\xi}f(\xi,u(\xi))+\frac{\partial}{\partial u}f(\xi,u(\xi))u'(\xi)\right|d\xi\\ &\leq C\left\{\tau_i+\int_{t_{i-1}}^{t_i}\left|u'(\xi)\right|d\xi\right\},\quad i=1,2,\ldots,N.\end{aligned}\tag{4.8}$$

For the remainder term (3.5), since $\left|\frac{\partial K_1}{\partial\xi}\right| \leq C$ and $|u(t)|\leq C$, it follows that

$$\begin{aligned}\left|R_i^{(2)}\right| &\leq \tau_i^{-1}\int_{t_{i-1}}^{t_i}(\xi-t_{i-1})\left|K_1(\xi,t,u(t))+\int_0^t\frac{\partial}{\partial\xi}K_1(\xi,s,u(s))ds\right|d\xi\\ &\leq C\tau_i,\quad i=1,2,\ldots,N.\end{aligned}\tag{4.9}$$

Next, from the remainder term (3.6), we get

$$\begin{aligned}\left|R_i^{(3)}\right| &\leq \sum_{j=1}^{i}\int_{t_{j-1}}^{t_j}(\xi-t_{j-1})\left|\frac{\partial}{\partial\xi}K_1(t,\xi,u(\xi))+\frac{\partial}{\partial u}K_1(t,\xi,u(\xi))u'(\xi)\right|d\xi\\ &\leq C\left\{\tau_i+\int_{t_{i-1}}^{t_i}\left|u'(\xi)\right|d\xi\right\}.\end{aligned}\tag{4.10}$$

For the term (3.7), we can write

$$\begin{aligned}\left|R_i^{(4)}\right| &\leq \lambda\tau_i^{-1}\int_{t_{i-1}}^{t_i}(\xi-t_{i-1})\left|K_2(\xi,t,u(t))+\int_0^T\frac{\partial}{\partial\xi}K_2(\xi,s,u(s))ds\right|d\xi\\ &\leq C\tau_i,\quad i=1,2,\ldots,N.\end{aligned}\tag{4.11}$$

Next, for the remainder term (3.8), we have

$$\left|R_i^{(5)}\right| \leq \lambda \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} (\xi - t_{j-1}) \left| \frac{\partial}{\partial \xi} K_2(t,\xi,u(\xi)) + \frac{\partial}{\partial u} K_2(t,\xi,u(\xi))u'(\xi) \right| d\xi$$

$$\leq C \left\{ \tau_i + \int_{t_{i-1}}^{t_i} \left|u'(\xi)\right| d\xi \right\}. \tag{4.12}$$

Substituting the inequalities (4.8), (4.9), (4.10), (4.11) and (4.12) in (3.10), and considering (2.5), we obtain

$$|R_i| \leq C \left\{ \tau_i + \frac{1}{\varepsilon} \int_{t_{i-1}}^{t_i} e^{-\frac{\alpha t}{\varepsilon}} dt \right\}. \tag{4.13}$$

Now, we estimate the error approximations according to the node points of Shishkin type mesh. Firstly, we consider the case $\sigma = \frac{T}{2}$. Then $\frac{T}{2} \leq \alpha^{-1}\varepsilon \ln N$, $\tau^{(1)} = \tau^{(2)} = \tau = TN^{-1}$. From (4.13), we get

$$|R_i| \leq C\left\{N^{-1} + \varepsilon^{-1}TN^{-1}\right\} \leq C\left\{N^{-1} + \alpha^{-1}N^{-1}\ln N\right\}.$$

Then, we obtain

$$|R_i| \leq CN^{-1}\ln N, \quad i = 1,2,\ldots,N.$$

Now, we consider the case $\sigma = \alpha^{-1}\varepsilon\ln N$, so that $\alpha^{-1}\varepsilon\ln N < \frac{T}{2}$. We estimate seperately R_i on $[0,\sigma]$ and $[\sigma,T]$. In the layer region $[0,\sigma]$, the relation (4.13) degrades as

$$|R_i| \leq C\left(1+\varepsilon^{-1}\right)\tau^{(1)} \leq C\left(1+\varepsilon^{-1}\right)\frac{2\alpha^{-1}\varepsilon\ln N}{N} \leq CN^{-1}\ln N, \quad i = 1,2,\ldots,\frac{N}{2}.$$

To estimate R_i in the layer region $[\sigma,T]$, the inequality (4.13) can be written in the form

$$|R_i| \leq C\left\{\tau^{(2)} + \alpha^{-1}\left(e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}}\right)\right\}, \quad i = \frac{N}{2}+1,\ldots,N. \tag{4.14}$$

On account of $t_i = \alpha^{-1}\varepsilon\ln N + \left(i - \frac{N}{2}\right)\tau^{(2)}$, it is found that

$$e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}} = \frac{1}{N}e^{\frac{-\alpha\left(i-1-\frac{N}{2}\right)\tau^{(2)}}{\varepsilon}}\left(1 - e^{\frac{-\alpha\tau^{(2)}}{\varepsilon}}\right) \leq N^{-1}.$$

This relation and (4.14) yield the estimate $|R_i| \leq CN^{-1}$ [3, 6, 19, 29, 33].

b) Now, we consider the node points of Bakhvalov type mesh. For the interval $[0,\sigma]$, we find

$$\tau_i = t_i - t_{i-1} = \alpha^{-1}\varepsilon\left\{-\ln\left[1-(1-\varepsilon)\frac{2i}{N}\right] + \ln\left[1-(1-\varepsilon)\frac{2(i-1)}{N}\right]\right\}$$
$$\leq 2\alpha^{-1}(1-\varepsilon)N^{-1}.$$

Considering (4.13), we have

$$|R_i| \leq C\left\{\tau_i + \int\limits_{t_{i-1}}^{t_i} |u'(t)|\, dt\right\} \leq C\left\{\tau_i + \alpha^{-1}\left[e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}}\right]\right\}.$$

From here, it can be seen clearly that

$$e^{-\frac{\alpha t_{i-1}}{\varepsilon}} - e^{-\frac{\alpha t_i}{\varepsilon}} = 2\,(T-\varepsilon)\,N^{-1}. \tag{4.15}$$

By virtue of (4.15), it is obtained $|R_i| \leq 4\alpha^{-1}CN^{-1}$. For the interval $[\sigma, T]$, using $\tau = 2(T-\sigma)/N = TN^{-1}$, $\sigma = \frac{T}{2}$ and $\frac{T}{2} < \alpha^{-1}\varepsilon \ln \varepsilon$, we get

$$\max \int\limits_{t_{i-1}}^{t_i} \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}}\, dt \leq \frac{2e^{-1}}{\alpha T}\tau = 2e^{-1}\alpha^{-1}CN^{-1},$$

$$|R_i| \leq C\left(T + \frac{2}{e\alpha}\right)N^{-1}.$$

Hence, we can show that $|R_i| \leq CN^{-1}$ [4, 10, 29].

□

5. Numerical Experiments

TABLE 1. Error approximations e^N and order of convergence p^N on S-Mesh

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-1}	0.01514971	0.00865547	0.00492091	0.00276883	0.00154652
	0.8076	0.8147	0.8296	0.8403	
2^{-2}	0.02951389	0.01708056	0.00968299	0.00548280	0.00308931
	0.7890	0.8188	0.8205	0.8276	
2^{-3}	0.05611796	0.03327546	0.01929541	0.01097674	0.00619335
	0.7540	0.7862	0.8138	0.8257	
2^{-4}	0.08652760	0.05297076	0.03169212	0.01827619	0.01042275
	0.7079	0.7411	0.7942	0.8102	
2^{-5}	0.10183001	0.06172058	0.03619636	0.02075972	0.01175285
	0.7223	0.7699	0.8021	0.8208	
2^{-6}	0.10948122	0.06609549	0.03874848	0.02220149	0.01254775
	0.7280	0.7704	0.8035	0.8232	
2^{-7}	0.11330682	0.06828295	0.03967454	0.02258739	0.01268525
	0.7306	0.7833	0.8127	0.8324	
...	...	...	...	...	...
2^{-10}	0.11665423	0.07019697	0.04062234	0.02305427	0.01292431
	0.7327	0.7891	0.8172	0.8349	
e^N	0.11665423	0.07019697	0.04062234	0.02305427	0.01292431
p^N	0.7079	0.7411	0.7942	0.8102	

In this section, we present the numerical results. Because of the existence of non-linear terms in the discretization (3.11), we apply the quasilinearization technique [11, 29]. Hence, we obtain

$$
\begin{aligned}
&\varepsilon y_{\bar{t},i}^{(n)} + f(t_i, y_i^{(n-1)}) + \frac{\partial}{\partial y} f(t_i, y_i^{(n-1)}) \left(y_i^{(n)} - y_i^{(n-1)} \right) \\
&+ \sum_{j=1}^{i} \tau_j \left[K_{1,ij}(t_i, t_j, y_j^{(n-1)}) + \frac{\partial}{\partial y} K_{1,ij}(t_i, t_j, y_j^{(n-1)}) \left(y_j^{(n)} - y_j^{(n-1)} \right) \right] \\
&+ \lambda \sum_{j=1}^{N} \tau_j \left[K_{2,ij}(t_i, t_j, y_j^{(n-1)}) + \frac{\partial}{\partial y} K_{2,ij}(t_i, t_j, y_j^{(n-1)}) \left(y_j^{(n)} - y_j^{(n-1)} \right) \right] = 0, \\
&y_0^{(n)} = A, \quad i = 1, 2, \ldots, N.
\end{aligned}
$$

TABLE 2. Error approximations e^N and order of convergence p^N on B-Mesh

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-1}	0.00822594	0.00434577	0.00227957	0.00119146	0.00061615
	0.9205	0.9309	0.9360	0.9514	
2^{-2}	0.01627748	0.00864498	0.00454739	0.00237997	0.00123259
	0.9129	0.9268	0.9341	0.9493	
2^{-3}	0.02598728	0.01367026	0.00708582	0.00366041	0.00187382
	0.9267	0.9480	0.9529	0.9660	
2^{-4}	0.03670097	0.01915414	0.00988129	0.00506712	0.00259023
	0.9381	0.9549	0.9635	0.9681	
2^{-5}	0.05057400	0.02636907	0.01357648	0.00694575	0.00352693
	0.9395	0.9577	0.9669	0.9777	
2^{-6}	0.06205511	0.03233567	0.01668547	0.00849394	0.00429500
	0.9404	0.9545	0.9741	0.9838	
2^{-7}	0.07308114	0.03814559	0.01965310	0.00989449	0.00496710
	0.9379	0.9568	0.9901	0.9942	
…	…	…	…	…	…
2^{-10}	0.09727740	0.05117653	0.02646298	0.01350693	0.00686821
	0.9266	0.9515	0.9703	0.9757	
e^N	0.09727740	0.05117653	0.02646298	0.01350693	0.00686821
p^N	0.9129	0.9268	0.9341	0.9493	

Example 1. Consider the first problem

$$
\begin{aligned}
&\varepsilon u'(t) + u^3(t) + u(t) + e^{-3t/\varepsilon} \left(\frac{\varepsilon - 6}{6} \right) \\
&+ \frac{\varepsilon}{4} e^{\frac{-2}{\varepsilon}} - \frac{5\varepsilon}{12} + \frac{1}{2} \int_0^t u^3(s) ds + \frac{1}{2} \int_0^1 u^2(s) ds = 0, \\
&u(0) = 1.
\end{aligned}
$$

whose exact solution is $u(t) = e^{-t/\varepsilon}$. Maximum pointwise errors are determined as $e^N = |y_i - u_i|$. Also, the convergence rates are calculated with $p^N = \frac{\ln(e^N/e^{2N})}{\ln 2}$. The computed results are presented in Tables 1-2.

TABLE 3. Error approximations e^N and order of convergence p^N on S-Mesh

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-1}	0.03373796	0.01899680	0.01059502	0.00588290	0.00325679
	0.8286	0.8424	0.8488	0.8531	
2^{-2}	0.06498652	0.03724032	0.02099584	0.01169480	0.00645516
	0.8032	0.8268	0.8442	0.8573	
2^{-3}	0.11815619	0.07164096	0.04123813	0.02298536	0.01291668
	0.7218	0.7968	0.8433	0.8315	
2^{-4}	0.17177238	0.09885501	0.05604937	0.03204937	0.01799512
	0.7971	0.8186	0.8064	0.8327	
2^{-5}	0.17288072	0.10097806	0.05762941	0.03262288	0.01820037
	0.7757	0.8092	0.8209	0.8419	
2^{-6}	0.17959856	0.10635639	0.06117739	0.03508942	0.01969521
	0.7559	0.7978	0.8020	0.8332	
2^{-7}	0.18295747	0.10845881	0.06242714	0.03549438	0.01987633
	0.7544	0.7969	0.8146	0.8365	
...	...	...	...	...	...
2^{-10}	0.18589653	0.11255785	0.06436884	0.03642489	0.02025114
	0.7238	0.8062	0.8214	0.8469	
e^N	0.18589653	0.11255785	0.06436884	0.03642489	0.02025114
p^N	0.7218	0.7968	0.8020	0.8315	

Example 2. Take into account the second problem

$$\varepsilon u'(t) + 2u(t) - e^{-u(t)} + \int_0^t e^{sin(u(s))} ds + \frac{1}{4}\int_0^1 \sin(u(s))\, ds = 0$$

with $u(0) = 1$. The exact solution of this problem is unknown. Since the exact solution is unknown, we use the double-mesh principle [6, 12, 13]. The error estimates are denoted by $e^N = \left|y_i^N - y_i^{2N}\right|$ and the order of convergence is computed as $p^N = \frac{\ln(e^N/e^{2N})}{\ln 2}$. The obtained results are shown in Tables 3-4.

From Tables 1-4, we observe that the presented method produces better results on B-mesh. Because of the effect of logarithmic factor on S-mesh, the rate of the convergence on S-mesh is lower than B-mesh. In spite of the fact that the proposed method provides reliable results on layer-adapted meshes, it is almost the first-order convergent.

6. CONCLUSION

In this paper, we have designed a new difference scheme for the nonlinear integro-differential equations with layer behavior. Error approximations have been analyzed

TABLE 4. Error approximations e^N and order of convergence p^N on B-Mesh

ε	$N=64$	$N=128$	$N=256$	$N=512$	$N=1024$
2^{-1}	0.02940317	0.01564298	0.00819606	0.00426697	0.00218826
	0.9104	0.9325	0.9417	0.9634	
2^{-2}	0.05665633	0.03066933	0.01563324	0.00789362	0.00396636
	0.8854	0.9722	0.9859	0.9929	
2^{-3}	0.10560228	0.05901346	0.03264993	0.01762837	0.00929240
	0.8395	0.8540	0.8892	0.9238	
2^{-4}	0.15490687	0.08742185	0.04807278	0.02567700	0.01350556
	0.8253	0.8628	0.9047	0.9269	
2^{-5}	0.15916537	0.08662951	0.04628234	0.02487489	0.01299633
	0.8776	0.9044	0.8958	0.9366	
2^{-6}	0.16095246	0.08824835	0.04720203	0.02507356	0.01320159
	0.8670	0.9027	0.9127	0.9255	
2^{-7}	0.16772393	0.09322601	0.04970150	0.02628564	0.01377238
	0.8473	0.9074	0.9190	0.9325	
...	...	...	...	...	...
2^{-10}	0.17329378	0.09491387	0.05044516	0.02664005	0.01392799
	0.8685	0.9119	0.9211	0.9356	
e^N	0.17329378	0.09491387	0.05044516	0.02664005	0.01392799
p^N	0.8253	0.8540	0.8892	0.9238	

on both B-type and S-type meshes. Because of the nonlinear terms in the iteration, the quasilinearization technique has been used. To show the suitability of the method, we have tested it on two examples and the computed results have been summarized in Tables 1-4. We can deduce that for majority values of N, exact errors decrase and the order of convergence is close to 1. Numerical investigations can be sustained for more sophisticated types such as partial integro-differential equations, delay form, higher dimensional, etc.

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