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ON MODELS OF THE LIE ALGEBRA $\mathscr{K}_5$ AND LAURICELLA FUNCTIONS USING AN INTEGRAL TRANSFORMATION

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Abstract. We construct new $(n+1)$-variable models of irreducible representations of the Lie algebra $\mathscr{K}_5$. An n-fold integral transformation is used to obtain a new set of models of $\mathscr{K}_5$ in terms of difference-differential operators. These models are further exploited to obtain recurrence relations, generating functions and addition theorems.

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1. INTRODUCTION

There is a close connection between models of Lie algebras and identities in special function theory. Miller [8] and Vilenkin [19] have developed this connection systematically from its basics. Manocha [7], Govil and Manocha [3] have studied this relationship via certain integral transformations thereby obtaining various recurrence relations and identities in the theory of special functions. Sahai [11, 12], Sahai and Srivastava [13] have extended this approach to obtain numerous results in the theory of special functions of one and several variables. The Lie algebras considered for obtaining these results mainly include the special linear complex algebra $sl(2,\mathbb{C})$, the oscillator algebra $\mathscr{G}(0,1)$ and the 3-dimensional Euclidean algebra $\mathscr{T}_3$. In the present paper, we extend this study to the 5-dimensional Lie algebra $\mathscr{K}_5$. We construct new $(n+1)$-variable models of irreducible representations of $\mathscr{K}_5$ in terms of differential operators. Using an integral transformation, we obtain models of $\mathscr{K}_5$ which are explored for recurrence relations, generating functions and addition theorems. These results are believed to be new.

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Earlier, notable contributions in relating the Lie algebra $\mathscr{K}_5$ and special functions have come from the works of Khan and Ali [6], Pathan et al [9], Srivastava et al [18] and Yadav and Rani [20]. The multi-variable Lauricella series as well as their Srivastava-Daoust generalizations are extensively described in the monograph by Srivastava and Karlsson [16]. Some developments in the usage of Lie algebraic techniques to study the theory of multi-variable and multi-index Hermite polynomials have been studied in [2]. Decomposition formulas, linearization problems and certain interesting properties associated with generalized Lauricella functions also exist in literature, see [4, 5, 15]. Moreover, the incomplete Lauricella functions and their associated properties including integral representations, finite summation formulas, transformation and derivative formulas have been discussed in [1]. Recently, Srivastava [14] has listed some important developments in the theory and applications of hypergeometric and related functions.

The paper is organized as follows. In Section 2, we give a brief review of the Lie algebra $\mathscr{K}_5$ and its representations $R'(\omega, m_0, \mu)$ and $\uparrow'_{\omega,\mu}$. One variable models of representations $R'(0, m_0, \mu)$ and $\uparrow'_{\omega,\mu}$ are given following Miller [8]. An r-fold integral transformation is defined and transforms of certain operator expressions are given that are needed for our discussion [3]. In Section 3, we give $(n+1)$-variable models of representations of $\mathscr{K}_5$ in which the representation spaces have basis functions appearing as ${}_1F_0\left(-\lambda; -; \sum_{i=1}^{n} u_i x_i\right) t^{\lambda}$ and ${}_1F_0\left(-\lambda-\omega; -; \sum_{i=1}^{n} u_i x_i\right) t^{\lambda+\omega}$, respectively. In Section 4, we obtain the transformed models of Section 3 in terms of difference-differential operators with basis functions as Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$, respectively. Further, we utilize these models to obtain some recurrence relations and generating functions. Finally, in Section 5, all the models of Sections 3 are utilized to obtain interesting addition theorems.

2. Preliminaries

The Lie algebra $\mathscr{K}_5$ is the Lie algebra of 5-dimensional complex Lie group K_5, given by:

$$K_5 = \left\{ g(q,a,b,c,\tau) = \begin{pmatrix} 1 & ce^{\tau} & be^{-\tau} & 2a-bc & \tau \\ 0 & e^{\tau} & 2qe^{-\tau} & b-2qc & 0 \\ 0 & 0 & e^{-\tau} & -c & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, q,a,b,c,\tau \in \mathbb{C} \right\},$$

where the group operation is matrix multiplication [8], satisfying the multiplication law:

$$\begin{aligned} &g_1(q,a,b,c,\tau)\, g_2(q', a', b', c', \tau') \\ &= g(q+q'e^{2\tau}, a+a'+e^{\tau}cb'+c^2q'e^{2\tau}, b+b'e^{\tau}+2cq'e^{2\tau}, c+c'e^{-\tau}, \tau+\tau'), \quad (2.1) \end{aligned}$$

where q, a, b, c, τ, q', a', b', c', $\tau' \in \mathbb{C}$ and g, g_1, $g_2 \in K_5$.

The identity element of K_5 is $g(0,0,0,0,0)$ and the inverse of $g(q,a,b,c,\tau)$ is $g(-qe^{-2\tau}, -a+bc-c^2q, -be^{-\tau}+2cqe^{-\tau}, -ce^{\tau}, -\tau)$. The 5-dimensional complex Lie algebra $\mathscr{K}_5$ has basis $\{\mathscr{J}^3, \mathscr{J}^{\pm}, \mathscr{E}, \mathscr{Q}\}$ satisfying the commutation relations:

$$\begin{aligned} &[\mathscr{J}^3, \mathscr{J}^{\pm}] = \pm\mathscr{J}^{\pm}, \quad [\mathscr{J}^3, \mathscr{Q}] = 2\mathscr{Q}, \quad [\mathscr{J}^-, \mathscr{J}^+] = \mathscr{E}, \\ &[\mathscr{J}^-, \mathscr{Q}] = 2\mathscr{J}^+, \quad [\mathscr{J}^+, \mathscr{Q}] = 0, \\ &[\mathscr{J}^{\pm}, \mathscr{E}] = [\mathscr{J}^3, \mathscr{E}] = [\mathscr{Q}, \mathscr{E}] = \mathscr{O}, \end{aligned} \tag{2.2}$$

where $\mathscr{O}$ is the 5×5 zero matrix.

Let ρ be an irreducible representation of $\mathscr{K}_5$ on a vector space V and let

$$J^{\pm} = \rho(\mathscr{J}^{\pm}), \quad J^3 = \rho(\mathscr{J}^3), \quad E = \rho(\mathscr{E}), \quad Q = \rho(\mathscr{Q})$$

be operators on V. Clearly, these operators obey the commutation relations identical to (2.2).

Let S denote the spectrum of J^3 and let the irreducible representation ρ satisfies the following conditions:

(1) Each eigenvalue of J^3 has multiplicity equal to one.
(2) There is a enumerable basis for V consisting of all the eigenvectors of J^3.

This ensures that S is enumerable and that there exists a basis for V consisting of vectors $\{f_m \mid m \in S\}$ such that $J^3 f_m = m f_m$. It is well known that the oscillator algebra $\mathscr{G}(0,1)$ is a subalgebra of Lie algebra $\mathscr{K}_5$ and an irreducible representation of $\mathscr{G}(0,1)$ satisfying (1)–(2) is isomorphic to one of the irreducible representations $R(\omega, m_0, \mu)$ or $\uparrow_{\omega,\mu}$ of $\mathscr{G}(0,1)$. In the following, we study such representations of $\mathscr{K}_5$. Indeed, we have [8].

Theorem 1. *Every irreducible representation ρ of $\mathscr{K}_5$ is isomorphic to a representation in the following list:*

(1) *The representation $R'(\omega, m_0, \mu)$ defined for all $\omega, m_0, \mu \in \mathbb{C}$ such that $\mu \neq 0$, $0 \leq Re\, m_0 < 1$ and $\omega + m_0$ is not an integer. The spectrum of J^3 is the set $S = \{m_0 + n : n \text{ is an integer}\}$.*
(2) *The representation $\uparrow'_{\omega,\mu}$ defined for all $\omega, \mu \in \mathbb{C}$ such that $\mu \neq 0$. The spectrum of J^3 is the set $S = \{-\omega + n : n \text{ is non-negative integer}\}$.*

For each of the above cases, there is a basis of V consisting of vectors $\{f_m \mid m \in S\}$ such that

$$\begin{aligned} &J^3 f_m = m f_m, \quad E f_m = \mu f_m, \quad Q f_m = \mu f_{m+2}, \\ &J^+ f_m = \mu f_{m+1}, \quad J^- f_m = (m+\omega) f_{m-1}. \end{aligned} \tag{2.3}$$

On the right hand side of (2.3), $f_m = 0$ if $m \notin S$.

One variable models of the irreducible representations of $\mathscr{K}_5$ are given by:

Representation $R'(0,m_0,\mu)$**:**

$$J^3 = z\frac{d}{dz}, \quad J^+ = \mu z, \quad J^- = \frac{d}{dz},$$
$$E = \mu, \quad Q = \mu z^2, \quad f_\lambda(z) = z^\lambda,$$

where $\lambda \in S = \{m_0 + n \mid m_0 \in \mathbb{C} - \{0\}, 0 \leq Re\, m_0 < 1, n = 0, \pm 1, \ldots\}$.

Representation $\uparrow'_{\omega,\mu}$**:**

$$J^3 = -\omega + z\frac{d}{dz}, \quad J^+ = \mu z, \quad J^- = \frac{d}{dz},$$
$$E = \mu, \quad Q = \mu z^2, \quad f_\lambda(z) = z^{\lambda+\omega},$$

where $\lambda \in S = \{-\omega + n \mid n = 0, 1, \ldots\}$.

2.1. *Integral Transformation*

Let V be a complex vector space consisting of all analytic functions $f(z_1,\ldots,z_r)$, analytic at $(z_1,\ldots,z_r) = (0,\ldots,0)$. We define

$$\begin{aligned}
h(\beta_i,\gamma_j,\gamma,x_i) &= I[f(z_1,\ldots,z_r)] \\
&= \left(\prod_{i=1}^{k} \frac{\Gamma(\gamma_i)}{\Gamma(\beta_i)\Gamma(\gamma_i-\beta_i)}\right) \frac{\Gamma(\gamma)}{\Gamma(\beta_{k+1})\ldots\Gamma(\beta_r)\Gamma\left(\gamma-\sum_{i=k+1}^{m}\beta_i\right)} \\
&\times \underbrace{\int\cdots\int}_{r-fold} \prod_{i=1}^{r} u_i^{\beta_i-1} \prod_{i=1}^{k}(1-u_i)^{\gamma_i-\beta_i-1}\left(1-\sum_{j=k+1}^{m}u_j\right)^{\gamma-\sum_{j=k+1}^{m}\beta_j-1} \\
&\times f(z_1,\ldots,z_r)\, du_1\ldots du_r,
\end{aligned}$$

where,

$$z_i = u_i x_i, \quad i = 1,\ldots,r;$$
$$Re\,\gamma_i > Re\,\beta_i > 0, \quad i = 1,\ldots,k;$$
$$Re\,\beta_j > 0, \quad j = k+1,\ldots,m;$$
$$Re\left(\gamma - \sum_{j=k+1}^{m}\beta_j\right) > 0;$$

and the path of integration is $0 \leq u_i \leq 1, i = 1,\ldots,k$, $u_j \geq 0$, $j = k+1,\ldots,m,\ldots,r$ and $\sum_{j=k+1}^{m} u_j \leq 1$ [3]. Then $W = IV$ is an isomorphic image of V under the transformation I: $f(z_1,\ldots,z_r) \to h(\beta_i,\gamma_j,\gamma,x_i)$. Next, we obtain transforms of certain expressions under the transformation I in terms of difference operators and differential operators defined as follows:

$$E_{\gamma_i}h(\beta_i,\gamma_i) = h(\beta_i,\gamma_i+1),$$

$$L_{\gamma_i} h(\beta_i,\gamma_i) = h(\beta_i,\gamma_i - 1),$$
$$\Delta_{\gamma_i} h(\beta_i,\gamma_i) = (E_{\gamma_i} - 1) h(\beta_i,\gamma_i),$$
$$E_{\beta_i\gamma_i} h(\beta_i,\gamma_i) = E_{\beta_i}\left[E_{\gamma_i} h(\beta_i,\gamma_i)\right],$$

where $h(\beta_i,\gamma_i \pm 1) = h(\beta_1,\gamma_1,\ldots,\beta_{i-1},\gamma_{i-1},\beta_i,\gamma_i \pm 1,\ldots,\beta_r,\gamma_r)$.

We obtain the following transforms under I:

$$I[u_i f] = \begin{cases} \frac{\beta_i}{\gamma_i} E_{\beta_i\gamma_i} h, & 1 \le i \le k; \\ \\ \frac{\beta_i}{\gamma} E_{\beta_i\gamma} h, & k+1 \le i \le m. \end{cases}$$

$$I\left[u_i \frac{\partial}{\partial u_i} f\right] = \beta_i \Delta_{\beta_i} h = x_i \frac{\partial}{\partial x_i} h, \quad 1 \le i \le r.$$

Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$ are defined as follows [17]:

$$\begin{aligned} & F_A^{(n)} := F_A^{(n)}(a;b_1,\ldots,b_n;c_1,\ldots,c_n;z_1,\ldots,z_n) \\ & = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c_1)_{m_1}\ldots(c_n)_{m_n}} \frac{z_1^{m_1}}{m_1\,!}\cdots\frac{z_n^{m_n}}{m_n\,!}, \\ & |z_1+\cdots+z_n| < 1; \end{aligned} \tag{2.4}$$

$$\begin{aligned} & F_D^{(n)} := F_D^{(n)}(a;b_1,\ldots,b_n;c;z_1,\ldots,z_n) \\ & = \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{(a)_{m_1+\cdots+m_n}(b_1)_{m_1}\ldots(b_n)_{m_n}}{(c)_{m_1+\cdots+m_n}} \frac{z_1^{m_1}}{m_1\,!}\cdots\frac{z_n^{m_n}}{m_n!}, \\ & |z_i| < 1,\ i = 1,\ldots,n; \end{aligned}$$

and $F_A^{(n)}(a+i)$ stands for $F_A^{(n)}$ in (2.4) with a replaced by $(a+i)$, *etc.*.

The integral representations of Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$ are given by [3]:

$$\begin{aligned} & F_A^{(n)}[\alpha;\beta_1,\ldots,\beta_n;\gamma_1,\ldots,\gamma_n;x_1,\ldots,x_n] \\ & = \prod_{i=1}^{n} \frac{\Gamma(\gamma_i)}{\Gamma(\beta_i)\Gamma(\gamma_i-\beta_i)} \underbrace{\int_0^1 \cdots \int_0^1}_{n-fold} \prod_{i=1}^{n} u_i^{\beta_i-1}(1-u_i)^{\gamma_i-\beta_i-1} \\ & \quad \times (1-\sum_{i=1}^{n} u_i\, x_i)^{-\alpha} du_1 \ldots du_n, \end{aligned}$$

$Re\,\gamma_i > Re\,\beta_i > 0, \quad i = 1,\ldots,n, \quad \sum_{i=1}^{n}|x_i| < 1;$

$$F_D^{(n)}[\alpha;\beta_1,\ldots,\beta_n;\gamma;x_1,\ldots,x_n]$$

$$= \prod_{i=1}^{n} \frac{\Gamma(\gamma)}{\Gamma(\beta_i)\,\Gamma(\gamma-\beta_i)} \underbrace{\int \cdots \int}_{n-fold} \prod_{i=1}^{n} u_i^{\beta_i - 1} (1-\sum_{i=1}^{n} u_i)^{\gamma - \sum_{i=1}^{n}\beta_i - 1}$$

$$\times (1-\sum_{i=1}^{n} u_i\, x_i)^{-\alpha} du_1 \ldots du_n,$$

$Re\,\beta_i > 0,\quad i = 1,\ldots,n;\ Re\,(\gamma - \sum_{i=1}^{n}\beta_i) > 0,\ |x_i| < 1,\ i = 1,\ldots,n;$ and the path of integration is $u_i \geq 0,\ i = 1,\ldots,n;\ \sum_{i=1}^{n} u_i \leq 1$.

3. $(n+1)$-Variable Models

Theorem 2. *Let $u = \phi(z)$ and $z = t\,(1-\sum_{i=1}^{n} z_i)$ then $z\frac{du}{dz} = t\frac{\partial u}{\partial t}$ and $\frac{du}{dz} = \frac{\partial u}{\partial t} - t^{-1}\sum_{i=1}^{n} z_i \frac{\partial u}{\partial z_i}$.*

Proof. We have $z = t\,(1-\sum_{i=1}^{n} z_i)$, which leads to

$$\frac{\partial u}{\partial t} = (1-\sum_{i=1}^{n} z_i)\,\frac{du}{dz},$$

$$\frac{\partial u}{\partial z_i} = -t\,\frac{du}{dz},\quad i = 1,\ldots,n.$$

From this follows the conclusion of the theorem. □

Directed by Theorems 1 and 2, we give below $(n+1)$-variable models of the representations $R'(0,m_0,\mu)$ and $\uparrow'_{\omega,\mu}$ of the Lie algebra $\mathscr{K}_5$. Let $\mathscr{F}$ be the space of all analytic and single-valued functions for all $z_i = u_i x_i \neq 1$, $i = 1,2,\ldots,n$ and $t \neq 0$. Multiplier representations of the Lie group K_5 induced by the J-operators are also given.

Model IA:
Representation $R'(0,m_0,\mu)$:

$$J^3 = t\,\frac{\partial}{\partial t},\quad J^+ = \mu t\,(1-\sum_{i=1}^{n} u_i x_i),\quad J^- = t^{-1}\left[t\,\frac{\partial}{\partial t} - \sum_{i=1}^{n} u_i\,\frac{\partial}{\partial u_i}\right],$$

$$E = \mu,\quad Q = \mu t^2 (1-\sum_{i=1}^{n} u_i x_i)^2,\quad f_\lambda(u_1,\ldots,u_n,t) = t^\lambda\,(1-\sum_{i=1}^{n} u_i x_i)^\lambda, \tag{3.1}$$

where $\lambda \in S = \{m_0 + n \mid m_0 \in \mathbb{C} - \{0\},\ 0 \leq Re\,m_0 < 1,\ n = 0,\pm 1,\ldots\}$.

The multiplier representation $T_1(g)f$ of the Lie group K_5 induced by the operators (3.1) on $\mathscr{F}$ is

$$[T_1(g)f](u_1,\ldots,u_n,t) = \exp\left\{\mu\left[qt^2(1-\sum_{i=1}^{n} u_i x_i)^2 + a + bt(1-\sum_{i=1}^{n} u_i x_i)\right]\right\}$$

$$\times f\left[\frac{u_1}{(1+\frac{c}{t})},\ldots,\frac{u_n}{(1+\frac{c}{t})}, t\,e^{\tau}\left(1+\frac{c}{t}\right)\right], \tag{3.2}$$

where $\left|\frac{c}{t}\right| < 1$ and $g = g(q,a,b,c,\tau) \in K_5$.

Model IIA:

Representation $\uparrow'_{\omega,\mu}$:

$$J^3 = -\omega + t\,\frac{\partial}{\partial t}, \quad J^+ = \mu t\,(1-\sum_{i=1}^{n} u_i x_i), \quad J^- = t^{-1}\left[t\,\frac{\partial}{\partial t} - \sum_{i=1}^{n} u_i\,\frac{\partial}{\partial u_i}\right],$$

$$E = \mu, \quad Q = \mu t^2(1-\sum_{i=1}^{n} u_i x_i)^2, \quad f_\lambda(u_1,\ldots,u_n,t) = t^{\lambda+\omega}\,(1-\sum_{i=1}^{n} u_i x_i)^{\lambda+\omega}, \tag{3.3}$$

where $\lambda \in S = \{-\omega + n \mid n = 0,1,\ldots\}$.

The multiplier representation $T_2(g)f$ of the Lie group K_5 induced by the operators (3.3) on $\mathscr{F}$ is

$$[T_2(g)f](u_1,\ldots,u_n,t) = \exp\left\{\mu\left[qt^2(1-\sum_{i=1}^{n} u_i x_i)^2 + a + bt(1-\sum_{i=1}^{n} u_i x_i)\right] - \omega\tau\right\}$$
$$\times f\left[\frac{u_1}{(1+\frac{c}{t})},\ldots,\frac{u_n}{(1+\frac{c}{t})}, t\,e^{\tau}\left(1+\frac{c}{t}\right)\right],$$

where $\left|\frac{c}{t}\right| < 1$ and $g = g(q,a,b,c,\tau) \in K_5$.

4. Transformed $(n+1)$-Variable Models of $\mathscr{K}_5$

To obtain models in terms of difference-differential operators with the basis functions appearing as Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$ respectively, we utilize a theorem from Govil and Manocha [3]:

Theorem 3. *Let ρ be an irreducible representation of the Lie algebra $\mathscr{K}_5$ in terms of operators $\{J^3, J^\pm, E, Q\}$ on a representation space V with basis functions $\{f_\lambda \mid \lambda \in S\}$. Then the transformation I induces another irreducible representation σ of $\mathscr{K}_5$ on the representation space $W = IV$ having basis functions $\{h_\lambda \mid \lambda \in S\}$ in terms of operators $\{K^3, K^\pm, E', Q'\}$, where*

$$K^3 = IJ^3I^{-1}, \quad K^\pm = IJ^\pm I^{-1}, \quad E' = IEI^{-1}, \quad Q' = IQI^{-1},$$
$$h_\lambda = If_\lambda, \quad \lambda \in S.$$

That is, ρ and σ are isomorphic. Indeed, the behavior of the commutation relations satisfied by K-operators is same as that of J-operators.

We give below the transforms of Models IA and IIA discussed above. The new models are in terms of difference-differential operators.

Model IB*(i)*:

$$K^3 = t\,\frac{\partial}{\partial t}, \quad K^+ = \mu t\left(1 - \sum_{i=1}^{n} \frac{\beta_i x_i}{\gamma_i} E_{\beta i \gamma i}\right), \quad K^- = t^{-1}\left(t\,\frac{\partial}{\partial t} - \sum_{i=1}^{n} \beta_i \Delta_{\beta_i}\right),$$

$$E' = \mu, \quad Q' = \mu t^2\left(1 - \sum_{i=1}^{n} \frac{\beta_i xi}{\gamma_i} E_{\beta i \gamma_i}\right)^2,$$

$$h_\lambda(x_1,\ldots,x_n,t) = F_A^{(n)}(-\lambda, \beta_1,\ldots,\beta_n; \gamma_1,\ldots,\gamma_n; x_1,\ldots,x_n)\,t^\lambda,$$

where $\lambda \in S = \{m_0 + n \mid m_0 \in \mathbb{C} - \{0\},\, 0 \le Re\, m_0 < 1,\, n = 0, \pm 1, \ldots\}$.

Model IB*(ii)*:

$$K^3 = t\,\frac{\partial}{\partial t}, \quad K^+ = \mu t\left(1 - \sum_{i=1}^{n} \frac{\beta_i x_i}{\gamma} E_{\beta i \gamma}\right), \quad K^- = t^{-1}\left(t\,\frac{\partial}{\partial t} - \sum_{i=1}^{n} \beta_i \Delta_{\beta_i}\right),$$

$$E' = \mu, \quad Q' = \mu t^2\left(1 - \sum_{i=1}^{n} \frac{\beta_i xi}{\gamma} E_{\beta i \gamma}\right)^2,$$

$$h_\lambda(x_1,\ldots,x_n,t) = F_D^{(n)}(-\lambda, \beta_1,\ldots,\beta_n; \gamma; x_1,\ldots,x_n)\,t^\lambda,$$

where $\lambda \in S = \{m_0 + n \mid m_0 \in \mathbb{C} - \{0\},\, 0 \le Re\, m_0 < 1,\, n = 0, \pm 1, \ldots\}$.

Model IIB*(i)*:

$$K^3 = -\omega + t\,\frac{\partial}{\partial t}, \quad K^+ = \mu t\left(1 - \sum_{i=1}^{n} \frac{\beta_i x_i}{\gamma_i} E_{\beta i \gamma i}\right), \quad K^- = t^{-1}\left(t\,\frac{\partial}{\partial t} - \sum_{i=1}^{n} \beta_i \Delta_{\beta_i}\right),$$

$$E' = \mu, \quad Q' = \mu t^2\left(1 - \sum_{i=1}^{n} \frac{\beta_i xi}{\gamma_i} E_{\beta i \gamma_i}\right)^2,$$

$$h_\lambda(x_1,\ldots,x_n,t) = F_A^{(n)}(-\lambda - \omega, \beta_1,\ldots,\beta_n; \gamma_1,\ldots,\gamma_n; x_1,\ldots,x_n)\,t^{\lambda+\omega},$$

where $\lambda \in S = \{-\omega + n \mid n = 0, 1, \ldots\}$.

Model IIB*(ii)*:

$$K^3 = -\omega + t\,\frac{\partial}{\partial t}, \quad K^+ = \mu t\left(1 - \sum_{i=1}^{n} \frac{\beta_i x_i}{\gamma} E_{\beta i \gamma}\right), \quad K^- = t^{-1}\left(t\,\frac{\partial}{\partial t} - \sum_{i=1}^{n} \beta_i \Delta_{\beta_i}\right),$$

$$E' = \mu, \quad Q' = \mu t^2\left(1 - \sum_{i=1}^{n} \frac{\beta_i xi}{\gamma} E_{\beta i \gamma}\right)^2,$$

$$h_\lambda(x_1,\ldots,x_n,t) = F_D^{(n)}(-\lambda - \omega, \beta_1,\ldots,\beta_n; \gamma; x_1,\ldots,x_n)\,t^{\lambda+\omega},$$

where $\lambda \in S = \{-\omega + n \mid n = 0, 1, \ldots\}$.

The models given above satisfy the following:

$$\left[K^3, K^{\pm}\right] = \pm K^{\pm}, \quad \left[K^3, Q'\right] = 2Q', \quad \left[K^-, K^+\right] = E',$$
$$\left[K^-, Q'\right] = 2K^+, \quad \left[K^+, Q'\right] = 0, \quad \left[K^{\pm}, E'\right] = \left[K^3, E'\right] = \left[Q', E'\right] = 0,$$

and thus lead to a representation of $\mathscr{K}_5$. Also, Model IB*(i)*, Model IB*(ii)* satisfy

$$K^3 f_\lambda = \lambda f_\lambda, \quad K^+ f_\lambda = \mu f_{\lambda+1}, \quad K^- f_\lambda = \lambda f_{\lambda-1},$$
$$E' f_\lambda = \mu f_\lambda, \quad Q' f_\lambda = \mu f_{\lambda+2},$$

and Model IIB*(i)*, Model IIB*(ii)* satisfy

$$K^3 f_\lambda = \lambda f_\lambda, \quad K^+ f_\lambda = \mu f_{\lambda+1}, \quad K^- f_\lambda = (\lambda+\omega) f_{\lambda-1},$$
$$E' f_\lambda = \mu f_\lambda, \quad Q' f_\lambda = \mu f_{\lambda+2}.$$

4.1. *Recurrence relations and generating functions*

We shall be utilizing Models IA and IIA for obtaining generating functions. Transformed Models IB*(i)*, IB*(ii)* and IIB*(i)*, IIB*(ii)* are further exploited for obtaining recurrence relations. To obtain generating functions, we follow the method given in Sahai [12]. We leave details and present the results only as follows:

4.1.1. *Recurrence relations*

We obtain the following two term and three term recurrence relations using raising and lowering operators, respectively of Model IB*(i)*:

$$F_A^{(n)} - \sum_{i=1}^{n} \frac{\beta_i x_i}{\gamma_i} F_A^{(n)}(\beta_i+1, \gamma_i+1) - F_A^{(n)}(-\lambda-1) = 0,$$

$$(\lambda + \sum_{i=1}^{n} \beta_i) F_A^{(n)} - \sum_{i=1}^{n} \beta_i F_A^{(n)}(\beta_i+1) - \lambda F_A^{(n)}(-\lambda+1) = 0,$$

$$F_A^{(n)} - \sum_{i=1}^{n} \frac{2\beta_i x_i}{\gamma_i} F_A^{(n)}(\beta_i+1, \gamma_i+1) + \sum_{j=1}^{n} \frac{\beta_j^2 x_j^2}{\gamma_j^2} F_A^{(n)}(\beta_j+2, \gamma_j+2)$$
$$+ \sum_{1\le i\ne j\le n} \left(\frac{\beta_i x_i}{\gamma_i}\right)\left(\frac{\beta_j x_j}{\gamma_j}\right) F_A^{(n)}(\beta_i+1, \gamma_i+1, \beta_j+1, \gamma_j+1) - F_A^{(n)}(-\lambda-2) = 0,$$

$$F_A^{(n)}(-\lambda-1) - \sum_{i=1}^{n} \frac{\beta_i x_i}{\gamma_i} F_A^{(n)}(\beta_i+1, \gamma_i+1) + \sum_{j=1}^{n} \frac{\beta_j^2 x_j^2}{\gamma_j^2} F_A^{(n)}(\beta_j+2, \gamma_j+2)$$
$$+ \sum_{1\le i\ne j\le n} \left(\frac{\beta_i x_i}{\gamma_i}\right)\left(\frac{\beta_j x_j}{\gamma_j}\right) F_A^{(n)}(\beta_i+1, \gamma_i+1, \beta_j+1, \gamma_j+1) - F_A^{(n)}(-\lambda-2) = 0.$$

Similarly, we obtain the following recurrence relations using Model IB*(ii)*:

$$F_D^{(n)} - \sum_{i=1}^{n} \frac{\beta_i x_i}{\gamma} F_D^{(n)}(\beta_i+1, \gamma+1) - F_D^{(n)}(-\lambda-1) = 0,$$

$$(\lambda + \sum_{i=1}^{n} \beta_i) F_D^{(n)} - \sum_{i=1}^{n} \beta_i F_D^{(n)}(\beta_i + 1) - \lambda F_D^{(n)}(-\lambda + 1) = 0,$$

$$F_D^{(n)} - \sum_{i=1}^{n} \frac{2\beta_i x_i}{\gamma} F_D^{(n)}(\beta_i + 1, \gamma + 1) + \sum_{j=1}^{n} \frac{\beta_j^2 x_j^2}{\gamma^2} F_D^{(n)}(\beta_j + 2, \gamma + 2)$$
$$+ \sum_{1 \le i \ne j \le n} \frac{(\beta_i x_i)(\beta_j x_j)}{\gamma^2} F_D^{(n)}(\beta_i + 1, \beta_j + 1, \gamma_j + 2) - F_D^{(n)}(-\lambda - 2) = 0,$$

$$F_D^{(n)}(-\lambda - 1) - \sum_{i=1}^{n} \frac{\beta_i x_i}{\gamma} F_D^{(n)}(\beta_i + 1, \gamma + 1) + \sum_{j=1}^{n} \frac{\beta_j^2 x_j^2}{\gamma^2} F_D^{(n)}(\beta_j + 2, \gamma + 2)$$
$$+ \sum_{1 \le i \ne j \le n} \frac{(\beta_i x_i)(\beta_j x_j)}{\gamma^2} F_D^{(n)}(\beta_i + 1, \beta_j + 1, \gamma + 2) - F_D^{(n)}(-\lambda - 2) = 0.$$

Similarly, other recurrence relations can be obtained using Models IIB*(i) and IIB*(ii).

4.1.2. *Generating functions*

The matrix elements $A_{lk}(g)$ of $T_1(g)f$ of the Lie group K_5 with respect to the basis functions $\{f_{m_0+k} \mid m_0 \in \mathbb{C} \setminus \{0\}, 0 \le Re\, m_0 < 1, k = 0, \pm 1, \ldots\}$ are defined by:

$$[T_1(g) f_{m_0+k}](u_1, \ldots, u_n, t) = \sum_{l=-\infty}^{\infty} A_{lk}(g) f_{m_0+l}(u_1, \ldots, u_n, t), \qquad (4.1)$$
$$g = g(q, a, b, c, \tau) \in K_5.$$

The special case of (4.1) for the particular case $q = 0$ is obtained by putting an expression for $T_1(g)$ from (3.2) in (4.1) and then computing the matrix elements $A_{lk}(g)$ by comparing the coefficients of t^l on both sides of (4.1). This leads to the following generating function:

$$\exp\left[\mu b t (1 - \sum_{i=1}^{n} u_i x_i)\right] \left(1 - \sum_{i=1}^{n} \frac{u_i x_i}{1 + \frac{c}{t}}\right)^{m_0+k} \left(1 + \frac{c}{t}\right)^{m_0+k} t^k \qquad (4.2)$$
$$= \sum_{l=-\infty}^{\infty} c^{k-l} L_{m_0+l}^{k-l}(-\mu bc)(1 - \sum_{i=1}^{n} u_i x_i)^{m_0+l}\, t^l,$$

where $k = 0, \pm 1, \ldots$ such that $k \ge l$, $|\sum_{i=1}^{n} u_i x_i| < 1$, $\left|\sum_{i=1}^{n} \frac{u_i x_i}{1+\frac{c}{t}}\right| < 1$ and $L_{m_0+l}^{k-l}(-\mu bc)$ are Laguerre polynomials defined by [8, 10]:

$$L_\nu^\alpha(z) = \frac{\Gamma(\nu + \alpha + 1)}{\Gamma(\alpha + 1)\Gamma(\nu + 1)}\, {}_1F_1(-\nu; \alpha + 1; z). \qquad (4.3)$$

Similarly, the following generating function is obtained by putting $a = c = \tau = 0$ in (4.1):

$$\exp\left\{\mu\left[qt^2(1-\sum_{i=1}^{n}u_ix_i)^2+bt(1-\sum_{i=1}^{n}u_ix_i)\right]\right\}(1-\sum_{i=1}^{n}u_ix_i)^k t^k$$
$$=\sum_{l=-\infty}^{\infty}\frac{(-\mu q)^{(l-k)/2}}{(l-k)\,!}H_{l-k}\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right)(1-\sum_{i=1}^{n}u_ix_i)^l t^l, \tag{4.4}$$

where $k = 0, \pm 1, \ldots$ such that $l \geq k$, $|\sum_{i=1}^{n} u_i x_i| < 1$ and $H_{l-k}\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right)$ are Hermite polynomials defined by [8, 10]:

$$\exp(2xy-y^2)=\sum_{l=0}^{\infty}\frac{y^l}{l\,!}H_l(x). \tag{4.5}$$

Another set of generating function is obtained by putting $a = b = \tau = 0$ in (4.1):

$$\exp\left\{\mu qt^2(1-\sum_{i=1}^{n}u_ix_i)^2\right\}\left(1+\frac{c}{t}\right)^{m_0+k}\left(1-\sum_{i=1}^{n}\frac{u_ix_i}{1+\frac{c}{t}}\right)^{m_0+k} t^k$$
$$=\sum_{l=-\infty}^{\infty}c^{k-l}\,\Gamma(m_0+k+1)\sum_{m}\frac{(\mu qc^2)^m}{m\,!(2m+k-l)\,!\,\Gamma(m_0+l-2m+1)}$$
$$\times(1-\sum_{i=1}^{n}u_ix_i)^{m_0+l}t^l, \tag{4.6}$$

where m ranges over all integral values so that summand makes sense, $k = 0, \pm 1, \ldots$, $|\sum_{i=1}^{n} u_i x_i| < 1$ and $\left|\sum_{i=1}^{n} \frac{u_i x_i}{1+\frac{c}{t}}\right| < 1$.

Similarly, the following generating function is obtained by putting $q = a = c = \tau = 0$ in (4.1):

$$\exp\left[\mu bt(1-\sum_{i=1}^{n}u_ix_i)\right](1-\sum_{i=1}^{n}u_ix_i)^k t^k=\sum_{l=-\infty}^{\infty}\frac{(\mu b)^{l-k}}{(l-k)\,!}(1-\sum_{i=1}^{n}u_ix_i)^l t^l, \tag{4.7}$$

where $k = 0, \pm 1, \ldots$ such that $l \geq k$ and $|\sum_{i=1}^{n} u_i x_i| < 1$. We also obtain the following generating function by putting $q = a = b = \tau = 0$ in (4.1):

$$\left(1+\frac{c}{t}\right)^{m_0+k}\left(1-\sum_{i=1}^{n}\frac{u_ix_i}{1+\frac{c}{t}}\right)^{m_0+k} t^k$$
$$=\sum_{l=-\infty}^{\infty}c^{k-l}\binom{m_0+k}{m_0+l}(1-\sum_{i=1}^{n}u_ix_i)^{m_0+l}t^l, \tag{4.8}$$

where $k = 0, \pm 1, \ldots$ such that $k \geq l$, $|\sum_{i=1}^{n} u_i x_i| < 1$ and $\left|\sum_{i=1}^{n} \frac{u_i x_i}{1+\frac{c}{t}}\right| < 1$.

Similarly, the following generating function is obtained by putting $b = c = \tau = 0$ in (4.1):

$$\exp\left[\mu q t^2 (1 - \sum_{i=1}^{n} u_i x_i)^2\right] (1 - \sum_{i=1}^{n} u_i x_i)^k t^k$$
$$= \sum_{l=-\infty}^{\infty} \frac{(\mu q)^{(l-k)/2}}{\Gamma\left(\frac{l-k}{2}+1\right)} (1 - \sum_{i=1}^{n} u_i x_i)^l t^l, \tag{4.9}$$

where $k = 0, \pm 1, \ldots$ such that $l \geq k$ and $|\sum_{i=1}^{n} u_i x_i| < 1$.

Another set of generating function is obtained by putting $a = \tau = 0$ in (4.1):

$$\exp\left\{\mu\left[q t^2 (1 - \sum_{i=1}^{n} u_i x_i)^2 + b t (1 - \sum_{i=1}^{n} u_i x_i)\right]\right\} \left(1 + \frac{c}{t}\right)^{m_0+k}$$
$$\times \left(1 - \sum_{i=1}^{n} \frac{u_i x_i}{1 + \frac{c}{t}}\right)^{m_0+k} t^k$$
$$= \sum_{l=-\infty}^{\infty} c^{k-l} \binom{m_0+k}{m_0+l} \sum_{m=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^m}{m!} \frac{(-m_0 - l)_m}{(k-l+1)_m} H_m\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right)$$
$$\times (1 - \sum_{i=1}^{n} u_i x_i)^{m_0+l} t^l, \tag{4.10}$$

where $k = 0, \pm 1, \ldots$ such that $k \geq l$, $|\sum_{i=1}^{n} u_i x_i| < \left|1 + \frac{c}{t}\right|$, $\left|\frac{c}{t}\right| < 1$ and $|\sum_{i=1}^{n} u_i x_i| < 1$.

The matrix elements $B_{lk}(g)$ of the $T_2(g)f$ of the Lie group K_5 with respect to the basis functions $\left\{f_k = t^k (1 - \sum_{i=1}^{n} u_i x_i)^k \mid k = 0, 1, \ldots\right\}$ are defined by:

$$[T_2(g) f_k](u_1, \ldots, u_n, t) = \sum_{l=0}^{\infty} B_{lk}(g) f_l(u_1, \ldots, u_n, t), \quad g = g(q, a, b, c, \tau) \in K_5. \tag{4.11}$$

We obtain the following generating functions under special cases of (4.11), similar to (4.2)–(4.10). We list a few of them and remaining can be obtained similarly.

$$\exp\left[\mu b t (1 - \sum_{i=1}^{n} u_i x_i)\right] \left(1 - \sum_{i=1}^{n} \frac{u_i x_i}{1 + \frac{c}{t}}\right)^k \left(1 + \frac{c}{t}\right)^k t^k$$
$$= \sum_{l=0}^{\infty} c^{k-l} L_l^{k-l}(-\mu b c) (1 - \sum_{i=1}^{n} u_i x_i)^l t^l,$$

where $k = 0, 1, \ldots$ such that $k \geq l \geq 0$, $|\sum_{i=1}^{n} u_i x_i| < \left|1 + \frac{c}{t}\right|$, $\left|\frac{c}{t}\right| < 1$;

$$\exp\left\{\mu\left[q t^2 (1 - \sum_{i=1}^{n} u_i x_i)^2 + b t (1 - \sum_{i=1}^{n} u_i x_i)\right]\right\} (1 - \sum_{i=1}^{n} u_i x_i)^k t^k$$

$$= \sum_{l=0}^{\infty} \frac{(-\mu q)^{(l-k)/2}}{(l-k)\,!} H_{l-k}\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right)(1-\sum_{i=1}^{n} u_i x_i)^l t^l,$$

where $k=0,1,\ldots$ such that $l \geq k \geq 0$, $|\sum_{i=1}^{n} u_i x_i| < 1$.

5. Addition theorems

The matrix elements $A_{lk}(g)$ of the multiplier representations $T_1(g)f$ of the Lie group K_5 satisfy the addition theorem:

$$A_{lk}(g_1 g_2) = \sum_{j=-\infty}^{\infty} A_{lj}(g_1) A_{jk}(g_2), \quad l,k = 0,\pm 1,\pm 2,\ldots. \tag{5.1}$$

We now enumerate some special cases of (5.1) when $g_1 = g_1(q,a,b,c,\tau)$ and $g_2 = g_2(q',a',b',c',\tau')$ are chosen suitably. For example, if we assign $q=a=b=\tau=q'=a'=c'=\tau'=0$ and $b'=b$ then (5.1) leads to

$$e^{\mu bc} c^{k-l} L_{m_0+l}^{k-l}(-\mu bc) = \sum_{j=k}^{\infty} c^{j-l} \binom{m_0+j}{m_0+l} \frac{(\mu b)^{j-k}}{(j-k)\,!}. \tag{5.2}$$

Similarly, assigning $b=c=\tau=q'=a'=c'=\tau'=0$ and $b'=b$ in (2.1) gives the following addition theorem

$$\frac{(-\mu q)^{(l-k)/2}}{(l-k)\,!} H_{l-k}\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right) = \sum_{j=k}^{\infty} \frac{(\mu q)^{(l-j)/2}}{\Gamma(\frac{l-j}{2}+1)} \frac{(\mu b)^{j-k}}{(j-k)\,!}. \tag{5.3}$$

Another addition theorem can be obtained by putting $a=\tau=q'=a'=c'=\tau'=0$ and $b'=b$. This yields

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{[-(-\mu q)^{1/2}c]^n}{n\,!} \frac{(-m_0-l)_n}{(k-l+1)_n} H_n\left(\frac{\mu b}{(-\mu q)^{1/2}}\right) \\ &= \sum_{j=k}^{\infty} (\mu bc)^{j-k} \binom{m_0+j}{m_0+k} \frac{(k-l)\,!}{(j-l)\,!} \sum_{m=0}^{\infty} \frac{[-(-\mu q)^{1/2}c]^m}{m\,!} \frac{(-m_0-l)_m}{(j-l+1)_m} \\ &\times H_m\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right).\end{aligned} \tag{5.4}$$

Finally, we put $a=\tau=q'=a'=c'=\tau'=0$ and $c'=c$ resulting in the following addition theorem

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{[-2(-\mu q)^{1/2}c]^n}{n\,!} \frac{(-m_0-l)_n}{(k-l+1)_n} 2^{k-l} H_n\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right) \\ &= \sum_{j=l}^{k} \binom{k-l}{j-l} \sum_{m=0}^{\infty} \frac{[-(-\mu q)^{1/2}c]^m}{m\,!} \frac{(-m_0-l)_m}{(j-l+1)_m} H_m\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right).\end{aligned} \tag{5.5}$$

Further, the matrix elements $B_{lk}(g)$ of the multiplier representations $T_2(g)f$ of the Lie group K_5 satisfy the addition theorem:

$$B_{lk}(g_1 g_2) = \sum_{j=0}^{\infty} B_{lj}(g_1) B_{jk}(g_2), \quad l,k = 0,1,2,\ldots. \tag{5.6}$$

Using the same choices for $g_1 = g_1(q,a,b,c,\tau)$ and $g_2 = g_2(q',a',b',c',\tau')$ as in (5.2)–(5.5), we obtain the following addition theorems from (5.6)

$$e^{\mu bc} c^{k-l} L_l^{k-l}(-\mu bc) = \sum_{j=k}^{\infty} c^{j-l} \binom{j}{l} \frac{(\mu b)^{j-k}}{(j-k)!}, \text{ provided } j \geq l \geq 0;$$

$$\frac{(-\mu q)^{n/2}}{n!} H_n\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right) = \sum_{m=0}^{n} \frac{(\mu q)^{(n-m)/2}}{\Gamma(\frac{n-m}{2}+1)} \frac{(\mu b)^m}{m!},$$

$$\sum_{n=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^n}{n!} \frac{(-l)_n}{(k-l+1)_n} H_n\left(\frac{\mu b}{(-\mu q)^{1/2}}\right)$$
$$= \sum_{j=k}^{\infty} (\mu bc)^{j-k} \binom{j}{k} \frac{(k-l)!}{(j-l)!} \sum_{m=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^m}{m!} \frac{(-l)_m}{(j-l+1)_m} H_m\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right),$$

$$\sum_{n=0}^{\infty} \frac{[-2(-\mu q)^{1/2} c]^n}{n!} \frac{(-l)_n}{(k-l+1)_n} 2^{k-l} H_n\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right)$$
$$= \sum_{j=l}^{k} \binom{k-l}{j-l} \sum_{m=0}^{\infty} \frac{[-(-\mu q)^{1/2} c]^m}{m!} \frac{(-l)_m}{(k-l+1)_m} H_m\left(\frac{\mu b}{2(-\mu q)^{1/2}}\right),$$

respectively.

6. Conclusion

We have obtained recurrence relations and generating functions involving basis functions as Lauricella functions $F_A^{(n)}$ and $F_D^{(n)}$ respectively, for $(n+1)$-variable models of $\mathscr{K}_5$. Results for $n=1$ and $n=2$ will lead to the corresponding relations involving basis functions as Gauss hypergeometric function ${}_2F_1$ and Appell functions F_2, F_1 respectively.

References

[1] J. Choi, R. K. Parmar, and H. M. Srivastava, "The incomplete Lauricella functions of several variables and associated properties and formulas." *Kyungpook Math. J.*, vol. 58, no. 1, pp. 19–35, 2018, doi: 10.5666/KMJ.2018.58.1.19.

[2] G. Dattoli, H. M. Srivastava, and S. Khan, "Operational versus Lie-algebraic methods and the theory of multi-variable Hermite polynomials." *Integral Transforms Spec. Funct.*, vol. 16, no. 1, pp. 81–91, 2005, doi: 10.1080/10652460412331270616.

[3] N. K. Govil and H. L. Manocha, "The Lie theory of hypergeometric functions arising from $(1-\sum_{i=1}^{k} z_i)^{\lambda}(1-\sum_{j=k+1}^{r} z_j)^{-\lambda}$." *J. Math. Anal. Appl*, vol. 171, no. 1, pp. 171–204, 1992, doi: 10.1016/0022-247X(92)90384-P.

[4] A. Hasanov and H. M. Srivastava, "Decomposition formulas associated with the Lauricella multivariable hypergeometric functions." *Comput. Math. Appl.*, vol. 53, no. 7, pp. 1119–1128, 2007, doi: 10.1016/j.camwa.2006.07.007.

[5] S. J. Liu, S. D. Lu, and H. M. Srivastava, "Linearization of the products of the generalized Lauricella polynomials and the multivariate Laguerre polynomials via their integral representations." *Studia Sci. Math. Hungar.*, vol. 50, no. 3, pp. 373–391, 2013, doi: 10.1556/SScMath.50.2013.3.1248.

[6] S. Khan and M. Ali, "Lie algebra $\mathscr{K}_5$ and 3-variable Laguerre-Hermite polynomials." *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, vol. 113, no. 2, pp. 831–843, 2019, doi: 10.1007/s13398-018-0509-9.

[7] H. L. Manocha, "Lie algebras of difference-differential operators and Appell functions F_1," *J. Math. Anal. Appl.*, vol. 138, no. 2, pp. 491–510, 1989, doi: 10.1016/0022-247X(89)90305-3.

[8] W. Miller, Jr., *Lie theory and special functions.* New York: Academic Press, 1968.

[9] M. A. Pathan, S. Khan, and G. Yasmin, "Lie theory and two variables generalized Hermite polynomials," *Ital. J. Pure Appl. Math.*, no. 14, pp. 129–136, 2003.

[10] E. D. Rainville, *Special functions*. New York: Chelsea Publishing Co., 1971.

[11] V. Sahai, "Lie theory of polynomials of the form F_D via Mellin transformation." *Indian J. Pure Appl. Math.*, vol. 22, no. 11, pp. 927–941, 1991.

[12] V. Sahai, "On certain models of irreducible representations of $\mathrm{sl}(2,\mathbf{C})$ and Euler integral transformation," *Indian J. Pure Appl. Math.*, no. 12, pp. 1177–1189, 1995.

[13] V. Sahai and S. Srivastava, "On models of Lie algebra $\mathscr{T}_3$ and Euler integral transformation," *J. Indian Math. Soc. (N.S.)*, vol. 69, no. 1-4, pp. 185–192, 2002.

[14] H. M. Srivastava, "A survey of some recent developments on higher transcendental functions of analytic number theory and applied mathematics." *Symmetry*, vol. 13, pp. 1–22, 2021, doi: 10.3390/sym13122294.

[15] H. M. Srivastava, R. K. Parmar, and M. M. Joshi, "Extended Lauricella and Appell functions and their associated properties." *Adv. Stud. Contemp. Math.*, vol. 58, no. 2, pp. 151–165, 2015, doi: 10.17777/ascm.2015.25.2.151.

[16] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian hypergeometric series*. New York: Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], 1985.

[17] H. M. Srivastava and H. L. Manocha, *A treatise on generating functions*. New York: Ellis Horwood Ltd., Chichester; Halsted Press [John Wiley & Sons, Inc.], 1984.

[18] H. M. Srivastava, G. Yasmin, and A. Muhy, "Lie algebra representations and 2-index 4-variable 1-parameter Hermite polynomials." *Note Mat.*, vol. 39, no. 1, pp. 65–87, 2019, doi: 10.1285/i15900932v39n1p65.

[19] N. J. Vilenkin, *Special functions and the theory of group representations.* Providence, R. I.: American Mathematical Society,, 1968.

[20] S. Yadav and G. Rani, "Certain models of the Lie algebra $\mathscr{K}_5$ and their connection with special functions." *Kyungpook Math. J.*, vol. 58, no. 4, pp. 615–625, 2018, doi: 10.1080/09720510.2018.1466967.

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