Miskolc Mathematical Notes
Vol. 25 (2024), No. 1, pp. 509–522

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4267

ON n-1-ABSORBING PRIMARY IDEALS

GULSEN ULUCAK, SAKINEH BABAEI, AND ESRA SENGELEN SEVIM

Received 31 May, 2022

Abstract. The paper aims to present a new primary ideal in a commutative ring R with nonzero identity element. We introduce the new primary ideal as an n-1-absorbing primary ideal that is a generalization of both primary and 1-absorbing primary ideals. We propose to achieve two goals with this paper. Firstly, we study and characterize some essential properties of an n-1-absorbing primary ideals and figure out the relations between the other types of ideals such as prime, 1-absorbing primary and irreducible ideals. Then, we classify some special rings that admit an n-1-absorbing primary ideal. We provide the results by introducing some examples.

2010 *Mathematics Subject Classification:* 13A15; 13C05

Keywords: primary ideal, 1-absorbing primary ideal, n-1-absorbing primary ideal

1. Introduction

Throughout this study, we assume that R is a commutative ring with identity ($1 \neq 0$) and n is a positive integer.

Firstly, we introduce some well-known facts for ideals in a commutative ring that we need to use in this paper: An ideal I is called proper, if $I \neq R$. The radical of I, denoted by $\sqrt{I}$, is defined by $\{r \in R \,:\, r^n \in I \text{ for some positive integer } n\}$. A proper ideal I of R and $x \in R - I$, the residual of I by x, $(I :_R x)$ is defined by $\{r \in R \,:\, rx \in I\}$. A proper ideal I of a commutative ring is called irreducible, if I cannot be represented as an intersection of two ideals, properly contained by I. A prime ideal of a ring is called a divided prime ideal if it is comparable to every principal ideal of a ring. Moreover, we need to introduce some well-known rings that we use to study in this paper: A ring R is called a chained ring if $a|b$ or $b|a$ for every $a, b \in R$. Note that every chained ring is a divided ring. An integral domain R is called a divided domain if every prime ideal of R is a divided prime ideal. An integral domain R is called a valuation domain if either $x|y$ or $y|x$ for all $0 \neq x, y \in R$, [2].

Prime ideals in commutative algebra always deserve more attention. Obtaining some important new prime ideals provides the concepts and tools to effect a treatment of rings in a direct and elegant way so that the structures of rings are significantly figured out. This not only gives a better understanding of the structure of rings but opens a vista comparable to the developments of commutative algebra. In [2], Anderson and Badawi have introduced and studied the n-absorbing ideals, namely, the generalization of the prime ideals. According to the definition in [2], a proper ideal I of R is called an n-absorbing ideal, if $x_1,\ldots,x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I$, then there are n of the x_i' s whose product is in I. The mentioned study has led to be introduced the new ideal which is a generalization of both n-absorbing ideals and primary ideals. It has been defined by Becker and it is called n-absorbing primary ideal, [4]. Moreover, in [6], Ulucak, Koc, and Tekir have defined an n-1-absorbing prime ideal. We define a new ideal which is called an n-1-absorbing primary ideal that is a generalization of n-1-absorbing prime ideal as well. A proper ideal I of R is defined as n-absorbing primary ideal if $x_1,\ldots,x_{n+1} \in R$ and $x_1 \cdots x_{n+1} \in I$ imply either $x_1 \cdots x_n \in I$ or there are n of the x_i's whose product (except for $x_1 \cdots x_n$) is in $\sqrt{I}$, [4]. Very recently, in [3], Badawi and Celikel have introduced another primary ideals that is called 1-absorbing primary ideal. A proper ideal I of R is called a 1-absorbing primary ideal if non-units $x_1,x_2,x_3 \in R$ and $x_1x_2x_3 \in I$ imply either $x_1x_2 \in I$ or $x_3 \in \sqrt{I}$, [3].

This study allows us to classify and construct a new class of prime ideals which is generalization of the primary ideals. A proper ideal I of R is called an n-1-absorbing primary ideal if $x_1 \cdots x_{n+1} \in I$ for some non-units $x_1,\ldots,x_{n+1} \in R$, then, either $x_1x_2 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$.

In the first part the focus will be on the discussion to characterize the n-1-absorbing primary ideals introducing some useful and essential properties to figure out the behaviour of the n-1-absorbing primary ideals in a commutative ring R. Moreover, we provide the results by finding some examples.

The final goal of this paper is to study on the n-1-absorbing primary ideals to characterize and find out the structure of some special rings ,e.i., Dedekind domain, valuation domain and quasi local rings. Explicitly, with the obtaining examples we will explain the structure of rings in commutative algebra.

2. CHARACTERIZATION OF n-1-ABSORBING PRIMARY IDEALS

In this section of this study we give the definition of an n-1-absorbing primary ideal. Moreover, we characterize and study some essential properties of n-1-absorbing primary ideal in a commutative ring R to understand the structure of R. Finally, we explain the structure of a commutative ring R presenting some examples for the n-1-absorbing primary ideal of R.

Definition 1. Let n be a positive integer. A proper ideal I of a ring R is called an n-1-absorbing primary ideal if $x_1 \cdots x_{n+1} \in I$ for some non-units $x_1, \ldots, x_{n+1} \in R$, then either $x_1 x_2 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$.

By the definition of n-1-absorbing primary ideal, it can be easily seen that every primary ideal of R is an n-1-absorbing primary ideal, and every 1-absorbing primary ideal of R is n-1-absorbing primary ideal for $n \geq 2$. The converses of these implications can not be true generally. Therefore, to make the implication clear we give the following example. Obviously, the concepts of 1-1-absorbing primary ideal and primary ideal of R coincide for $n = 1$. Also, when we take $n = 2$, then by the definition of n-1-absorbing primary ideal, we see that the definition allows us to interpret that a 1-absorbing primary ideal and a 2-1-absorbing primary ideal coincide. Consequently, an n-1-absorbing primary ideal assert its importance that it is a generalization of both primary ideals and 1-absorbing primary ideals.

Example 1. Let $S = F[x, y]$, where F is a field. Consider $M = (x, y)S$ and $R = S_M$. It can be seen that R is a quasi local ring with maximal ideal M_M. Then $I = yM_M = (xy, y^2)R$ is a 2-1-absorbing primary ideal of R, [3, Theorem 5]. However, I is not a primary ideal of R since $xy \in I$, but neither $y \in I$ nor $x \in \sqrt{I} = yR$.

Now, we give some basic and useful results of an n-1-absorbing primary ideal as follows:

Theorem 1. *Let I be a proper ideal of a ring R and n be a positive integer. Then, the following statements are satisfied:*

(1) *Every n-1-absorbing prime ideal of R is an n-1-absorbing primary ideal.*
(2) *Every n-1-absorbing primary ideal of R is an n-absorbing primary ideal.*
(3) *Let $m \geq n$ be a positive integer. If I is an n-1-absorbing primary ideal of R, then I is an m-1-absorbing ideal of R.*
(4) *Let I be an n-1-absorbing primary ideal of R. Then $\sqrt{I}$ is a prime ideal of R.*

Proof.

(1) It is clear from the definitions of n-1-absorbing prime ideal and n-1-absorbing primary ideal.
(2) Assume that I is an n-1-absorbing primary ideal of R and $x_1 \cdots x_{n+1} \in I$ for some $x_1, \ldots, x_{n+1} \in R$. If at least one of the x_i's is unit, then the proof is done. So, we assume that $x_1, \ldots, x_{n+1} \in R$ are non-unit. Since I is an n-1-absorbing primary ideal of R, then we have either $x_1 x_2 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$. This implies that $x_1 x_2 \cdots x_n \in I$ or a product of n of the x_i's which contains x_{n+1} is in $\sqrt{I}$. Thus, I is an n-absorbing primary ideal of R.
(3) We use mathematical induction on n, m. To show that the claim is true, it is sufficient to prove that I is an $(n+1)$-1-absorbing primary ideal if I is an n-1-absorbing primary ideal of R. Hence, we assume that I is an n-1-absorbing primary ideal of R and $x_1 x_2 \cdots x_{n+1} x_{n+2} = (x_1 x_2) x_3 \cdots x_{n+1} x_{n+2} \in I$ for some

non-units $x_1, x_2, \ldots, x_{n+2} \in R$. Since I is an n-1-absorbing primary ideal of R, we have $(x_1x_2)x_3\cdots x_{n+1} \in I$ or $x_{n+2} \in \sqrt{I}$. As a result, I is an $(n+1)$-1-absorbing primary ideal of R.

(4) Let $xy \in \sqrt{I}$ and $x \notin \sqrt{I}$ for some $x, y \in R$. Then, there exists a positive integer k such that $x^k y^k \in I$. If $k \leq n$, then we get $x^n y^k = x^{n-k}x^k y^k \in I$. Since $x \notin \sqrt{I}$, we conclude that $x^n \notin I$ and hence $y^k \in \sqrt{I}$, namely, $y \in \sqrt{I}$. Now, we assume $k \geq n$. By (3), I is k-1-absorbing primary ideal and since $x^k \notin I$, we conclude that $y^k \in \sqrt{I}$ and so $y \in \sqrt{I}$. Then, $\sqrt{I}$ is a prime ideal of R.

□

Now, we give the following examples to show that the converses of Theorem 1 (1), Theorem 1 (2) are not true generally. Moreover, we provide the Example 1 verifies that the converse of the Theorem 1 (4) is not true in general.

Consider the Example 1 for the converse of Theorem 1 (3): $I = yM_M = (xy, y^2)R$ is a 2-1-absorbing primary ideal of R, however it is not a 1-1-absorbing primary.

Example 2. Consider the ideal (8) of the ring $\mathbb{Z}$. It is clear that (8) is a 2-1-absorbing primary ideal but not a 2-1-absorbing prime ideal since $2\cdot 2\cdot 2 \in (8)$, but neither $2.2 \in (8)$ nor $2 \in (8)$.

Example 3. Consider the ideal (15) of the ring $\mathbb{Z}$. It is clear that (15) is a 2-absorbing primary ideal but not a 2-1-absorbing primary ideal since $2\cdot 3\cdot 5 \in (15)$, but neither $2.3 \in (15)$ nor $5 \in \sqrt{(15)}$.

Theorem 2. *Let I be an n-1-absorbing primary ideal of R and $x \in R - I$ be a non-unit element. Then $(I :_R x)$ is an $(n-1)$-1-absorbing primary ideal of R.*

Proof. Let $x_1\cdots x_n \in (I :_R x)$ for some non-units $x_1, \ldots, x_n \in R$. Then $xx_1\cdots x_n \in I$. By the assumption, we have either $xx_1\cdots x_{n-1} \in I$ or $x_n \in \sqrt{I} \subseteq \sqrt{(I :_R x)}$ and so $x_1x_2\cdots x_{n-1} \in (I :_R x)$ or $x_n \in \sqrt{(I :_R x)}$. Therefore, $(I :_R x)$ is an $(n-1)$-1-absorbing primary ideal of R. □

Theorem 3. *Let I be an n-1-absorbing primary ideal of a ring R with $I \subset \sqrt{I}$ and $n \geq 2$. If $m \geq 2$ is the least positive integer with $x^m \in I$ for $x \in \sqrt{I} - I$, then $(I :_R x^{m-1})$ is an $(n-m+1)$-1-absorbing primary ideal of R.*

Proof. We see that $m \leq n$ since I is an n-1-absorbing primary ideal of R and thus it must be $n-m+2 \geq 2$. Let $x_1x_2\cdots x_{n-m+2} \in (I :_R x^{m-1})$ for some non-units $x_1, \ldots, x_{n-m+2} \in R$. Then, $x^{m-1}x_1x_2\cdots x_{n-m+2} \in I$. Hence, by the assumption, we obtain that $x^{m-1}x_1x_2\cdots x_{n-m+1} \in I$ or $x_{n-m+2} \in \sqrt{I} \subseteq \sqrt{(I :_R x^{m-1})}$. Therefore, we have $x_1x_2\cdots x_{n-m+1} \in (I :_R x^{m-1})$ or $x_{n-m+2} \in \sqrt{(I :_R x^{m-1})}$. Thus, $(I :_R x^{m-1})$ is an $(n-m+1)$-1-absorbing primary ideal of R. □

Corollary 1. *Let I be an n-1-absorbing primary ideal of a ring R with $I \subset \sqrt{I}$ and $n \geq 2$ be the smallest positive integer with $x^n \in I$ for $x \in \sqrt{I} - I$. Then $(I :_R x^{n-1})$ is a primary ideal of R.*

Proof. Let us choose $n = m$ in Theorem 3. So $(I :_R x^{n-1})$ is an $(n-n+2)$-1-absorbing primary ideal of R, namely, $(I :_R x^{n-1})$ is a 1-1-absorbing primary ideal of R. Thus, $(I :_R x^{n-1})$ is a primary ideal of R. □

Theorem 4. *A proper ideal I of a ring R is an n-1-absorbing primary ideal if and only if $\sqrt{(I :_R x_1 \cdots x_n)} = \sqrt{I}$ where $x_1 \cdots x_n \notin I$ for all non-units $x_1, \ldots, x_n \in R$.*

Proof.

(⇒)**:** Let I be an n-1-absorbing primary ideal of a ring R and $x_1 \cdots x_n \notin I$ for some non-units $x_1, \ldots, x_n \in R$. It is obvious that $\sqrt{I} \subseteq \sqrt{(I :_R x_1 \cdots x_n)}$. Thus, we assume that $y \in \sqrt{(I :_R x_1 \cdots x_n)}$. Then, there is a positive integer k with $y^k \in (I :_R x_1 \cdots x_n)$. Since $x_1 \cdots x_n y^k \in I$ and $x_1 \cdots x_n \notin I$, then, by the assumption, we have $y^k \in \sqrt{I}$. Thus, $y \in \sqrt{I}$ and so we have that $\sqrt{(I :_R x_1 \cdots x_n)} \subseteq \sqrt{I}$ and thus we obtain $\sqrt{(I :_R x_1 \cdots x_n)} = \sqrt{I}$.

(⇐)**:** Let $y_1 \cdots y_{n+1} \in I$ and $y_1 \cdots y_n \notin I$ for some non-units $y_1, \ldots, y_{n+1} \in R$. Since $y_{n+1} \in (I :_R y_1 \cdots y_n) \subseteq \sqrt{(I :_R y_1 \cdots y_n)}$, by the assumption we get $y_{n+1} \in \sqrt{I}$. Thus, I is an n-1-absorbing primary ideal.

□

By the Theorem 4, we give the following corollary:

Corollary 2. *Let I be an n-1-absorbing primary ideal of a ring R and $x \in R - \sqrt{I}$. Then $\sqrt{(I :_R x^n)} = \sqrt{I}$.*

Theorem 5. *Let I be a proper ideal of R. The following statements are equivalent.*

(1) *I is an n-1-absorbing primary ideal of R.*

(2) *If $x_1 x_2 \cdots x_n J \subseteq I$ for some non-units $x_1, x_2, \ldots, x_n \in R$ and a proper ideal J of R, then $x_1 x_2 \cdots x_n \in I$ or $J \subseteq \sqrt{I}$.*

(3) *If $I_1 I_2 \cdots I_{n+1} \subseteq I$ for some proper ideals $I_1, I_2, \ldots, I_{n+1}$ of R, then $I_1 I_2 \cdots I_n \subseteq I$ or $I_{n+1} \subseteq \sqrt{I}$.*

Proof.

(1) ⇒ (2)**:** Let I be an n-1-absorbing primary ideal of R. Assume $x_1 x_2 \cdots x_n J \subseteq I$ and $x_1 x_2 \cdots x_n \notin I$ for some non-units $x_1, x_2, \ldots, x_n \in R$ and a proper ideal J of R. Then $x_1 x_2 \cdots x_n j \subseteq I$ and $x_1 x_2 \cdots x_n \notin I$ for every $j \in J$. By the assumption, $j \in \sqrt{I}$ and so $J \subseteq \sqrt{I}$. Thus, we complete the proof.

(2) ⇒ (3)**:** Suppose that $I_1 I_2 \cdots I_{n+1} \subseteq I$ and $I_1 I_2 \cdots I_n \nsubseteq I$ for some proper ideals $I_1, I_2, \ldots, I_{n+1}$ of R. Then, there are $x_1 \in I_1, x_2 \in I_2, \ldots, x_n \in I_n$ such that $x_1 x_2 \cdots x_n \notin I$. Since $x_1 x_2 \cdots x_n I_{n+1} \subseteq I$, then, by the assumption, we have $I_{n+1} \subseteq \sqrt{I}$.

(3) ⇒ (1)**:** It is straightforward.

□

Theorem 6. *Let $f : R \to R'$ be a ring homomorphism such that $f(x)$ is a non-unit in R' for all non-unit elements x in R and $f(1) = 1$. Then the followings are satisfied:*

(1) *If J is an n-1-absorbing primary ideal of R', then $f^{-1}(J)$ is an n-1-absorbing primary ideal of R.*
(2) *If f is an epimorphism and I is a proper ideal of R with $\ker(f) \subseteq I$, then I is an n-1-absorbing primary ideal of R if and only if $f(I)$ is an n-1-absorbing primary ideal of R'.*

Proof.

(1) Let $x_1 \cdots x_{n+1} \in f^{-1}(J)$ for some non-unit elements $x_{1,} \ldots, x_{n+1} \in R$. Then, $f(x_1 \cdots x_{n+1}) = f(x_1) \cdots f(x_{n+1}) \in \sqrt{J}$. By the assumption, we obtain $f(x_1) \cdots f(x_n) \in J$ or $f(x_{n+1}) \in \sqrt{J}$. Hence $x_1 \cdots x_n \in f^{-1}(J)$ or $x_{n+1} \in f^{-1}(\sqrt{J}) = \sqrt{f^{-1}(J)}$. Consequently, $f^{-1}(J)$ is an n-1-absorbing primary ideal of R.
(2) Let f be a surjective homomorphism and I is a proper ideal of R with $\ker(f) \subseteq I$. Then $I = f^{-1}(f(I))$. So, we sssume that $f(I)$ is an n-1-absorbing primary ideal of R'. Thus, I is an n-1-absorbing primary ideal of R by (1). For the converse of (2), assume that I is an n-1-absorbing primary ideal of R and $y_1 y_2 \cdots y_{n+1} \in f(I)$ for some non-unit elements $y_1, y_2, \ldots, y_{n+1} \in R$. By the assumption, we get $f(x_i) = y_i$ for each $1 \leq i \leq n$ and so $f(x_1) \cdots f(x_{n+1}) \in f(I)$. Then, we have $x_1 \cdots x_{n+1} \in I$ since $\ker(f) \subseteq I$. Since I is an n-1-absorbing primary ideal, we deduce that either $x_1 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$ and so, $y_1 \cdots y_n \in f(I)$ or $y_{n+1} \in f(\sqrt{I}) = \sqrt{f(I)}$, thus the proof is done.

□

Corollary 3. *Let I and J be proper ideals of a ring R with $I \subseteq J$ and $u(R/I) = \{x + I : x \in u(R)\}$. Then the followings are satisfied:*

(1) *J is an n-1-absorbing primary ideal of R if and only if J/I is an n-1-absorbing primary ideal of R/I.*
(2) *Let S be a multiplicatively closed subset of R. If I is an n-1-absorbing primary ideal of R with $I \cap S = \varnothing$, then I_S is an n-1-absorbing primary ideal of R_S.*

Proof.

(1) Consider the natural homomorphism $\pi : R \to R/I$, defined by $\pi(x) = x + I$ for each $x \in R$. Then (1) is obtained by Theorem 6 (2).
(2) Consider the injection $i : S \to R$, defined by $i(x) = x$ for each $x \in S$. Then, the result follows by Theorem 6 (2).

□

The converse of Corollary 3 (2) is not true generally. See the following example:

Example 4. Let $R = \mathbb{Z}$ and $I = p_1 p_2 \mathbb{Z}$ for some distinct prime numbers p_1, p_2. One can see that I is not an n-1-absorbing primary ideal since $p_1^n p_2 \in I$, $p_1^n \notin I$ and

$p_2 \notin \sqrt{I} = I$. Let $S = R - p_1\mathbb{Z}$. Thus, we get $I_S = (p_1\mathbb{Z})_S$ is a prime ideal of R_S and so I_S is an n-1-absorbing primary ideal of R_S.

Corollary 4. *Let S be a multiplicatively closed subset of R, and I be a proper ideal of R. If I_S is an n-1-absorbing primary ideal of R_S and $S \cap \mathbf{Z}_I(R) = \varnothing$, then I is an n-1-absorbing primary ideal of R.*

Proof. Let $x_1 \cdots x_{n+1} \in I$ for some non-units $x_1, \ldots, x_n \in R$. Then $\frac{x_1 \cdots x_{n+1}}{1} = \frac{x_1}{1} \cdots \frac{x_{n+1}}{1} \in I_S$. By the assumption, we obtain that either $\frac{x_1}{1} \cdots \frac{x_n}{1} \in I_S$ or $\frac{x_{n+1}}{1} \in \sqrt{I_S} = (\sqrt{I})_S$. If $\frac{x_1 \cdots x_n}{1} = \frac{x_1}{1} \cdots \frac{x_n}{1} \in I_S$, there is a $u \in S$ such that $ux_1 \cdots x_n \in I$. Since $u \notin \mathbf{Z}_I(R)$, we have $x_1 \cdots x_n \in I$. Now, we suppose that $\frac{x_{n+1}}{1} \in (\sqrt{I})_S$. Then, there is a $v \in S$ such that $vx_{n+1} \in \sqrt{I}$ and so $(vx_{n+1})^m \in I$ for some positive integer m. Since $v^m \notin \mathbf{Z}_I(R)$, we have $x_{n+1}^m \in I$ and thus, we have $x_{n+1} \in \sqrt{I}$, which implies that I is an n-1-absorbing primary ideal of R. □

Definition 2. Let I be an n-1-absorbing primary ideal of a ring R. We know that $P = \sqrt{I}$ is a prime ideal of R. Thus, I is defined as a P-n-1-absorbing primary ideal of R.

Using the definition, we give the following theorem:

Theorem 7. *Let $I_1, \ldots, I_m$ be P-n-1-absorbing primary ideals of a ring R and $I = \bigcap_{i=1}^{m} I_i$. Then, I is a P-n-1-absorbing primary ideal of R.*

Proof. Note that $\sqrt{I} = P$. Let $x_1 \cdots x_{n+1} \in I$ and $x_1 \cdots x_n \notin I$ for some non-units $x_1, \ldots, x_{n+1} \in R$. Without loss of generality, we suppose that $x_1 \cdots x_n \notin I_k$ for some $k \in \{1, 2, \ldots, m\}$. Since I_k is an n-1-absorbing primary ideal, then $x_{n+1} \in \sqrt{I_k} = P$. So, it implies that I is a P-n-1-absorbing primary ideal. □

Let I_1 and I_2 be n-1-absorbing primary ideals of a ring. However, when $\sqrt{I_1} \neq \sqrt{I_2}$, $I_1 \cap I_2$ might not be an n-1-absorbing primary ideal of R. For that reason, we give the following basic example.

Example 5. Consider the ideals $I_1 = (2)$ and $I_2 = (3)$ of the ring $\mathbb{Z}$. The ideals I_1 and I_2 are 2-1-absorbing primary ideals of $\mathbb{Z}$ but $I_1 \cap I_2 = (6)$ is not a 2-1-absorbing primary ideal since $2 \cdot 2 \cdot 3 \in (6)$, $2 \cdot 2 \notin (6)$ and $3 \notin \sqrt{(6)}$.

Recall that a proper ideal of a commutative ring is irreducible if it cannot be stated as an intersection of two ideals properly containing it. Then, we can give the following theorem:

Theorem 8. *Let I be an irreducible ideal of a ring R. Then, I is an n-1-absorbing primary ideal of R if and only if $(I :_R y^{n-1}) = (I :_R y^n)$ for all $y \in R - \sqrt{I}$.*

Proof.

($\Rightarrow$): Let I be an n-1-absorbing primary ideal of R and $y \in R - \sqrt{I}$. It is obvious that $(I :_R y^{n-1}) \subseteq (I :_R y^n)$. So, we assume $x \in (I :_R y^n)$. Then, $xy^n \in I$ and since $y \notin \sqrt{I}$, then $xy^{n-1} \in I$ and so $x \in (I :_R y^{n-1})$. Thus, $(I :_R y^{n-1}) = (I :_R y^n)$.

($\Leftarrow$): Let $(I :_R y^{n-1}) = (I :_R y^n)$ for all $y \in R - \sqrt{I}$ and $x_1 \cdots x_{n+1} \in I$ for some non-units x_i' s of R. Assume that $x_1 \cdots x_n \notin I$ and $x_{n+1} \notin \sqrt{I}$. By the assumption, we get $(I :_R x_{n+1}^{n-1}) = (I :_R x_{n+1}^n)$. Note that $I \subseteq (I + x_{n+1}^{n-1}R) \cap (I + x_1 \cdots x_n R)$. Suppose $a \in (I + x_{n+1}^{n-1}R) \cap (I + x_1 \cdots x_n R)$. Then, $a = i + r_1 x_{n+1}^{n-1} = j + r_2 x_1 \cdots x_n$ for some $i, j \in I$. Since $ax_{n+1} = i + r_1 x_{n+1}^n = j + r_2 x_1 \cdots x_n \cdot x_{n+1}$, then $r_1 x_{n+1}^n = ax_{n+1} - ix_{n+1} \in I$ and so $r_1 \in (I : Rx_{n+1}^n) = (I : Rx_{n+1}^{n-1})$. Thus $a = i + r_1 x_{n+1}^{n-1} \in I$ as $r_1 x_{n+1}^{n-1} \in I$. Therefore, we get $I = (I + x_{n+1}^{n-1}R) \cap (I + x_1 \cdots x_n R)$. Since I is an irreducible ideal, then either $I = (I + x_{n+1}^{n-1}R)$ or $I = (I + x_1 \cdots x_n R)$ and so $x_{n+1}^{n-1} \in I$, that is $x_{n+1} \in \sqrt{I}$, or $x_1 \cdots x_n \in I$. It leads to a contradiction. So we omit the claim " $x_1 \cdots x_n \notin I$ and $x_{n+1} \notin \sqrt{I}$. Thus, it must be $x_1 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$. Hence, I is an n-1-absorbing primary ideal of R.

□

Theorem 9. *Let $S = R \times R'$ be a decomposable ring and $I = I \times I'$ be an ideal of S. Then $J = I \times I'$ is an n-1-absorbing primary ideal of S if and only if I is an n-1-absorbing primary ideal of R and $I' = R'$ or I' is an n-1-absorbing primary ideal of R' and $I = R$.*

Proof.

($\Rightarrow$): Assume that $J = I \times I'$ is an n-1-absorbing primary ideal of S. By Theorem 1(4), we have $\sqrt{I \times I'} = \sqrt{I} \times \sqrt{I'}$ which is a prime ideal of S. Thus $\sqrt{I} = R$ or $\sqrt{I'} = R'$ and so $I = R$ or $I' = R'$. Assume that $I' = R'$. Let $x_1 \cdots x_{n+1} \in I$ for some non-units $x_1, \ldots, x_{n+1} \in R$. Then $(x_1, 0) \cdots (x_{n+1}, 0) \in I \times I'$ and $I \times I'$ is an n-1-absorbing primary ideal of S and so $(x_1, 0) \cdots (x_n, 0) \in I \times I'$ or $(x_{n+1}, 0) \in \sqrt{I \times I'} = \sqrt{I} \times \sqrt{I'}$. Therefore, $x_1 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$. It implies that I is an n-1-absorbing primary ideal of R. In this case $I = R$, by a similar argument we can show that I' is an n-1-absorbing primary ideal of R'.

($\Leftarrow$): Now, we assume that I is an n-1-absorbing primary ideal of R and $I' = R'$. Let $(x_1, y_1) \cdots (x_{n+1}, y_{n+1}) \in J = I \times R$. Then $x_1 \cdots x_{n+1} \in I$ and since I is a n-1-absorbing primary, then $x_1 \cdots x_n \in I$ or $x_{n+1} \in \sqrt{I}$. Thus $(x_1, y_1) \cdots (x_n, y_n) \in J = I \times R'$ or $(x_{n+1}, y_{n+1}) \in \sqrt{J} = \sqrt{I} \times R'$ and so J is an n-1-absorbing primary ideal of R and $I' = R'$. The other case is similar.

□

Theorem 10. *Let $S = R_1 \times \cdots \times R_k$ be a decomposable ring and $I = I_1 \times \cdots \times I_k$ be an ideal of S, where $(k \geq 2)$. Then, $J = I_1 \times I_2 \times \cdots \times I_k$ is an n-1-absorbing primary ideal of S if and only if there exists i, where $1 \leq i \leq n$ that I_i is an n-1-absorbing primary ideal of R_i and $I_j = R_j$, for all $i \neq j$.*

Proof. Assume that J is an n-1-absorbing primary ideal of S. We prove the claim by induction on k. Let $k = 2$. Then the result follows the Theorem 9. Now assume that $k \geq 2$ and the conclusion is true for all positive integer $t < k$. Let $S = R_1 \times R_2 \times \cdots \times R_k$ and $J = I_1 \times I_2 \times \cdots \times I_k$ is an n-1-absorbing primary ideal of R. Then, $\sqrt{J} = \sqrt{I_1 \times I_2 \times \cdots \times I_{k-1}} \times \sqrt{I_k}$. Thus $\sqrt{I_1 \times I_2 \times \cdots \times I_{k-1}} = R_1 \times R_2 \times \cdots \times R_{k-1}$ or $\sqrt{I_k} = R_k$. We conclude that the first case is done, and the case follows the induction. The converse is similar of the proof of Theorem 9. □

3. n-1-ABSORBING PRIMARY IDEALS ON SOME SPECIAL RINGS

In this section, we characterize some special rings that admit an n-1-absorbing primary ideals: We determine the conditions for a ring R that admits an n-1-absorbing primary to be a quasi local ring. Moreover, we classify the quasi local rings that admit an n-1-absorbing primary ideals. We provide the results with some example. Afterwards, we classify a ring R which admits n-1-absorbing primary ideals to be a chaind ring, a divided ring, a valuation domain, and a Dedekind domain. We present the results providing some examples.

Theorem 11. *Assume that I is an n-1-absorbing primary ideal of R that is not an $(n-1)$-1-absorbing primary. Then R is a quasi local ring.*

Proof. Assume that I is an n-1-absorbing primary ideal of R that is not an $(n-1)$-1-absorbing primary. Then, there are some non-units $x_1, \ldots, x_n \in R$ such that $x_1 \cdots x_n \in I$, $x_1 \cdots x_{n-1} \notin I$ and $x_n \notin \sqrt{I}$. Let y be a non-unit in R. Then $yx_1 \cdots x_n \in I$ and so $yx_1 \cdots x_{n-1} \in I$ since I is an n-1-absorbing primary ideal and $x_n \notin \sqrt{I}$. Consider a unit element u of R. Suppose that $y + u$ is not a unit in R. Then, we have $(y+u)x_1 \cdots x_n \in I$ and thus, $(y+u)x_1 \cdots x_{n-1} = yx_1 \cdots x_{n-1} + ux_1 \cdots x_{n-1} \in I$ since $x_n \notin \sqrt{I}$. Hence, $ux_1 \cdots x_{n-1} \in I$ and so $x_1 \cdots x_{n-1} \in I$, which implies a contradiction. Therefore, $y + u$ is a unit element in R and thus, R is a quasi local ring [3, Lemma 1]. □

By Theorem 11, we give the following corollary:

Corollary 5. *Let a ring R be not a quasi local ring. Then, a proper ideal I of R is an n-1-absorbing primary ideal if and only if I is a primary ideal.*

Theorem 12. *Let R be a Noetherian integral domain (not a field) and I a proper ideal of R. Then the following statements are equivalent:*

(1) *R is a Dedekind domain.*

(2) *I is an n-1-absorbing primary ideal of R if and only if $I = P^m$ for some prime ideal P of R and some positive integer m.*

Proof.

$(1) \Rightarrow (2)$**:** Assume that I is an n-1-absorbing primary ideal of R. By Theorem 1 (4), $\sqrt{I} = P$ is a non-zero prime ideal. By assumption "R is a Dedekind domain", we get that P is maximal and thus, P is primary. Consequently, $I = P^m$ for some positive integer m by [5, Theorem 6.20]. Now, assume the converse holds, that is, $I = P^m$ for some prime ideal P of R and some positive integer m. Then, we obtain that P is maximal since R is a Dedekind domain. Thus, I is primary and so I is an n-1-absorbing primary ideal.

$(2 \Rightarrow (1)$**:** Assume that a non-zero ideal I of R is an n-1-absorbing primary ideal if and only if $I = P^m$ for some prime ideal P of R and some positive integer m. Let M be a maximal ideal of R. Assume that there is an ideal J of R such that $M^2 \subset J \subset M$. Then J is a primary ideal. By assumption, J is an n-1-absorbing primary ideal. From (2), we get $J = M^m$ for some positive integer m, a contradiction. Because there doesn't exist such an ideal between M^2 and M. By [5, Theorem 6.20], R is a Dedekind domain.

□

We have the following result by Theorem 12 since a principal ideal domain is a Dedekind domain.

Corollary 6. *Let R be a principal ideal domain. Then, a non-zero proper ideal I of R is an n-1-absorbing primary ideal if and only if $I = p^m$ for some prime element p of R and some positive integer m.*

We prove the following theorems that allow us to give some examples of n-1-absorbing primary ideals of R.

Theorem 13. *Let R be a quasi local domain with a maximal ideal M. Assume x is a prime element of R and $xR \subset M$. Then, $x^{n-1}M$ is an n-1-absorbing primary ideal of R which is not an $(n-1)$-1-absorbing primary ideal.*

Proof. We consider the non-unit elements $x_1, x_2, \ldots, x_{n+1} \in R$, $x_1x_2\cdots x_{n+1} \in x^{n-1}M$ and $x_1x_2\cdots x_n \notin x^{n-1}M$. If $x \nmid x_i$ for all $1 \leq i \leq n$, then $x \mid x_{n+1}$ and so $x_{n+1} \in xR = \sqrt{x^{n-1}M}$, it follows that $x^{n-1}M$ is an n-1-absorbing primary. Thus, we assume that $x^k | x_1x_2\cdots x_n$ and $x^{k+1} \nmid x_1x_2\cdots x_n$ for some integer k. Since R is a domain and quasi local, we have $k < n-1$. Hence, $x_1x_2\cdots x_n = x^k z$ for some $z \in R$ that $x \nmid z$. Then, we have $x_1x_2\cdots x_{n+1} \in x^{n-1}M$ and so $x_1x_2\cdots x_{n+1} = x^{n-1}a$ for some $a \in M$. Therefore, $x^{n-1}a = x^k z x_{n+1}$. Thus $x^k(zx_{n+1} - x^{n-k-1}a) = 0$. Since R is a domain, we conclude that $zx_{n+1} = x^{n-k-1}a$ and so $x | zx_{n+1}$ and then $x | x_{n+1}$. Therefore $x^{n-1}M$ is an n-1-absorbing primary ideal of R. So $x^{n-1}M$ is n-1-absorbing primary. Now, we show that $x^{n-1}M$ is not an $(n-1)$-1-absorbing primary ideal: Let $m \in M \setminus xR$. Then

$x^{n-1}m \in x^{n-1}M$ when $x^{n-1} \notin x^{n-1}M$ and $m \notin xR = \sqrt{x^{n-1}M}$. Thus, $x^{n-1}M$ is not an $(n-1)$-1-absorbing primary ideal. □

Theorem 14. *Let (R,M) be a quasi local. Assume that $P \subseteq M$ is a prime ideal of R and $(P^iM : a) = P^{i-1}M$ for all positive integer i and $a \in P - P^{i-1}$. Then $P^{i-1}M$ is an i-1-absorbing primary ideal of R.*

Proof. We prove by induction on i. Let $i = 2$. Then, Theorem 8 in [3] implies that PM is a 2-1-absorbing primary ideal of R. Assume that the claim is true for all positive integer$< i$. Then, $P^{i-2}M$ is $(i-1)$-1-absorbing primary ideal of R. We show that $P^{i-1}M$ is i-1-absorbing primary ideal of R. Let the non-unit elements $x_1, x_2, \ldots, x_{i+1} \in R$, $x_1x_2\cdots x_{i+1} \in P^{i-1}M$ and $x_{i+1} \notin P = \sqrt{P^{i-1}M}$. Since $x_1x_2\cdots x_{i+1} \in P^{i-1}M \subseteq P$ and P is a prime ideal of R, we conclude that $x_k \in P$ for some $1 \leq k \leq i$. Without loss of the generality we can assume $x_1 \in P$. If $x_1 \in P^{i-1}$, then $x_1x_2\cdots x_i \in P^{i-1}M$ and so $P^{i-1}M$ is an i-1-absorbing primary ideal of R. Then, suppose that $x_1 \in P - P^{i-1}$. Then by the assumption, we obtain $(P^{i-1}M : a) = P^{i-2}M$. Thus, $x_2\cdots x_{i+1} \in P^{i-2}M$ and by induction $P^{i-2}M$ is an $(i-1)$-1-absorbing primary ideal of R. Then $x_2\cdots x_i \in P^{i-2}M$. Therefore, $x_1x_2\cdots x_i \in PP^{i-2}M = P^{i-1}M$ implies the proof. □

Example 6. Let $S = F[x,y,z]$, where F is a field. Consider $M = (x,y,z)S$ and $R = S_M$. It can be seen that R is a quasi local ring with maximal ideal M_M. Let $P = (x/1, y/1)S$ and $I = P^2M_M$. It is easy to see that we have $(P^2M : f) = PM$ for all $f \in P$. Thus I is a 3-1-absorbing primary ideal of R by Theorem 14.

Example 7. Let $R = F[x,y]$, where F is a field. Therefore R is a domain. Let $S = R - (x,y)R$. Then $S^{-1}R$ is a quasi local domain. $x/1$ is an irreducible element of S. Then we Consider that $I = \frac{x^{n-1}}{1}S^{-1}(x,y)$. So, Theorem 13 implies that I is an n-1-absorbing primary ideal and that is not an $(n-1)$-1-absorbing primary.

Note that $\sqrt{I} = \frac{x}{1}S^{-1}R$ and it is a prime ideal of $S^{-1}R$ and then it becomes an n-1-absorbing prime ideal for all $n \geq 1$. While it is not k-1-absorbing primary ideal for $k = 1, 2, \ldots, n-1$. It shows that the converse of Theorem 1(4) is not correct.

An ideal I of a ring R is called as semi-primary if $\sqrt{I}$ is prime ideal. A semi-primary ideal may not be a primary ideal (See [7, p.154]).

Theorem 15. *Let I be an n-1-absorbing primary ideal of a ring R. Then I is a semi-primary ideal.*

Proof. Let $xy \in \sqrt{I}$ and $x \notin \sqrt{I}$ for some $x, y \in R$. Then there exists a positive ineteger k such that $x^ky^k \in I$ and also $x^m \notin I$ for all positive integer m. Assume $k \leq n$. Then we get $x^ny^n = x\cdots x \cdot y^n \in I$ and since $x\cdots x = x^n \notin I$, then $y^n \in \sqrt{I}$, that is, $y \in \sqrt{I}$. Now, assume $k \geq n$. We have $\underbrace{x\cdots x}_{n-1 \text{ times}} x^{k-n+1}y^k \in I$. Since $\underbrace{x\cdots x}_{n-1 \text{ times}} x^{k-n+1} \notin I$, then $y^k \in \sqrt{I}$ and so $y \in \sqrt{I}$. Therefore, $\sqrt{I}$ is prime and thus I is semi-primary. □

Theorem 16. *Let R be a Dedekind domain and I be a proper ideal of R. Then I is an n-1-absorbing primary ideal if and only if $\sqrt{I}$ is prime ideal.*

Proof.

($\Rightarrow$): It is obvious, by Theorem 1(4).

($\Leftarrow$): Assume that I is a semi-primary ideal of R, namely, $\sqrt{I}$ is a prime ideal. We know that every nonzero prime ideal of R is maximal since R is a Dedekind domain. Hence, $\sqrt{I}$ is a maximal ideal and so I is a primary ideal. Therefore, I is an n-1-absorbing primary ideal.

□

Theorem 17. *Let R be a divided ring and I a proper ideal of R. Then, I is an n-1-absorbing primary ideal of R if and only if I is a primary ideal of R.*

Proof. The part "if I is a primary ideal of R, then I is an n-1-absorbing primary ideal" is straightforward. Thus, we assume that I is an n-1-absorbing primary ideal of R. Let $xy \in I$ and $x \notin \sqrt{I}$ for some $x, y \in R$. x and y might be accepted as non-units of R. Since $\sqrt{I}$ is prime, then $y \in \sqrt{I}$ and $y^m \notin \sqrt{I}$ for all positive integers m, and then, $y^{n-1} \notin \sqrt{I}$. As R is a divided ring, then $\sqrt{I} \subset (x^{n-1})R$. Hence, $y \in (x^{n-1})R$ and so, there is an element r in R with $y = rx^{n-1}$ since $y \in \sqrt{I}$. Since I is n-1-absorbing primary ideal , $xy = xrx^{n-1} \in I$ and $x \notin \sqrt{I}$, then $rx^{n-1} \in I$ and so $y \in I$. Therefore, I is a primary ideal of R. □

A ring R is called as chained ring if $a|b$ or $b|a$ for every $a, b \in R$. Note that every chained ring is divided. Thus, we give the following corollary as a result of Theorem 17:

Corollary 7. *Let R be a chain ring and I a proper ideal of R. Then, I is an n-1-absorbing primary ideal of R if and only if I is a primary ideal of R.*

Recall that an integral domain R is a valuation domain if R is a chained ring and then we obtain that any valuation domain is a divided domain.

Theorem 18. *Let I be a proper ideal of a valuation domain R with $\sqrt{I} = P$. Then, the following are equivalent:*

(1) *I is an n-1-absorbing primary ideal of R.*
(2) *I is a primary ideal of R.*
(3) *$I = P^m$ for some positive integer $m \geq 1$ where $P \neq P^2$.*

Proof. Firstly, note that $\sqrt{I} = P$ is a prime ideal of R.

(1) $\Rightarrow$ (2): By Theorem 17, It is trivial.
(2) $\Rightarrow$ (3): By [5, Theorem 5.11], we have obviously the result.
(3) $\Rightarrow$ (1): By [3, Theorem 11], It is obvious.

□

Let M be an R-module. The idealization defined by $R(+)M = \{(a,m) : a \in R, m \in M\}$. Assume that $a_1, a_2 \in R$ and $m_1, m_2 \in M$, then $R(+)M$ is a commutative ring by the addition operator $(a_1,m_1)+(a_2,m_2) = (a_1+a_2, m_1+m_2)$ and the multiplication operator $(a_1,m_1)(a_2,m_2) = (a_1a_2, a_2m_1+a_1m_2)$. Let I be an ideal of R and N be a submodule of M. Then $I(+)N$ is an ideal of $R(+)M$ if and only if $IM \subseteq N$, this ideal is called homogeneous. For more properties of the idealization refer to [1].

Theorem 19. *Assume that $I(+)N$ is a homogeneous ideal of $R(+)M$. Then*

(1) *If $I(+)N$ is an n-1-absorbing primary ideal of $R(+)M$, then I is an n-1-absorbing primary ideal of R.*
(2) *If $\sqrt{I}M \subseteq N$, $(N :_M a) = N$ for all $a \notin \sqrt{I}$ and I is an n-1-absorbing primary ideal of R, Then $I(+)N$ is an n-1-absorbing primary ideal of $R(+)M$.*

Proof.

(1) Let $I(+)N$ be an n-1-absorbing primary ideal of $R(+)M$. Assume that $a_1,\dots,a_{n+1} \in R$ and $a_1\cdots a_{n+1} \in I$. Then $(a_1,0)\cdots(a_{n+1},0) \in I(+)N$ and therefore $(a_1,0)\cdots(a_n,0) = (a_1\cdots a_n, 0) \in I(+)N$ or $(a_{n+1},0) \in \sqrt{I(+)N} = \sqrt{I}(+)N$. Thus $a_1\cdots a_n \in I$ or $a_{n+1} \in \sqrt{I}$. It implies that I is an n-1-absorbing primary ideal of R.
(2) Assume that $\sqrt{I}M \subseteq N$, $(N :_M a) = N$ for all $a \notin \sqrt{I}$ and I is an n-1-absorbing primary ideal of R. Let $(a_1,m_1)\cdots,(a_{n+1},m_{n+1}) \in I(+)N$ for some elements $a_1,\cdots,a_{n+1} \in R$ and $m_1\cdots m_{n+1} \in I$. It implies that
$$a_1\cdots a_{n+1} \in I, \quad \textstyle\sum_{i=1}^{n+1} a_1\cdots\hat{a}_i\cdots a_{n+1}m_i \in N.$$
Then we have $a_1\cdots a_n \in I$ or $a_{n+1} \in \sqrt{I}$. Firstly, we assume that $a_{n+1} \in \sqrt{I}$, then $a_{n+1}^k \in I$ for some integer k. Therefore $(a_{n+1},m_{n+1})^{(k+1)} = (a_{n+1}^{(k+1)}, (k+1)a_{n+1}^k m_{n+1}) \in I(+)N$. It means that $(a_{n+1},m_{n+1}) \in \sqrt{I(+)N}$. Thus, we assume that $a_{n+1} \notin \sqrt{I}$. Then $a_1\cdots a_n \in I$ implies that $a_1\cdots a_n m_{n+1} \in IM \subseteq N$. Hence $\sum_{i=1}^{n} a_1\cdots\hat{a}_i\cdots a_n a_{n+1} m_i \in N$ and so $\sum_{i=1}^{n} a_1\cdots\hat{a}_i\cdots a_n m_i \in N : a_{n+1}$. Now, since $N : a_{n+1} = N$, we conclude that $\sum_{i=1}^{n} a_1\cdots\hat{a}_i\cdots a_n m_i \in N$. It shows that $(a_1,m_1)\cdots,(a_n,m_n) \in I(+)N$ and therefore $I(+)N$ is an n-1-absorbing primary ideal of $R(+)M$.

□

REFERENCES

[1] D. D. Anderson and M. Winders, "Idealization of a module." *J. Commut. Algebra* , vol. 1, no. 1, pp. 3–56, 2009, doi: 10.1216/JCA-2009-1-1-3.
[2] D. F. Anderson and A. Badawi, "On n-absorbing ideals of commutative rings," *Commun. Algebra*, vol. 39, no. 5, pp. 1646–1672, 2011, doi: 10.1080/00927871003738998.
[3] A. Badawi and E. Y. Celikel, "On 1-absorbing primary ideals of commutative rings," *J. Algebra Appl.*, vol. 19, no. 06, p. 2050111, 2020, doi: 10.1142/S021949882050111X.
[4] A. E. Becker, "Results on n-absorbing ideals of commutative rings," Ph.D. dissertation, The University of Wisconsin-Milwaukee, 2015.

[5] E. A. Behrens and C. Reis, *Multiplicative theory of ideals*. Academic press, 1971.
[6] G. Ulucak, S. Koc, and U. Tekir, “On n-1-absorbing ideals of commutative rings,” *J. Algebra Appl.*, 2022, doi: 10.1142/S0219498823502079.
[7] O. Zariski and P. Samuel, “Commutative algebra. Vol. 1. With the cooperation of I. S. Cohen,” *Inc., Princeton, New Jersey*, 1958.

Authors’ addresses

Gulsen Ulucak
Department of Mathematics, Faculty of Science, Gebze Technical University, Gebze Kocaeli, Turkey
E-mail address: gulsenulucak@gtu.edu.tr

Sakineh Babaei
Imam Khomeini International University, Postal Code: 34149-1-6818, Qazvin, Iran
E-mail address: sakinehbabaei@gmail.com

Esra Sengelen Sevim
(**Corresponding author**) Department of Mathematics, Istanbul Bilgi University, 34060, Eski Silahtaraga Elektrik Santrali, Kazim Karabekir Cad. No: 2/13 Eyupsultan, Istanbul, Turkey
E-mail address: esra.sengelen@bilgi.edu.tr