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COMPARISON OF BLUPS UNDER MULTIPLE PARTITIONED LINEAR MODEL AND ITS CORRECTLY-REDUCED MODELS

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Abstract. In this study, we investigate the relations between a multiple partitioned linear model and its correctly-reduced models. We consider the comparison problem of covariance matrices of the best linear unbiased predictors (BLUPs) of all unknown vectors including partial parameters under these models by using the block matrix rank and inertia formulas. We derive various inequalities and equalities for covariance matrices of BLUPs under some general assumptions. Also, results for special cases are given.

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1. INTRODUCTION

In the present paper, we consider the comparison problem of covariance matrices of predictors in the context of a multiple partitioned linear model and its reduced models by using the inertia and rank formulas together with elementary block matrix operations. Before proceeding, we introduce the notations used in this paper. $\mathbb{R}^{m\times n}$ stands for the set of all $m\times n$ real matrices. $\mathbf{A}'$, $r(\mathbf{A})$, $\mathscr{C}(\mathbf{A})$ and $\mathbf{A}^+$ denote the transpose, the rank, the column space, and the Moore–Penrose generalized inverse of $\mathbf{A}\in\mathbb{R}^{m\times n}$, respectively. $\mathbf{I}_m$ denotes the identity matrix of order m. $\mathbf{E}_\mathbf{A}=\mathbf{A}^\perp=\mathbf{I}_m-\mathbf{A}\mathbf{A}^+$ stands for the orthogonal projector. $i_+(\mathbf{A})$ and $i_-(\mathbf{A})$ denote the positive and the negative inertias of symmetric matrix $\mathbf{A}$, respectively, and for both $i_\pm(\mathbf{A})$ and $i_\mp(\mathbf{A})$ are used. The inequality $\mathbf{A}_1-\mathbf{A}_2\prec(\preccurlyeq,\succ,\succcurlyeq)\,\mathbf{0}$ or $\mathbf{A}_1\prec(\preccurlyeq,\succ,\succcurlyeq)\,\mathbf{A}_2$ mean that the difference $\mathbf{A}_1-\mathbf{A}_2$ is negative definite (negative semi-definite, positive definite, positive semi-definite) matrix in the Löwner partial ordering for the symmetric matrices $\mathbf{A}_1$ and $\mathbf{A}_2$ of same size.

Linear regression models are one of the most commonly used tools in statistical theory and their applications to analyze data and to develop new methods. These models may need to be converted to their different forms such as certain partitioned

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forms to meet the requirements of the analysis. In this way, by using various linear transformations in linear partitioned models, reduced linear models of original models can be obtained for making statistical inferences of general parametric functions of partial parameters. In this case, original linear models and their reduced models become competing linear regression models for estimation and prediction problems on all unknown vectors with general parametric functions of partial parameters. In this paper, we consider the following linear regression model with its multiple partitioned form:

$$\begin{gathered}\mathcal{M}: \mathbf{y} = \mathbf{X}\alpha + \varepsilon = \left[\mathbf{X}_1, \ldots, \mathbf{X}_t\right] \left[\alpha_1', \ldots, \alpha_t'\right]' + \varepsilon = \mathbf{X}_1\alpha_1 + \cdots + \mathbf{X}_t\alpha_t + \varepsilon \\ \text{with } \mathrm{E}(\varepsilon) = \mathbf{0} \text{ and } \mathrm{cov}(\varepsilon, \varepsilon) = \mathrm{D}(\varepsilon) = \sigma^2\Sigma,\end{gathered} \tag{1.1}$$

where $\mathbf{y} \in \mathbb{R}^{n\times 1}$ is a vector of observable response variables, $\mathbf{X}_i \in \mathbb{R}^{n\times k_i}$ is a known matrix of arbitrary rank with $\mathbf{X} = \left[\mathbf{X}_1, \ldots, \mathbf{X}_t\right] \in \mathbb{R}^{n\times k}$, $\alpha_i \in \mathbb{R}^{k_i\times 1}$ is a vector of fixed but unknown parameters with $\alpha = \left[\alpha_1', \ldots, \alpha_t'\right]' \in \mathbb{R}^{k\times 1}$, $\varepsilon \in \mathbb{R}^{n\times 1}$ is an unobservable vector of random errors, σ^2 is a positive unknown parameter, and $\Sigma \in \mathbb{R}^{n\times n}$ is a known positive semi-definite matrix of arbitrary rank, $k_1 + \cdots + k_t = k$ and $i = 1, \ldots, t$. The matrix $\mathbf{X}$ in (1.1) can also be written as

$$\mathbf{X} = \widehat{\mathbf{X}}_i + \mathbf{R}_i = \widehat{\mathbf{X}}_1 + \cdots + \widehat{\mathbf{X}}_t,$$

where $\widehat{\mathbf{X}}_i = \left[\mathbf{0}, \ldots, \mathbf{X}_i, \ldots, \mathbf{0}\right]$ and $\mathbf{R}_i = \left[\mathbf{X}_1, \ldots, \mathbf{X}_{i-1}, \mathbf{0}, \mathbf{X}_{i+1}, \ldots, \mathbf{X}_t\right]$, $i = 1, \ldots, t$, and thereby $\mathcal{M}$ can be written as $\mathbf{y} = (\widehat{\mathbf{X}}_i + \mathbf{R}_i)\alpha + \varepsilon = \mathbf{X}_i\alpha_i + \mathbf{R}_i\alpha + \varepsilon$. By pre-multiplying $\mathbf{R}_i^{\perp}$ on the both sides of $\mathcal{M}$,

$$\mathcal{R}_i : \mathbf{R}_i^{\perp}\mathbf{y} = \mathbf{R}_i^{\perp}\mathbf{X}_i\alpha_i + \mathbf{R}_i^{\perp}\varepsilon \tag{1.2}$$

is obtained, $i = 1, \ldots, t$. The model $\mathcal{R}_i$ in (1.2) encompasses one of the partial unknown parameters in the multiple partitioned linear model $\mathcal{M}$. This model is a reduced model of $\mathcal{M}$, which is also known as correctly-reduced models of $\mathcal{M}$; see, e.g., [2, 4] among others.

Consideration of $\mathcal{M}$ and $\mathcal{R}_i$ together is meaningful for obtaining the results separately or simultaneously for making estimation or prediction on joint unknown vectors α_i and ε. Thus, we construct the following vector that consists all unknown vectors with partial parameters in the considered models:

$$\mathbf{r}_i = \mathbf{K}_i\alpha_i + \mathbf{H}\varepsilon = \left[\mathbf{0}, \ldots, \mathbf{K}_i, \ldots, \mathbf{0}\right]\alpha + \mathbf{H}\varepsilon := \widehat{\mathbf{K}}_i\alpha + \mathbf{H}\varepsilon \tag{1.3}$$

in accordance with the partition considered in (1.1) for given matrices $\mathbf{K}_i \in \mathbb{R}^{s\times k_i}$ and $\mathbf{H} \in \mathbb{R}^{s\times n}$, where $\widehat{\mathbf{K}}_i = \left[\mathbf{0}, \ldots, \mathbf{K}_i, \ldots, \mathbf{0}\right]$, $i = 1, \ldots, t$. According to assumptions in (1.1), we can write

$$\begin{gathered}\mathrm{D}(\mathbf{y}) = \sigma^2\Sigma, \ \ \mathrm{D}(\mathbf{R}_i^{\perp}\mathbf{y}) = \sigma^2\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}, \ \ \mathrm{D}(\mathbf{r}_i) = \sigma^2\mathbf{H}\Sigma\mathbf{H}', \\ \mathrm{cov}(\mathbf{r}_i, \mathbf{y}) = \sigma^2\mathbf{H}\Sigma, \quad \text{and} \quad \mathrm{cov}(\mathbf{r}_i, \mathbf{R}_i^{\perp}\mathbf{y}) = \sigma^2\mathbf{H}\Sigma\mathbf{R}_i^{\perp}.\end{gathered} \tag{1.4}$$

The best linear unbiased predictors (BLUPs) defined from the minimum covariance matrix requirement in the Löwner partial ordering are one of the fundamental predictors in statistical analysis. Their covariance matrices are usually used as comparison criteria to determine optimal predictors among other types of unbiased predictors. In this paper, we investigate relations between multiple partitioned linear model $\mathcal{M}$ and its correctly-reduced models $\mathcal{R}_i$, which can be considered as competing models for making statistical inference on partial parameters, by comparing the covariance matrices of BLUPs. We establish variety of inequalities and equalities for covariance matrices of BLUPs of $\mathbf{r}_i$ in (1.3) by considering the general assumptions given in (1.4) under models $\mathcal{M}$ and $\mathcal{R}_i$. Statistical inference problems on BLUPs involve complicated matrix operations. Therefore, we use formulas of inertia and rank for block matrices with elementary block matrix operations while derivation of BLUPs, characterizations of their properties and establish inequalities and equalities for covariance matrices of BLUPs under considered models. These kinds of mathematical tools simplify matrix expressions when we face with heavy mathematical computations with matrix expressions including Moore-Penrose inverses of matrices. For more details on inertias and ranks of symmetric matrices and relations between inertias and Löwner partial ordering of symmetric matrices, see, e.g., [15, 18, 19, 24]. As further reference for comparison of covariance matrix of predictors/estimators by using matrix rank/inertia method, we may mention [6, 22, 23] among others. More related work on prediction/estimation problems under partitioned linear models can be found in; see, e.g., [5, 7–12, 14, 17, 25].

2. Preliminary results on BLUPs

In this section, we review the concepts related the BLUPs of unknown vectors under $\mathcal{M}$ and $\mathcal{R}_i$, $i = 1, \ldots, t$. We give the following definitions regarding many fundamental results and facts about consistency of the models, predictability and estimability of unknown parameters under the models, matrix equations and formulas for BLUPs; see e.g., [1, 3, 16, 21].

Definition 1. The consistency requirements are given as follows:

(1) The model $\mathcal{M}$ is said to be consistent if and only if $\mathbf{y} \in \mathscr{C}\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix}$ holds with probability 1.

(2) $\mathcal{R}_i$ is said to be consistent if and only if $\mathbf{R}_i^{\perp}\mathbf{y} \in \mathscr{C}\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\end{bmatrix}$ holds with probability 1, $i = 1, \ldots, t$.

Note that $\mathcal{R}_i$ is consistent under the assumption of consistency of $\mathcal{M}$. In what follows, we will assume that the considered models are consistent.

Definition 2. The predictability requirements are given as follows for $i = 1, \ldots, t$.

(1) $\mathbf{r}_i$ is predictable under $\mathcal{M} \Leftrightarrow \mathscr{C}(\widehat{\mathbf{K}}_i') \subseteq \mathscr{C}(\mathbf{X}') \Leftrightarrow r\begin{bmatrix}\mathbf{X} \\ \widehat{\mathbf{K}}_i\end{bmatrix} = r(\mathbf{X}) \Leftrightarrow \widehat{\mathbf{K}}_i\alpha$ is estimable under $\mathcal{M}$.

(2) $\widehat{\mathbf{X}}_i\alpha+\varepsilon$ is predictable under $\mathcal{M} \Leftrightarrow \mathscr{C}(\widehat{\mathbf{X}}_i') \subseteq \mathscr{C}(\mathbf{X}') \Leftrightarrow r(\mathbf{X}_i)+r(\mathbf{R}_i)=r(\mathbf{X})$ $\Leftrightarrow \widehat{\mathbf{X}}_i\alpha$ is estimable under $\mathcal{M}$, where $\widehat{\mathbf{X}}_i=\left[\mathbf{0},\ldots,\mathbf{X}_i,\ldots,\mathbf{0}\right]$.

(3) $\mathbf{R}_i^{\perp}\widehat{\mathbf{X}}_i\alpha+\varepsilon$ is always predictable under $\mathcal{M}$, $\mathbf{R}_i^{\perp}\widehat{\mathbf{X}}_i\alpha$ is always estimable under $\mathcal{M}$, and ε is always predictable under $\mathcal{M}$.

(4) $r(\mathbf{R}_i^{\perp}\widehat{\mathbf{X}}_i)=k_i \Leftrightarrow \alpha_i$ is estimable under $\mathcal{M}$.

(5) $\mathbf{r}_i$ is predictable under $\mathcal{R}_i \Leftrightarrow \mathscr{C}(\mathbf{K}_i') \subseteq \mathscr{C}(\mathbf{X}_i'\mathbf{R}_i^{\perp}) \Leftrightarrow r\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i\\ \mathbf{K}_i\end{bmatrix}=r(\mathbf{R}_i^{\perp}\mathbf{X}_i) \Leftrightarrow$ $\mathbf{K}_i\alpha_i$ is estimable under $\mathcal{R}_i$.

(6) $\mathbf{X}_i\alpha_i+\varepsilon$ is predictable under $\mathcal{R}_i \Leftrightarrow \mathscr{C}(\mathbf{X}_i') \subseteq \mathscr{C}(\mathbf{X}_i'\mathbf{R}_i^{\perp}) \Leftrightarrow r(\mathbf{X}_i)=r(\mathbf{R}_i^{\perp}\mathbf{X}_i)$ $\Leftrightarrow \mathbf{X}_i\alpha_i$ is estimable under $\mathcal{R}_i$.

(7) $\mathbf{R}_i^{\perp}\mathbf{X}_i\alpha_i+\varepsilon$ is always predictable under $\mathcal{R}_i$, $\mathbf{R}_i^{\perp}\mathbf{X}_i\alpha_i$ is always estimable under $\mathcal{R}_i$, and ε is always predictable under $\mathcal{R}_i$.

(8) $r(\mathbf{R}_i^{\perp}\mathbf{X}_i)=k_i \Leftrightarrow \alpha_i$ is estimable under $\mathcal{R}_i$.

Note that if $\mathbf{r}_i$ is predictable under $\mathcal{R}_i$, $i=1,\ldots,t$, then it is predictable under $\mathcal{M}$.

Definition 3. The BLUP and the best linear unbiased estimator (BLUE) expressions for models $\mathcal{M}$ and $\mathcal{R}_i$ are given as follows, $i=1,\ldots,t$.

(1) Let $\mathbf{r}_i$ be predictable under $\mathcal{M}$. If there exists $\mathbf{L}_i\mathbf{y}$ such that

$$\mathrm{D}(\mathbf{L}_i\mathbf{y}-\mathbf{r}_i)=\ \min \text{ s.t. } \mathrm{E}(\mathbf{L}_i\mathbf{y}-\mathbf{r}_i)=\mathbf{0}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{L}_i\mathbf{y}$ is defined to be the BLUP of $\mathbf{r}_i$ and is denoted by $\mathbf{L}_i\mathbf{y}=\mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)=\mathrm{BLUP}_{\mathcal{M}}(\widehat{\mathbf{K}}_i\alpha+\mathbf{H}\varepsilon)$. If $\mathbf{H}=\mathbf{0}$ in $\mathbf{r}_i$, $\mathbf{L}_i\mathbf{y}$ corresponds the BLUE of $\widehat{\mathbf{K}}_i\alpha$, denoted by $\mathrm{BLUE}_{\mathcal{M}}(\widehat{\mathbf{K}}_i\alpha)$, under $\mathcal{M}$.

(2) Let $\mathbf{r}_i$ be predictable under $\mathcal{R}_i$. If there exists $\mathbf{G}_i\mathbf{R}_i^{\perp}\mathbf{y}$ such that

$$\mathrm{D}(\mathbf{G}_i\mathbf{R}_i^{\perp}\mathbf{y}-\mathbf{r}_i)=\ \min \text{ s.t. } \mathrm{E}(\mathbf{G}_i\mathbf{R}_i^{\perp}\mathbf{y}-\mathbf{r}_i)=\mathbf{0}$$

holds in the Löwner partial ordering, the linear statistic $\mathbf{G}_i\mathbf{R}_i^{\perp}\mathbf{y}$ is defined to be the BLUP of $\mathbf{r}_i$ and is denoted by $\mathbf{G}_i\mathbf{R}_i^{\perp}\mathbf{y}=\mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)=\mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i+\mathbf{H}\varepsilon)$. If $\mathbf{H}=\mathbf{0}$ in $\mathbf{r}_i$, $\mathbf{G}_i\mathbf{R}_i^{\perp}\mathbf{y}$ corresponds the best linear unbiased estimator (BLUE) of $\mathbf{K}_i\alpha_i$, denoted by $\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)$, under $\mathcal{R}_i$.

The fundamental BLUP equations of $\mathbf{r}_i$ and their covariance matrices under $\mathcal{M}$ and $\mathcal{R}_i$, $i=1,\ldots,t$, are collected in the following theorem.

Theorem 1. *Let $\mathcal{M}$ and $\mathcal{R}_i$ be as given in* (1.1) *and* (1.2)*, respectively, and assume that $\mathbf{r}_i$ is predictable under $\mathcal{R}_i$ (also predictable under $\mathcal{M}$), $i=1,\ldots,t$. Then the following results hold.*

(1) *BLUP of $\mathbf{r}_i$ under $\mathcal{M}$ is*

$$\mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)=\mathbf{L}_i\mathbf{y}=\left(\left[\widehat{\mathbf{K}}_i,\quad \mathbf{H}\Sigma\mathbf{X}^{\perp}\right]\mathbf{W}^{+}+\mathbf{U}\mathbf{W}^{\perp}\right)\mathbf{y}, \tag{2.1}$$

where $\mathbf{U}\in\mathbb{R}^{s\times n}$ is an arbitrary matrix and $\mathbf{W}=\left[\mathbf{X},\quad \Sigma\mathbf{X}^{\perp}\right]$. In particular,

(a) $\mathbf{L}_i$ *is unique* $\Leftrightarrow r\begin{bmatrix}\mathbf{X}, & \Sigma\mathbf{X}^{\perp}\end{bmatrix} = n$.
(b) $\mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)$ *is unique* $\Leftrightarrow \mathcal{M}$ *is consistent.*
(c) $r\begin{bmatrix}\mathbf{X}, & \Sigma\mathbf{X}^{\perp}\end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix} = r\begin{bmatrix}\mathbf{X}, & \mathbf{X}^{\perp}\Sigma\end{bmatrix}$.
(d) $\mathscr{C}\begin{bmatrix}\mathbf{X}, & \Sigma\mathbf{X}^{\perp}\end{bmatrix} = \mathscr{C}\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix} = \mathscr{C}\begin{bmatrix}\mathbf{X}, & \mathbf{X}^{\perp}\Sigma\end{bmatrix}$.
(e) *The following covariance matrix equalities hold.*

$$\mathrm{D}[\mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] = \sigma^2\begin{bmatrix}\widehat{\mathbf{K}}_i, & \mathbf{H}\Sigma\mathbf{X}^{\perp}\end{bmatrix}\mathbf{W}^{+}\Sigma\left(\begin{bmatrix}\widehat{\mathbf{K}}_i, & \mathbf{H}\Sigma\mathbf{X}^{\perp}\end{bmatrix}\mathbf{W}^{+}\right)', \tag{2.2}$$

$$\begin{aligned}\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] = &\sigma^2\left(\begin{bmatrix}\widehat{\mathbf{K}}_i, & \mathbf{H}\Sigma\mathbf{X}^{\perp}\end{bmatrix}\mathbf{W}^{+} - \mathbf{H}\right)\Sigma \\ &\times\left(\begin{bmatrix}\widehat{\mathbf{K}}_i, & \mathbf{H}\Sigma\mathbf{X}^{\perp}\end{bmatrix}\mathbf{W}^{+} - \mathbf{H}\right)'.\end{aligned} \tag{2.3}$$

(2) BLUP *of* $\mathbf{r}_i$ *under* $\mathcal{R}_i$ *is*

$$\begin{aligned}\mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i) &= \mathbf{G}_i\mathbf{R}_i^{\perp}\mathbf{y} \\ &= \left(\begin{bmatrix}\mathbf{K}_i, & \mathbf{H}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\end{bmatrix}\mathbf{W}_i^{+} + \mathbf{U}_i\mathbf{W}_i^{\perp}\right)\mathbf{R}_i^{\perp}\mathbf{y},\end{aligned} \tag{2.4}$$

where $\mathbf{U}_i \in \mathbb{R}^{s\times n}$ *is an arbitrary matrix and* $\mathbf{W}_i = \begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\end{bmatrix}$.
In particular,
(a) $\mathbf{G}_i$ *is unique* $\Leftrightarrow r\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\end{bmatrix} = n$.
(b) $\mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)$ *is unique* $\Leftrightarrow \mathcal{R}_i$ *is consistent.*
(c) $r\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\end{bmatrix} = r\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\end{bmatrix}$
$= r\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & (\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\end{bmatrix}$.
(d) $\mathscr{C}\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\end{bmatrix} = \mathscr{C}\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\end{bmatrix}$
$= \mathscr{C}\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & (\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\end{bmatrix}$.
(e) *The following covariance matrix equalities hold.*

$$\begin{aligned}\mathrm{D}[\mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] = &\sigma^2\begin{bmatrix}\mathbf{K}_i, & \mathbf{H}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\end{bmatrix}\mathbf{W}_i^{+}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} \\ &\times\left(\begin{bmatrix}\mathbf{K}_i, & \mathbf{H}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\end{bmatrix}\mathbf{W}_i^{+}\right)',\end{aligned} \tag{2.5}$$

$$\begin{aligned}\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] = &\sigma^2\left(\begin{bmatrix}\mathbf{K}_i, & \mathbf{H}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\end{bmatrix}\mathbf{W}_i^{+}\mathbf{R}_i^{\perp} - \mathbf{H}\right)\Sigma \\ &\times\left(\begin{bmatrix}\mathbf{K}_i, & \mathbf{H}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\end{bmatrix}\mathbf{W}_i^{+}\mathbf{R}_i^{\perp} - \mathbf{H}\right)'.\end{aligned} \tag{2.6}$$

3. MAIN RESULTS

In this section, some results on the comparison of covariance matrices of predictors under multiple partitioned linear model $\mathcal{M}$ and its reduced models $\mathcal{R}_i$, $i = 1,\ldots,t$, are derived and related conclusions are established for special cases by using block matrices' rank and inertia formulas with elementary block matrix operations.

Theorem 2. *Let $\mathcal{M}$ and $\mathcal{R}_i$ be as given in* (1.1) *and* (1.2)*, respectively, and assume that $\mathbf{r}_i$ is predictable under $\mathcal{R}_i$ (also predictable under $\mathcal{M}$), $i = 1,\ldots,t$. Denote*

$$\mathbf{M} = \begin{bmatrix} \Sigma & \Sigma\mathbf{R}_i^{\perp} & \Sigma\mathbf{H}' & \mathbf{0} & \mathbf{X} \\ \mathbf{R}_i^{\perp}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^{\perp}\mathbf{X}_i & \mathbf{0} \\ \mathbf{H}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{K}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then

$$i_{+}\left(\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] - \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)]\right)$$
$$= i_{+}(\mathbf{M}) - r\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix} - r(\mathbf{R}_i^{\perp}\mathbf{X}_i), \tag{3.1}$$
$$i_{-}\left(\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] - \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)]\right)$$
$$= i_{-}(\mathbf{M}) - r\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\end{bmatrix} - r(\mathbf{X}), \tag{3.2}$$
$$r\left(\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)] - \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)]\right)$$
$$= r(\mathbf{M}) - r\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix} - r(\mathbf{R}_i^{\perp}\mathbf{X}_i) - r\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\end{bmatrix} - r(\mathbf{X}). \tag{3.3}$$

In consequence, the following results hold.

(1) $\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] \succ \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$
$\Leftrightarrow i_{-}(\mathbf{M}) = r\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\end{bmatrix} + r(\mathbf{X}) + s.$

(2) $\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] \prec \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$
$\Leftrightarrow i_{+}(\mathbf{M}) = r\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix} + r(\mathbf{R}_i^{\perp}\mathbf{X}_i) + s.$

(3) $\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] \succcurlyeq \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$
$\Leftrightarrow i_{+}(\mathbf{M}) = r\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix} + r(\mathbf{R}_i^{\perp}\mathbf{X}_i).$

(4) $\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] \preccurlyeq \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$
$\Leftrightarrow i_{-}(\mathbf{M}) = r\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\end{bmatrix} + r(\mathbf{X}).$

(5) $\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{R}_i}(\mathbf{r}_i)] = \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathcal{M}}(\mathbf{r}_i)]$
$\Leftrightarrow r(\mathbf{M}) = r\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix} + r(\mathbf{R}_i^{\perp}\mathbf{X}_i) + r\begin{bmatrix}\mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\end{bmatrix} + r(\mathbf{X}).$

Many consequences can be derived from Theorem 2 for different choices of the matrices $\mathbf{K}_i$ and $\mathbf{H}$ in $\mathbf{r}_i$, $i = 1,\ldots,t$. By setting special choices of the matrices $\mathbf{K}_i$ and $\mathbf{H}$ in this theorem, the block matrix $\mathbf{M}$ in Theorem 2 can be reduced to some simpler forms. Some of them are given below.

Corollary 1. *Let $\mathcal{M}$ and $\mathcal{R}_i$ be as given in* (1.1) *and* (1.2)*, respectively, $i = 1,\ldots,t$.*

(1) *Assume that $\mathbf{K}_i\alpha_i$ is predictable under $\mathcal{R}_i$ (also predictable under $\mathcal{M}$). Denote*

$$\mathbf{M} = \begin{bmatrix} \Sigma & \Sigma\mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{X} \\ \mathbf{R}_i^{\perp}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^{\perp}\mathbf{X}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

Then, the following results hold.

(a) $D[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)] \succ D[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\alpha_i)]$
$\Leftrightarrow i_-(\mathbf{M}) = r\begin{bmatrix}\mathbf{R}_i^\perp\mathbf{X}_i, & \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp\end{bmatrix} + r(\mathbf{X}) + s.$

(b) $D[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)] \prec D[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\alpha_i)]$
$\Leftrightarrow i_+(\mathbf{M}) = r\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix} + r(\mathbf{R}_i^\perp\mathbf{X}_i) + s.$

(c) $D[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)] \succcurlyeq D[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\alpha_i)]$
$\Leftrightarrow i_+(\mathbf{M}) = r\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix} + r(\mathbf{R}_i^\perp\mathbf{X}_i).$

(d) $D[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)] \preccurlyeq D[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\alpha_i)]$
$\Leftrightarrow i_-(\mathbf{M}) = r\begin{bmatrix}\mathbf{R}_i^\perp\mathbf{X}_i, & \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp\end{bmatrix} + r(\mathbf{X}).$

(e) $D[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{K}_i\alpha_i)] = D[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{K}_i\alpha_i)]$
$\Leftrightarrow r(\mathbf{M}) = r\begin{bmatrix}\mathbf{X}, & \Sigma\end{bmatrix} + r(\mathbf{R}_i^\perp\mathbf{X}_i) + r\begin{bmatrix}\mathbf{R}_i^\perp\mathbf{X}_i, & \mathbf{R}_i^\perp\Sigma\mathbf{R}_i^\perp\end{bmatrix} + r(\mathbf{X}).$

(2) *Assume that* $\mathbf{X}_i\alpha_i$ *is predictable under* $\mathcal{R}_i$ *(also predictable under* $\mathcal{M}$*). Then*

$$\begin{aligned} &i_\pm(D[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\alpha_i)] - D[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{X}_i\alpha_i)]) \\ &\quad = r(D[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\alpha_i)] - D[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{X}_i\alpha_i)]) = 0. \end{aligned}$$

(3) $\mathbf{R}_i^\perp\mathbf{X}_i\alpha_i$ *is always predictable under* $\mathcal{R}_i$ *(also always predictable under* $\mathcal{M}$*). Then*

$$\begin{aligned} &i_\pm(D[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{R}_i^\perp\mathbf{X}_i\alpha_i)] - D[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{R}_i^\perp\mathbf{X}_i\alpha_i)]) \\ &\quad = r(D[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{R}_i^\perp\mathbf{X}_i\alpha_i)] - D[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{R}_i^\perp\mathbf{X}_i\alpha_i)]) = 0. \end{aligned}$$

4. Concluding Remarks

In this study, we consider comparison problems of predictors under a multiple partitioned linear model and its correctly-reduced models. We present inertia and rank relations between covariance matrices of BLUPs of unknown vectors under considered models by using various inertia and rank formulas of block matrices with elementary matrix operations. In order to establish the general results on the predictors, we consider the general linear function of all unknown vectors under general assumptions. The results obtained in this paper can present useful aspects for determining relation between multiple partitioned linear model and its correctly-reduced models regarding statistical inference on partial parameters by comparing performances of BLUPs/BLUEs under the considered models.

Appendix

We collect some fundamental results of block matrices in the following lemmas used in the proofs of the main results in the paper.

Lemma 1 ([18])**.** *Let* $\mathbf{A}_1$, $\mathbf{A}_2 \in \mathbb{R}^{m\times n}$*, or, let* $\mathbf{A}_1 = \mathbf{A}_1'$, $\mathbf{A}_2 = \mathbf{A}_2' \in \mathbb{R}^{m\times m}$*. Then,*

(1) $\mathbf{A}_1 = \mathbf{A}_2 \Leftrightarrow r(\mathbf{A}_1 - \mathbf{A}_2) = 0.$

(2) $\mathbf{A}_1 \succ \mathbf{A}_2 \Leftrightarrow i_+(\mathbf{A}_1 - \mathbf{A}_2) = m$ *and* $\mathbf{A}_1 \prec \mathbf{A}_2 \Leftrightarrow i_-(\mathbf{A}_1 - \mathbf{A}_2) = m.$

(3) $\mathbf{A}_1 \succcurlyeq \mathbf{A}_2 \Leftrightarrow i_-(\mathbf{A}_1 - \mathbf{A}_2) = 0$ *and* $\mathbf{A}_1 \preccurlyeq \mathbf{A}_2 \Leftrightarrow i_+(\mathbf{A}_1 - \mathbf{A}_2) = 0.$

Lemma 2 ([18]). *Let* $\mathbf{A}_1 = \mathbf{A}_1' \in \mathbb{R}^{m\times m}$, $\mathbf{A}_2 = \mathbf{A}_2' \in \mathbb{R}^{n\times n}$, $\mathbf{Q} \in \mathbb{R}^{m\times n}$, *and* $k \in \mathbb{R}$. *Then,*

$$r(\mathbf{A}_1) = i_+(\mathbf{A}_1) + i_-(\mathbf{A}_1). \tag{4.1}$$

$$i_\pm(k\mathbf{A}_1) = \begin{cases} i_\pm(\mathbf{A}_1) & if \;\; k > 0 \\ i_\mp(\mathbf{A}_1) & if \;\; k < 0 \end{cases}. \tag{4.2}$$

$$i_\pm \begin{bmatrix} \mathbf{A}_1 & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{A}_2 \end{bmatrix} = i_\pm \begin{bmatrix} \mathbf{A}_1 & -\mathbf{Q} \\ -\mathbf{Q}' & \mathbf{A}_2 \end{bmatrix} = i_\mp \begin{bmatrix} -\mathbf{A}_1 & \mathbf{Q} \\ \mathbf{Q}' & -\mathbf{A}_2 \end{bmatrix}. \tag{4.3}$$

$$i_\pm \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} = i_\pm(\mathbf{A}_1) + i_\pm(\mathbf{A}_2), \quad i_+ \begin{bmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{0} \end{bmatrix} = i_- \begin{bmatrix} \mathbf{0} & \mathbf{Q} \\ \mathbf{Q}' & \mathbf{0} \end{bmatrix} = r(\mathbf{Q}). \tag{4.4}$$

Lemma 3 ([18]). *Let* $\mathbf{A}_1 = \mathbf{A}_1' \in \mathbb{R}^{m\times m}$, $\mathbf{B} = \mathbf{B}' \in \mathbb{R}^{n\times n}$, *and* $\mathbf{A}_2 \in \mathbb{R}^{m\times n}$. *Then,*

$$i_\pm \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{0} \end{bmatrix} = r(\mathbf{A}_2) + i_\pm(\mathbf{E}_{\mathbf{A}_2}\mathbf{A}_1\mathbf{E}_{\mathbf{A}_2}). \tag{4.5}$$

$$i_+ \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{0} \end{bmatrix} = r\begin{bmatrix} \mathbf{A}_1, & \mathbf{A}_2 \end{bmatrix} \;\; and \;\; i_- \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{0} \end{bmatrix} = r(\mathbf{A}_2) \;\; if \;\; \mathbf{A}_1 \succcurlyeq \mathbf{0}. \tag{4.6}$$

$$i_\pm \begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 \\ \mathbf{A}_2' & \mathbf{B} \end{bmatrix} = i_\pm(\mathbf{A}_1) + i_\pm(\mathbf{B} - \mathbf{A}_2'\mathbf{A}_1^+\mathbf{A}_2) \;\; if \;\; \mathscr{C}(\mathbf{A}_2) \subseteq \mathscr{C}(\mathbf{A}_1). \tag{4.7}$$

Lemma 4 ([13]). *The linear matrix equation* $\mathbf{AX} = \mathbf{B}$ *is consistent if and only if* $r\begin{bmatrix} \mathbf{A}, & \mathbf{B} \end{bmatrix} = r(\mathbf{A})$, *or equivalently,* $\mathbf{AA}^+\mathbf{B} = \mathbf{B}$. *In this case, the general solution of* $\mathbf{AX} = \mathbf{B}$ *can be written in the following form* $\mathbf{X} = \mathbf{A}^+\mathbf{B} + (\mathbf{I} - \mathbf{A}^+\mathbf{A})\mathbf{U}$, *where* $\mathbf{U}$ *is an arbitrary matrix.*

Constrained quadratic matrix-valued function optimization problem related to minimization problems on covariance matrices of predictors is given in the following lemma.

Lemma 5 ([19]). *Let* $\mathbf{A} \in \mathbb{R}^{n\times p}$, $\mathbf{B} \in \mathbb{R}^{m\times p}$ *be given matrices, and* $\mathbf{P} \in \mathbb{R}^{n\times n}$ *symmetric positive semi-definite matrix. Also assume that there exists* $\mathbf{X}_0 \in \mathbb{R}^{m\times n}$ *such that* $\mathbf{X}_0\mathbf{A} = \mathbf{B}$. *Then the maximal positive inertia of* $\mathbf{X}_0\mathbf{P}\mathbf{X}_0' - \mathbf{X}\mathbf{P}\mathbf{X}'$ *subject to all solutions of* $\mathbf{XA} = \mathbf{B}$ *is*

$$\max_{\mathbf{XA}=\mathbf{B}} i_+(\mathbf{X}_0\mathbf{P}\mathbf{X}_0' - \mathbf{X}\mathbf{P}\mathbf{X}') = r\begin{bmatrix} \mathbf{X}_0\mathbf{P} \\ \mathbf{A}' \end{bmatrix} - r(\mathbf{A}) = r(\mathbf{X}_0\mathbf{P}\mathbf{A}^\perp). \tag{4.8}$$

Hence there exists solution $\mathbf{X}_0$ *of* $\mathbf{X}_0\mathbf{A} = \mathbf{B}$ *such that* $\mathbf{X}_0\mathbf{P}\mathbf{X}_0' \preccurlyeq \mathbf{X}\mathbf{P}\mathbf{X}'$ *holds for all solutions of* $\mathbf{XA} = \mathbf{B}$ *if and only if* $\mathbf{X}_0$ *satisfies both* $\mathbf{X}_0\mathbf{A} = \mathbf{B}$ *and* $\mathbf{X}_0\mathbf{P}\mathbf{A}^\perp = \mathbf{0}$.

Characterization of fundamental BLUP properties can be found in the statistical literature. The similar approach used in proof of Theorem 1 given below was used for transformed linear mixed models in [6], for the different approaches; see, e.g., [15, 21].

Proof of Theorem 1. Let $\mathbf{L}_i\mathbf{y}$ and $\mathbf{T}_i\mathbf{y}$ be two unbiased linear predictors for $\mathbf{r}_i$, $i = 1,\ldots,t$. Then, according to Definition 3, we can write the following expressions

$$\mathrm{E}(\mathbf{L}_i\mathbf{y}-\mathbf{r}_i)=\mathbf{0} \Leftrightarrow \mathbf{L}_i\mathbf{X}=\widehat{\mathbf{K}}_i, \text{ i.e., } \begin{bmatrix}\mathbf{L}_i, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{X}\\ \widehat{\mathbf{K}}_i\end{bmatrix}=\mathbf{0}, \tag{4.9}$$

$$\mathrm{D}(\mathbf{L}_i\mathbf{y}-\mathbf{r}_i)=\sigma^2(\mathbf{L}_i-\mathbf{H})\Sigma(\mathbf{L}_i-\mathbf{H})'=\sigma^2\begin{bmatrix}\mathbf{L}_i, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\mathbf{L}_i, & -\mathbf{I}_s\end{bmatrix}',$$

$$\mathrm{D}(\mathbf{T}_i\mathbf{y}-\mathbf{r}_i)=\sigma^2(\mathbf{T}_i-\mathbf{H})\Sigma(\mathbf{T}_i-\mathbf{H})'=\sigma^2\begin{bmatrix}\mathbf{T}_i, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\mathbf{T}_i, & -\mathbf{I}_s\end{bmatrix}'.$$

Then the matrix minimization problem for finding the BLUP under $\mathscr{M}$ characterized in Definition 3 can be accordingly expressed as to find solution $\mathbf{L}_i$ of the consistent linear matrix equation $\mathbf{L}_i\mathbf{X}=\widehat{\mathbf{K}}_i$ such that $\mathrm{D}(\mathbf{L}_i\mathbf{y}-\mathbf{r}_i)\preccurlyeq\mathrm{D}(\mathbf{T}_i\mathbf{y}-\mathbf{r}_i)$ s.t. $\mathbf{T}_i\mathbf{X}=\widehat{\mathbf{K}}_i$, i.e.,

$$\begin{aligned}&\begin{bmatrix}\mathbf{L}_i, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\mathbf{L}_i, & -\mathbf{I}_s\end{bmatrix}'\preccurlyeq\begin{bmatrix}\mathbf{T}_i, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\mathbf{T}_i, & -\mathbf{I}_s\end{bmatrix}'\\ &\qquad\text{s.t. } \mathbf{T}_i\mathbf{X}=\widehat{\mathbf{K}}_i.\end{aligned} \tag{4.10}$$

(4.10) is a standard constrained quadratic matrix-valued function optimization problem in the Lőwner partial ordering as given in Lemma 5. Applying (4.8) to (4.10), the maximal positive inertia of $\mathrm{D}(\mathbf{L}_i\mathbf{y}-\mathbf{r}_i)-\mathrm{D}(\mathbf{T}_i\mathbf{y}-\mathbf{r}_i)$ subject to $\mathbf{T}_i\mathbf{X}=\widehat{\mathbf{K}}_i$ is obtained as follows:

$$\begin{aligned}&\max_{\mathrm{E}(\mathbf{T}_i\mathbf{y}-\mathbf{r}_i)=\mathbf{0}} i_+(\mathrm{D}(\mathbf{L}_i\mathbf{y}-\mathbf{r}_i)-\mathrm{D}(\mathbf{T}_i\mathbf{y}-\mathbf{r}_i))\\ &=r\begin{bmatrix}\begin{bmatrix}\mathbf{L}_i, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}'\\ \begin{bmatrix}\mathbf{X}\\ \widehat{\mathbf{K}}_i\end{bmatrix}'\end{bmatrix}-r\begin{bmatrix}\mathbf{X}\\ \widehat{\mathbf{K}}_i\end{bmatrix}\\ &=r\left(\begin{bmatrix}\mathbf{L}_i, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\mathbf{X}\\ \widehat{\mathbf{K}}_i\end{bmatrix}^{\perp}\right).\end{aligned} \tag{4.11}$$

Combining (4.9) with (4.11), we conclude that $\mathrm{D}(\mathbf{L}_i\mathbf{y}-\mathbf{r}_i)=\min \Leftrightarrow$ there exists $\mathbf{L}_i$ satisfying both

$$\mathbf{L}_i\mathbf{X}=\widehat{\mathbf{K}}_i \text{ and } \begin{bmatrix}\mathbf{L}_i, & -\mathbf{I}_s\end{bmatrix}\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}\Sigma\begin{bmatrix}\mathbf{I}_n\\ \mathbf{H}\end{bmatrix}'\begin{bmatrix}\mathbf{X}\\ \widehat{\mathbf{K}}_i\end{bmatrix}^{\perp}=\mathbf{0},$$

i.e., $\mathbf{L}_i\mathbf{y}=\mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)\Leftrightarrow\mathbf{L}_i\begin{bmatrix}\mathbf{X}, & \Sigma\mathbf{X}^{\perp}\end{bmatrix}=\begin{bmatrix}\widehat{\mathbf{K}}_i, & \mathbf{H}\Sigma\mathbf{X}^{\perp}\end{bmatrix}$. This matrix equation is consistent. By using Lemma 4, we obtain (2.1). Results (a) and (b) follow directly from (2.1) and for (c) and (d), we refer [20, Lemma 2.1(a)]. (2.2) is seen from (1.4) and (2.1). We obtain

$$\operatorname{cov}\{\mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i),\mathbf{r}_i\} = \left[\widehat{\mathbf{K}}_i, \quad \mathbf{H}\Sigma\mathbf{X}^{\perp}\right]\left[\mathbf{X}, \quad \Sigma\mathbf{X}^{\perp}\right]^{+}\Sigma\mathbf{H}' \tag{4.12}$$

by using (1.4) and (2.1). (2.3) is seen from (2.2) and (4.12). Thus, the first part of the proof is completed.

The second part of the theorem is obtained in a similar way to the first part. □

Proof of Theorem 2. By using (2.6), and applying (4.7) to the difference between $\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)]$ and $\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{R}_i}(\mathbf{r}_i)]$, we obtain

$$\begin{aligned}
& i_{\pm}(\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)] - \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{R}_i}(\mathbf{r}_i)]) \\
&= i_{\pm}\left(\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)] - \left(\mathbf{Z}_i\mathbf{W}_i^{+}\mathbf{R}_i^{\perp} - \mathbf{H}\right)\Sigma\left(\mathbf{Z}_i\mathbf{W}_i^{+}\mathbf{R}_i^{\perp} - \mathbf{H}\right)'\right) \\
&= i_{\pm}\begin{bmatrix} \Sigma & \Sigma\left(\mathbf{Z}_i\mathbf{W}_i^{+}\mathbf{R}_i^{\perp} - \mathbf{H}\right)' \\ \left(\mathbf{Z}_i\mathbf{W}_i^{+}\mathbf{R}_i^{\perp} - \mathbf{H}\right)\Sigma & \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)] \end{bmatrix} - i_{\pm}(\Sigma) \\
&= i_{\pm}\left(\begin{bmatrix} \Sigma & -\Sigma\mathbf{H}' \\ -\mathbf{H}\Sigma & \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)] \end{bmatrix} + \begin{bmatrix} \Sigma\mathbf{R}_i^{\perp} & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_i \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{W}_i \\ \mathbf{W}_i' & \mathbf{0} \end{bmatrix}^{+}\begin{bmatrix} \mathbf{R}_i^{\perp}\Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_i' \end{bmatrix}\right) \\
&\quad - i_{\pm}(\Sigma),
\end{aligned} \tag{4.13}$$

where $\mathbf{Z}_i = \left[\mathbf{K}_i, \quad \mathbf{H}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\right]$ and $\mathbf{W}_i = \left[\mathbf{R}_i^{\perp}\mathbf{X}_i, \quad \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\right]$. We can reapply (4.7) to (4.13) since $\mathscr{C}(\mathbf{R}_i^{\perp}\Sigma) = \mathscr{C}(\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}) \subseteq \mathscr{C}(\mathbf{W}_i)$ and $\mathscr{C}(\mathbf{Z}_i') \subseteq \mathscr{C}(\mathbf{W}_i')$ hold. Then (4.13) is equivalently written as follows by setting the matrices $\mathbf{Z}_i$ and $\mathbf{W}_i$.

$$\begin{aligned}
& i_{\pm}\begin{bmatrix} \mathbf{0} & -\mathbf{R}_i^{\perp}\mathbf{X}_i & -\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp} & \mathbf{R}_i^{\perp}\Sigma & \mathbf{0} \\ -\mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_i' \\ -(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{0} & (\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{H}' \\ \Sigma\mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{0} & \Sigma & -\Sigma\mathbf{H}' \\ \mathbf{0} & \mathbf{K}_i & \mathbf{H}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp} & -\mathbf{H}\Sigma & \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)] \end{bmatrix} \\
&\quad - r\left[\mathbf{R}_i^{\perp}\mathbf{X}_i, \quad \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\right] - i_{\pm}(\Sigma) \\
&= i_{\pm}\begin{bmatrix} -\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} & -\mathbf{R}_i^{\perp}\mathbf{X}_i & -\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp} & \mathbf{R}_i^{\perp}\Sigma\mathbf{H}' \\ -\mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{0} & \mathbf{K}_i' \\ -(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{0} & (\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{H}' \\ \mathbf{H}\Sigma\mathbf{R}_i^{\perp} & \mathbf{K}_i & \mathbf{H}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp} & \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)] - \mathbf{H}\Sigma\mathbf{H}' \end{bmatrix} \\
&\quad - r\left[\mathbf{R}_i^{\perp}\mathbf{X}_i, \quad \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\right] \\
&= i_{\pm}\begin{bmatrix} -\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} & -\mathbf{R}_i^{\perp}\mathbf{X}_i & \mathbf{R}_i^{\perp}\Sigma\mathbf{H}' \\ -\mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{K}_i' \\ \mathbf{H}\Sigma\mathbf{R}_i^{\perp} & \mathbf{K}_i & \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)] - \mathbf{H}\Sigma\mathbf{H}' \end{bmatrix} \\
&\quad + i_{\pm}((\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}) - r\left[\mathbf{R}_i^{\perp}\mathbf{X}_i, \quad \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}\right]
\end{aligned}$$

$$= i_{\mp}\left(\begin{bmatrix} \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} & \mathbf{R}_i^{\perp}\Sigma\mathbf{H}' & \mathbf{R}_i^{\perp}\mathbf{X}_i \\ \mathbf{H}\Sigma\mathbf{R}_i^{\perp} & \mathbf{H}\Sigma\mathbf{H}' & \mathbf{K}_i \\ \mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} - \begin{bmatrix} \mathbf{0} \\ \mathbf{I}_s \\ \mathbf{0} \end{bmatrix} \mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)] \begin{bmatrix} \mathbf{0} & \mathbf{I}_s & \mathbf{0} \end{bmatrix}\right)$$
$$- r\begin{bmatrix} \mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} \end{bmatrix} + i_{\pm}((\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}). \tag{4.14}$$

We can apply (4.7) to (4.14) after setting $\mathrm{D}[\mathbf{r}_i - \mathrm{BLUP}_{\mathscr{M}}(\mathbf{r}_i)]$ in (2.3). Then in a similar way to obtaining (4.13), (4.14) is equivalently written as

$$i_{\mp}\left(\begin{bmatrix} \Sigma & \mathbf{0} & -\Sigma\mathbf{H}' & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} & \mathbf{R}_i^{\perp}\Sigma\mathbf{H}' & \mathbf{R}_i^{\perp}\mathbf{X}_i \\ -\mathbf{H}\Sigma & \mathbf{H}\Sigma\mathbf{R}_i^{\perp} & \mathbf{H}\Sigma\mathbf{H}' & \mathbf{K}_i \\ \mathbf{0} & \mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \Sigma & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \\ \mathbf{0} & [\widehat{\mathbf{K}}_i, \ \mathbf{H}\Sigma\mathbf{X}^{\perp}] \\ \mathbf{0} & \mathbf{0} \end{bmatrix}\right.$$
$$\left.\times \begin{bmatrix} \mathbf{0} & \mathbf{W} \\ \mathbf{W}' & \mathbf{0} \end{bmatrix}^{+} \begin{bmatrix} \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & [\widehat{\mathbf{K}}_i, \ \mathbf{H}\Sigma\mathbf{X}^{\perp}]' & \mathbf{0} \end{bmatrix}\right)$$
$$+ i_{\pm}((\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}) - r\begin{bmatrix} \mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} \end{bmatrix} - i_{\mp}(\Sigma). \tag{4.15}$$

We can apply (4.7) to (4.15) since $\mathscr{C}([\widehat{\mathbf{K}}_i, \ \mathbf{H}\Sigma\mathbf{X}^{\perp}]') \subseteq \mathscr{C}(\mathbf{W}')$ and $\mathscr{C}(\Sigma) \subseteq \mathscr{C}(\mathbf{W})$, where $\mathbf{W} = [\mathbf{X}, \ \Sigma\mathbf{X}^{\perp}]$. From Lemma 2 and 3, and by using elementary block matrix operations, (4.15) is equivalently written as

$$i_{\mp}\begin{bmatrix} \mathbf{0} & -\mathbf{X} & -\Sigma\mathbf{X}^{\perp} & \Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} \\ -\mathbf{X}^{\perp}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}^{\perp}\Sigma\mathbf{H}' & \mathbf{0} \\ \Sigma & \mathbf{0} & \mathbf{0} & \Sigma & \mathbf{0} & -\Sigma\mathbf{H}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} & \mathbf{R}_i^{\perp}\Sigma\mathbf{H}' & \mathbf{R}_i^{\perp}\mathbf{X}_i \\ \mathbf{0} & \widehat{\mathbf{K}}_i & \mathbf{H}\Sigma\mathbf{X}^{\perp} & -\mathbf{H}\Sigma & \mathbf{H}\Sigma\mathbf{R}_i^{\perp} & \mathbf{H}\Sigma\mathbf{H}' & \mathbf{K}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} \end{bmatrix}$$
$$- r\begin{bmatrix} \mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} \end{bmatrix} + i_{\pm}((\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}) - i_{\mp}(\Sigma) - r\begin{bmatrix} \mathbf{X}, & \Sigma\mathbf{X}^{\perp} \end{bmatrix}$$
$$= i_{\mp}\begin{bmatrix} -\Sigma & -\mathbf{X} & -\Sigma\mathbf{X}^{\perp} & \mathbf{0} & \Sigma\mathbf{H}' & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} \\ -\mathbf{X}^{\perp}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}^{\perp}\Sigma\mathbf{H}' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} & \mathbf{R}_i^{\perp}\Sigma\mathbf{H}' & \mathbf{R}_i^{\perp}\mathbf{X}_i \\ \mathbf{H}\Sigma & \widehat{\mathbf{K}}_i & \mathbf{H}\Sigma\mathbf{X}^{\perp} & \mathbf{H}\Sigma\mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{K}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} \end{bmatrix}$$
$$- r\begin{bmatrix} \mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} \end{bmatrix} + i_{\pm}((\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}) - r\begin{bmatrix} \mathbf{X}, & \Sigma \end{bmatrix}$$

$$= i_{\mp}\begin{bmatrix} -\Sigma & -\mathbf{X} & \mathbf{0} & \Sigma\mathbf{H}' & \mathbf{0} \\ -\mathbf{X}' & \mathbf{0} & \mathbf{0} & \widehat{\mathbf{K}}_i' & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} & \mathbf{R}_i^{\perp}\Sigma\mathbf{H}' & \mathbf{R}_i^{\perp}\mathbf{X}_i \\ \mathbf{H}\Sigma & \widehat{\mathbf{K}}_i & \mathbf{H}\Sigma\mathbf{R}_i^{\perp} & \mathbf{0} & \mathbf{K}_i \\ \mathbf{0} & \mathbf{0} & \mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} \end{bmatrix} - r\begin{bmatrix} \mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} \end{bmatrix}$$

$$+ i_{\pm}((\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}) - r\begin{bmatrix} \mathbf{X}, & \Sigma \end{bmatrix} + i_{\mp}(\mathbf{X}^{\perp}\Sigma\mathbf{X}^{\perp})$$

$$= i_{\pm}\begin{bmatrix} \Sigma & \Sigma\mathbf{R}_i^{\perp} & \Sigma\mathbf{H}' & \mathbf{0} & \mathbf{X} \\ \mathbf{R}_i^{\perp}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{R}_i^{\perp}\mathbf{X}_i & \mathbf{0} \\ \mathbf{H}\Sigma & \mathbf{0} & \mathbf{0} & \mathbf{K}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{X}_i'\mathbf{R}_i^{\perp} & \mathbf{K}_i' & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} - r\begin{bmatrix} \mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} \end{bmatrix}$$

$$+ i_{\pm}((\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}\mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp}(\mathbf{R}_i^{\perp}\mathbf{X}_i)^{\perp}) - r\begin{bmatrix} \mathbf{X}, & \Sigma \end{bmatrix} + i_{\mp}(\mathbf{X}^{\perp}\Sigma\mathbf{X}^{\perp}). \tag{4.16}$$

In consequence, by using (4.5) and (4.6), we obtain(3.1) and (3.2). According to (4.1), adding the equalities in (3.1) and (3.2) yields (3.3). Applying Lemma 1 to (3.1)-(3.3) yields (a)-(e). □

Proof of Corollary 1. The first part of the corollary is an immediate consequence of Theorem 2 by setting $\mathbf{H} = \mathbf{0}$ in the matrix $\mathbf{M}$.

For the second part, we set $\mathbf{K}_i = \mathbf{X}_i$ and $\mathbf{H} = \mathbf{0}$ in the matrix $\mathbf{M}$ and then this matrix reduces the simpler form. In this case, we obtain

$$i_{+}(\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\alpha_i)] - \mathrm{D}[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{X}_i\alpha_i)])$$

$$= i_{+}\begin{bmatrix} \Sigma & \Sigma\mathbf{R}_i^{\perp} & \mathbf{X} \\ \mathbf{R}_i^{\perp}\Sigma & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} \end{bmatrix} + r(\mathbf{X}_i) - r\begin{bmatrix} \mathbf{X}, & \Sigma \end{bmatrix} - r(\mathbf{R}_i^{\perp}\mathbf{X}_i)$$

and

$$i_{-}(\mathrm{D}[\mathrm{BLUE}_{\mathcal{M}}(\mathbf{X}_i\alpha_i)] - \mathrm{D}[\mathrm{BLUE}_{\mathcal{R}_i}(\mathbf{X}_i\alpha_i)])$$

$$= i_{-}\begin{bmatrix} \Sigma & \Sigma\mathbf{R}_i^{\perp} & \mathbf{X} \\ \mathbf{R}_i^{\perp}\Sigma & \mathbf{0} & \mathbf{0} \\ \mathbf{X}' & \mathbf{0} & \mathbf{0} \end{bmatrix} + r(\mathbf{X}_i) - r\begin{bmatrix} \mathbf{R}_i^{\perp}\mathbf{X}_i, & \mathbf{R}_i^{\perp}\Sigma\mathbf{R}_i^{\perp} \end{bmatrix} - r(\mathbf{X}).$$

By using (4.6) and the predictability requirement of $\mathbf{X}_i\alpha_i$ under the models given in Definition 2, and in view of well-known rank equality $r\begin{bmatrix} \mathbf{A}, & \mathbf{B} \end{bmatrix} = r(\mathbf{A}) + r(\mathbf{E_A}\mathbf{B})$ (for any conformable $\mathbf{A}$, $\mathbf{B}$), we obtain the required result.

The third part is obtained in a similar way to the second part by setting $\mathbf{K}_i = \mathbf{R}_i^{\perp}\mathbf{X}_i$ and $\mathbf{H} = \mathbf{0}$ in the matrix $\mathbf{M}$ in Theorem 2. □

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