



SOME PROPERTIES OF r -SUPPLEMENTED MODULES

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Abstract. In this work, r -supplemented modules are defined and some properties of these modules are investigated. It is proved that every factor module and every homomorphic image of an r -supplemented module are r -supplemented. Let M be an R -module and $M = M_1 + M_2 + \dots + M_n$. If M_i is r -supplemented for each $i = 1, 2, \dots, n$, then M is also r -supplemented. Let M be an r -supplemented module. Then every finitely M -generated R -module is r -supplemented.

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1. INTRODUCTION

Throughout this paper all rings will be associative with identity and all modules will be unital left modules.

Let R be a ring and M be an R -module. We will denote a submodule N of M by $N \leq M$. Let M be an R -module and $N \leq M$. If $L = M$ for every submodule L of M such that $M = N + L$, then N is called a *small* submodule of M and denoted by $N \ll M$. Let M be an R -module. M is called a *hollow* module if every proper submodule of M is small in M . M is said to be *local* if M has the largest submodule, i.e. a proper submodule which contains all other proper submodules. Let M be an R -module and $U, V \leq M$. If $M = U + V$ and V is minimal with respect to this property, or equivalently, $M = U + V$ and $U \cap V \ll V$, then V is called a *supplement* of U in M . M is called a *supplemented* module if every submodule of M has a supplement in M . Let M be an R -module and $U \leq M$. If for every $V \leq M$ such that $M = U + V$, U has a supplement V' with $V' \leq V$, we say U has *ample supplements* in M . If every submodule of M has ample supplements in M , then M is called an *amply supplemented* module. The intersection of maximal submodules of an R -module M is called the *radical* of M and denoted by $RadM$. If M have no maximal submodules, then we denote $RadM = M$. Let M be an R -module and $U, V \leq M$. If $M = U + V$

and $U \cap V \leq \text{Rad}V$, then V is called a *generalized (radical) supplement* (briefly, *Rad-supplement*) of U in M . M is said to be *generalized (radical) supplemented* (briefly, *Rad-supplemented*) if every submodule of M has a Rad-supplement in M . Let M be an R -module and $K \leq M$. If $K \ll \text{Rad}M$, then K is called an *r-small* submodule of M and denoted by $K \ll_r M$. Let M be an R -module. It is defined the relation ' β^* ' on the set of submodules of an R -module M by $X\beta^*Y$ if and only if $Y + K = M$ for every $K \leq M$ such that $X + K = M$ and $X + T = M$ for every $T \leq M$ such that $Y + T = M$. Let M be an R -module and $K \leq V \leq M$. We say V lies above K in M if $V/K \ll_r M/K$.

More informations about (amply) supplemented modules are in [2, 3, 8, 9]. More results about Rad-supplemented modules are in [7]. The definition of r-small submodules and some properties of them are in [4, 5]. The definition of β^* relation and some properties of this relation are in [1].

Lemma 1. *Let M be an R -module. The following assertions hold.*

- (i) *If $K \ll_r M$, then $K \ll M$.*
- (ii) *If $L \ll_r M$ and $K \leq L$, then $K \ll_r M$.*
- (iii) *If $K \ll_r L \leq M$, then $K \ll_r M$.*
- (iv) *If $K_i \ll_r L_i \leq M$ for $i = 1, 2, \dots, n$, then $K_1 + K_2 + \dots + K_n \ll_r L_1 + L_2 + \dots + L_n$.*
- (v) *If $K_i \ll_r M$ for $i = 1, 2, \dots, n$, then $K_1 + K_2 + \dots + K_n \ll_r M$.*
- (vi) *If $K \ll_r M$, then $(K + L)/L \ll_r M/L$ for every $L \leq M$.*
- (vii) *If $K \ll M$ and $\text{Rad}M$ is a supplement submodule in M , then $K \ll_r M$.*
- (viii) *Let N be an R -module and $f : M \rightarrow N$ be an R -module homomorphism. If $K \ll_r M$, then $f(K) \ll_r f(M)$.*
- (ix) $\text{Rad}(\text{Rad}M) = \sum_{K \ll_r M} K$.

Proof. See [4, 5]. □

2. R-SUPPLEMENTED MODULES

Definition 1. Let M be an R -module and $U, V \leq M$. If $M = U + V$ and $U \cap V \ll_r V$, then V is called an *r-supplement* of U in M . If every submodule of M has an r-supplement in M , then M is called an *r-supplemented module*. (See also [6])

Clearly we can see that every r-supplemented module is supplemented. But the converse is not true in general (See Example 1).

Lemma 2. *Let V be a Rad-supplement of U in M . Then V is an r-supplement of U in M if and only if $\text{Rad}V$ is a supplement of U in $U + \text{Rad}V$.*

Proof.

(\implies) Let V be an r-supplement of U in M . Then $U \cap V \ll_r \text{Rad}V$ and $U \cap \text{Rad}V = U \cap V \cap \text{Rad}V \ll_r \text{Rad}V$. Hence $\text{Rad}V$ is a supplement of U in $U + \text{Rad}V$.

(\Leftarrow) Let $RadV$ be a supplement of U in $U + RadV$. Since V is a Rad-supplement of U in M , $M = U + V$ and $U \cap V \leq RadV$. Then $U \cap V = U \cap V \cap RadV = U \cap RadV$. Since $RadV$ is a supplement of U in $U + RadV$, $U \cap RadV \ll RadV$. Hence $U \cap V = U \cap RadV \ll RadV$ and $U \cap V \ll_r V$. Therefore, V is an r -supplement of U in M . \square

Corollary 1. *Let V be an r -supplement of U in M . Then $Rad(RadV) = RadV \cap Rad(U + RadV)$.*

Proof. Since V is an r -supplement of U in M , by Lemma 2, $RadV$ is a supplement of U in $U + RadV$. Then by [8, 41.1 (5)], $Rad(RadV) = RadV \cap Rad(U + RadV)$, as desired. \square

Lemma 3. *Let V be an r -supplement of U in M and $U\beta^*X$ in $U + RadV$ with $X \leq U + RadV$. Then V is an r -supplement of X in M .*

Proof. Since V is an r -supplement of U in M , by Lemma 2, $RadV$ is a supplement of U in $U + RadV$. Since $U\beta^*X$ in $U + RadV$, by [1, Theorem 2.6 (ii)], $RadV$ is a supplement of X in $U + RadV$. Here $X + RadV = U + RadV$ and $X \cap RadV \ll RadV$. Hence $RadV$ is a supplement of X in $X + RadV$. Let $N = U + RadV$ and $U + K = M$ with $K \leq M$. Since $U + K = M$, $N = N \cap M = N \cap (U + K) = U + N \cap K$ and since $U\beta^*X$ in N , $X + N \cap K = N$. Here $X + K = X + N \cap K + K = N + K = U + N \cap K + K = U + K = M$. Interchanging the roles of U and X , we can see that $U + T = M$ for every $T \leq M$ with $X + T = M$. Hence $U\beta^*X$ in M . Since V is an r -supplement of U in M , V is a supplement of U in M and since $U\beta^*X$ in M , by [1, Theorem 2.6 (ii)], V is a supplement of X in M . Then V is a Rad-supplement of X in M and since $RadV$ is a supplement of X in $X + RadV$, by Lemma 2, V is an r -supplement of X in M . \square

Corollary 2. *Let V be an r -supplement of U in M and U lies above X in $U + RadV$. Then V is an r -supplement of X in M .*

Proof. Clear from Lemma 3. \square

Lemma 4. *Let V be a supplement of U in M . If $RadV$ is a supplement submodule in V , then V is an r -supplement of U in M .*

Proof. Since V is a supplement of U in M , $M = U + V$ and $U \cap V \ll V$. Since $RadV$ is a supplement submodule in V , by Lemma 1, $U \cap V \ll_r V$. Hence V is an r -supplement of U in M , as desired. \square

Corollary 3. *Let V be a supplement of U in M . If $RadV$ is a direct summand of V , then V is an r -supplement of U in M .*

Proof. Since $RadV$ is a direct summand of V , $RadV$ is a supplement submodule in V . Then by Lemma 4, V is an r -supplement of U in M . \square

Corollary 4. *Let V be a supplement of U in M . If $RadV$ is a supplement submodule in M , then V is an r -supplement of U in M .*

Proof. Let $RadV$ be a supplement of X in M . Then $M = X + RadV$ and $X \cap RadV \ll RadV$. Since $M = X + RadV$, by Modular law, $V = V \cap M = V \cap (X + RadV) = V \cap X + RadV$ and since $V \cap X \cap RadV = X \cap RadV \ll RadV$, $RadV$ is a supplement of $V \cap X$ in V . Then by Lemma 4, V is an r -supplement of U in M . \square

We can also prove this Corollary 4 as follows:

Proof. Since V is a supplement of U in M , $M = U + V$ and $U \cap V \ll V$. Then $U \cap V \leq RadV$ and $U \cap V \ll M$. Since $RadV$ is a supplement submodule in M , by [8, 41.1 (5)], $U \cap V = U \cap V \cap RadV \ll RadV$. Hence $U \cap V \ll_r V$ and V is an r -supplement of U in M . \square

Corollary 5. *Let V be a supplement of U in M . If $RadV$ is a direct summand of M , then V is an r -supplement of U in M .*

Proof. Clear from Corollary 4. \square

Lemma 5. *Let M be a supplemented module. If $RadV$ is a supplement submodule in V for every supplement submodule V in M , then M is r -supplemented.*

Proof. Let $U \leq M$. Since M is supplemented, U has a supplement V in M . By hypothesis, $RadV$ is a supplement submodule in V . Then by Lemma 4, V is an r -supplement of U in M . Hence M is r -supplemented, as desired. \square

Corollary 6. *Let M be a supplemented module. If $RadV$ is a direct summand of V for every supplement submodule V in M , then M is r -supplemented.*

Proof. Clear from Lemma 5. \square

Proposition 1. *Let M be an r -supplemented module. Then $RadM$ is a supplement submodule in M .*

Proof. Since M is r -supplemented, $RadM$ has an r -supplement V in M . Here $M = RadM + V$ and $V \cap RadM \ll_r V$. Since $V \cap RadM \ll_r V$, $V \cap RadM \ll RadV \leq RadM$. Hence $RadM$ is a supplement of V in M . \square

Remark 1. The converse of Proposition 1 is not true in general. Consider the \mathbb{Z} -module ${}_Z\mathbb{Q}$. Since $Rad_Z\mathbb{Q} = {}_Z\mathbb{Q}$, $Rad_Z\mathbb{Q}$ is a supplement submodule in ${}_Z\mathbb{Q}$. But ${}_Z\mathbb{Q}$ is not r -supplemented.

Proposition 2. *Let M be an r -supplemented module. Then $M/Rad(RadM)$ is semisimple.*

Proof. Let $\frac{K}{Rad(RadM)}$ be any submodule of $\frac{M}{Rad(RadM)}$. Since M is r -supplemented, K has an r -supplement V in M . Then $M = K + V$ and $K \cap V \ll_r V$. Since $M = K + V$, $\frac{M}{Rad(RadM)} = \frac{K}{Rad(RadM)} + \frac{V+Rad(RadM)}{Rad(RadM)}$. Since $K \cap V \ll_r V$, by Lemma 1, $K \cap V \leq$

$Rad(RadM)$. Then $\frac{K}{Rad(RadM)} \cap \frac{V+Rad(RadM)}{Rad(RadM)} = \frac{K \cap V + Rad(RadM)}{Rad(RadM)} = 0$ and $\frac{M}{Rad(RadM)} = \frac{K}{Rad(RadM)} \oplus \frac{V+Rad(RadM)}{Rad(RadM)}$. Hence every submodule of $\frac{M}{Rad(RadM)}$ is a direct summand of $\frac{M}{Rad(RadM)}$ and $\frac{M}{Rad(RadM)}$ is semisimple. \square

Lemma 6. *Let M be an R -module, $U \leq M$ and $M_1 \leq M$. If X is an r -supplement of $U + M_1$ in M and Y is an r -supplement of $(U + X) \cap M_1$ in M_1 , then $X + Y$ is an r -supplement of U in M .*

Proof. Since X is an r -supplement of $U + M_1$ in M , $M = U + M_1 + X$ and $X \cap (U + M_1) \ll_r X$. Since Y is an r -supplement of $(U + X) \cap M_1$ in M_1 , $M_1 = (U + X) \cap M_1 + Y$ and $(U + X) \cap Y = (U + X) \cap M_1 \cap Y \ll_r Y$. Then $M = U + M_1 + X = U + X + (U + X) \cap M_1 + Y = U + X + Y$ and $U \cap (X + Y) \leq (U + X) \cap Y + (U + Y) \cap X \leq (U + M_1) \cap X + (U + X) \cap Y \ll_r X + Y$. Hence $X + Y$ is a r -supplement of U in M . \square

Lemma 7. *Let M be an R -module, $U \leq M$ and $M_1 \leq M$. If M_1 is r -supplemented and $U + M_1$ has an r -supplement in M , then U has an r -supplement in M .*

Proof. Clear from Lemma 6. \square

Corollary 7. *Let M be an R -module, $U \leq M$ and $M_i \leq M$ for $i = 1, 2, \dots, n$. If M_i is r -supplemented for every $i = 1, 2, \dots, n$ and $U + M_1 + M_2 + \dots + M_n$ has an r -supplement in M , then U has an r -supplement in M .*

Proof. Clear from Lemma 7. \square

Lemma 8. *Let $M = M_1 + M_2$. If M_1 and M_2 are r -supplemented, then M is also r -supplemented.*

Proof. Let $U \leq M$. Then 0 is an r -supplement of $U + M_1 + M_2$ in M . Since M_2 is r -supplemented, by Lemma 7, $U + M_1$ has an r -supplement in M . Since M_1 is r -supplemented, by Lemma 7 again, U has an r -supplement in M . Hence M is r -supplemented. \square

Corollary 8. *Let $M = M_1 + M_2 + \dots + M_n$. If M_i is r -supplemented for each $i = 1, 2, \dots, n$, then M is also r -supplemented.*

Proof. Clear from Lemma 8. \square

Lemma 9. *Let V be an r -supplement of U in M and $K \leq U$. Then $\frac{V+K}{K}$ is an r -supplement of $\frac{U}{K}$ in $\frac{M}{K}$.*

Proof. Since V is an r -supplement of U in M , $M = U + V$ and $U \cap V \ll_r V$. Since $M = U + V$ and $K \leq U$, $\frac{M}{K} = \frac{U+V}{K} = \frac{U}{K} + \frac{V+K}{K}$. Since $U \cap V \ll_r V$, by Lemma 1, $\frac{U}{K} \cap \frac{V+K}{K} = \frac{U \cap V + K}{K} \ll_r \frac{V+K}{K}$. Hence $\frac{V+K}{K}$ is an r -supplement of $\frac{U}{K}$ in $\frac{M}{K}$. \square

Lemma 10. *Every factor module of an r -supplemented module is r -supplemented.*

Proof. Let M be an r -supplemented R -module and $\frac{M}{K}$ be any factor module of M . Let $\frac{U}{K} \leq \frac{M}{K}$. Since M is r -supplemented, U has an r -supplement V in M . Since $K \leq U$, by Lemma 9, $\frac{V+K}{K}$ is an r -supplement of $\frac{U}{K}$ in $\frac{M}{K}$. Hence $\frac{M}{K}$ is r -supplemented. \square

Corollary 9. *Every homomorphic image of an r -supplemented module is r -supplemented.*

Proof. Clear from Lemma 10. \square

Corollary 10. *Every direct summand of an r -supplemented module is r -supplemented.*

Proof. Clear from Lemma 10. \square

Lemma 11. *Let M be an r -supplemented module. Then every finitely M -generated R -module is r -supplemented.*

Proof. Let N be a finitely M -generated R -module. Then there exist a finite index set Λ and an R -module epimorphism $f : M^{(\Lambda)} \rightarrow N$. Since M is r -supplemented, by Corollary 8, $M^{(\Lambda)}$ is r -supplemented. Then by Corollary 9, N is r -supplemented. \square

Proposition 3. *Let R be a ring. Then ${}_R R$ is r -supplemented if and only if every finitely generated R -module is r -supplemented.*

Proof. Clear from Lemma 11. \square

Lemma 12. *Let M be a local module with $\text{Rad}M \neq 0$. Then M is not r -supplemented. But M is supplemented.*

Proof. Since M is local, M is the only supplement of $\text{Rad}M$ in M . Since $M \cap \text{Rad}M = \text{Rad}M$ is not small in $\text{Rad}M$, M is not an r -supplement of $\text{Rad}M$ in M . Hence M is not r -supplemented. But since M is local, M is supplemented. \square

Lemma 13. *Let M be a hollow module with $\text{Rad}M = M$. Then M is r -supplemented.*

Proof. Clear from definitions. \square

Example 1. Consider the \mathbb{Z} -module \mathbb{Z}_8 . Since \mathbb{Z}_8 is local and $\text{Rad}\mathbb{Z}_8 = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} \neq 0$, by Lemma 12, \mathbb{Z}_8 is supplemented, but not r -supplemented.

Example 2. Consider the \mathbb{Z} -module \mathbb{Z}_{p^∞} . Since \mathbb{Z}_{p^∞} is hollow and $\text{Rad}\mathbb{Z}_{p^\infty} = \mathbb{Z}_{p^\infty}$, by Lemma 13, \mathbb{Z}_{p^∞} is r -supplemented.

Example 3. Consider the \mathbb{Z} -module $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_5$. Since \mathbb{Z}_5 is simple, \mathbb{Z}_5 is r -supplemented. Since \mathbb{Z}_{p^∞} and \mathbb{Z}_5 are r -supplemented, by Lemma 8, $M = \mathbb{Z}_{p^\infty} \oplus \mathbb{Z}_5$ is r -supplemented. Here $\text{Rad}M = \text{Rad}\mathbb{Z}_{p^\infty} \oplus \text{Rad}\mathbb{Z}_5 = \mathbb{Z}_{p^\infty} \oplus 0 = \mathbb{Z}_{p^\infty} \neq M$.

REFERENCES

- [1] G. F. Birkenmeier, F. Takil Mutlu, C. Nebiyev, N. Sokmez, and A. Tercan, “Goldie*-supplemented modules.” *Glasg. Math. J.*, vol. 52A, pp. 41–52, 2010, doi: [10.1017/S0017089510000212](https://doi.org/10.1017/S0017089510000212).
- [2] J. Clark, C. Lomp, N. Vanaja, and R. Wisbauer, *Lifting modules. Supplements and projectivity in module theory.*, ser. Front. Math. Basel: Birkhäuser, 2006.
- [3] C. Nebiyev and A. Pancar, “On supplement submodules.” *Ukr. Math. J.*, vol. 65, no. 7, pp. 1071–1078, 2013, doi: [10.1007/s11253-013-0842-2](https://doi.org/10.1007/s11253-013-0842-2).
- [4] C. Nebiyev and H. H. Ökten, “ r -small submodules,” in *3rd International E-Conference on Mathematical Advances and Applications (ICOMAA-2020)*, 2020.
- [5] C. Nebiyev and H. H. Ökten, “Some properties of r -small submodules,” *Erzincan University Journal of Science and Technology*, vol. 15, no. 3, pp. 996–1001, 2022, doi: [10.18185/erzif-bed.876557](https://doi.org/10.18185/erzif-bed.876557).
- [6] C. Nebiyev and N. Sökmez, “ r -supplemented modules,” in *4th International E-Conference on Mathematical Advances and Applications (ICOMAA-2021)*, 2021.
- [7] Y. Wang and N. Ding, “Generalized supplemented modules.” *Taiwanese J. Math.*, vol. 10, no. 6, pp. 1589–1601, 2006, doi: [10.11650/twjm/1500404577](https://doi.org/10.11650/twjm/1500404577).
- [8] R. Wisbauer, *Foundations of module and ring theory. A handbook for study and research.*, ser. Algebra Log. Appl. Philadelphia etc.: Gordon and Breach Science Publishers, 1991, vol. 3.
- [9] H. Zöschinger, “Komplementierte Moduln über Dedekindringen,” *J. Algebra*, vol. 29, pp. 42–56, 1974, doi: [10.1016/0021-8693\(74\)90109-4](https://doi.org/10.1016/0021-8693(74)90109-4). [Online]. Available: epub.ub.uni-muenchen.de/4100/

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