Miskolc Mathematical Notes
Vol. 25 (2024), No. 1, pp. 21–34

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4300

ON SOME NEW AND GENERAL q-HERMITE–HADAMARD TYPE INEQUALITIES FOR CONVEX FUNCTIONS

ZOYA ABDULLAH, AWAIS YOUSAF, CHANON PROMSAKON, AND THANIN SITTHIWIRATTHAM

Received 22 June, 2022

Abstract. In this paper, we establish a new version of Hermite–Hadamard type inequality for convex functions. Moreover, we establish a general version of q-integral identity involving q-differentiable functions to prove some new q-midpoint and q-trapezoidal type inequalities for q-differentiable convex functions. It is also shown that the newly established inequalities can be converted into some existing inequalities within the literature. Finally, we add some mathematical examples to show the validation of newly established inequalities.

2010 *Mathematics Subject Classification:* 26D10; 26D15; 26A51

Keywords: Hermite-Hadamard inequality, midpoint inequalities, trapezoid inequalities, convex functions

1. INTRODUCTION

The Hermite–Hadamard (H–H) inequality, named after Charles Hermite and Jacques Hadamard and commonly known as Hadamard's inequality, says that if a function $f\colon [a,b] \to \mathbb{R}$ is convex, the following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

If f is a concave mapping, the above inequality holds in the opposite direction. There has been much research done in the direction of H–H for different kinds of convexities. For example, in [7, 10], the authors established some inequalities linked with midpoint and trapezoid formulas of numerical integration for convex functions.

On the other hand, in [3, 4], Alp et al. and Bermudo et al. used q-derivatives and integrals to prove two different versions of quantum Hermite–Hadamard (q-H–H) inequalities and some estimates. The q-H–H inequalities are described as:

Theorem 1 ([3,4])**.** *For a convex mapping $f\colon [a,b]\to\mathbb{R}$, the following inequalities hold for $q\in(0,1)$:*

$$f\left(\frac{qa+b}{[2]_q}\right)\le\frac{1}{b-a}I^q_{a+}f(b)\le\frac{qf(a)+f(b)}{[2]_q},\tag{1.2}$$

$$f\left(\frac{a+qb}{[2]_q}\right)\le\frac{1}{b-a}I^q_{b-}f(a)\le\frac{f(a)+qf(b)}{[2]_q}.\tag{1.3}$$

Remark 1. It is very easy to observe that by adding (1.2) and (1.3), we have the following q-H–H inequality for $q\in(0,1)$ (see, [4]):

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{2(b-a)}\left[I^q_{a+}f(b)+I^q_{b-}f(a)\right]\le\frac{f(a)+f(b)}{2}.\tag{1.4}$$

Recently, Ali et al. [1] and Sitthiwirattham et al. [13] used new techniques to prove the following two different and new versions of H–H type inequalities:

Theorem 2 ([1,13])**.** *For a convex mapping $f\colon [a,b]\to\mathbb{R}$, the following inequalities hold for $q\in(0,1)$:*

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\left[I^q_{\frac{a+b}{2}+}f(b)+I^q_{\frac{a+b}{2}-}f(a)\right]\le\frac{f(a)+f(b)}{2},\tag{1.5}$$

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\left[I^q_{a+}f\left(\frac{a+b}{2}\right)+I^q_{b-}f\left(\frac{a+b}{2}\right)\right]\le\frac{f(a)+f(b)}{2}.\tag{1.6}$$

Remark 2. By setting the limit as $q\to1^-$ in (1.2)-(1.6), we recapture the traditional H–H inequality (1.1).

There has been much research done in the direction of q-integral inequalities for different kinds of convexities. For example, in [5], some new midpoint and trapezoidal type inequalities for q-integrals and q-differentiable convex functions were established. The authors of [2, 6] used q-integral and established Simpson's type inequalities for q-differentiable convex and general convex functions. For more recent inequalities in q-calculus, one can consult [8, 11, 15–17].

As a result of these ongoing studies, we establish a new and general version of q-Hermite–Hadamard type inequality for convex functions. Moreover, we establish some new q-midpoint and q-trapezoidal type inequalities for q-differentiable convex functions. We also show that the newly established inequalities are generalizations of the inequalities proved in [4, 13].

2. Preliminaries of q-calculus

We shall recall some basics of quantum calculus in this section. For the sake of brevity, let $q\in(0,1)$ and we use the following notation (see, [9]):

$$[n]_q=\frac{1-q^n}{1-q}=1+q+q^2+...+q^{n-1}.$$

Definition 1 ([16])**.** The left or q_a-derivative of $f\colon [a,b]\to\mathbb{R}$ at $x\in[a,b]$ is expressed as:

$$ {}_aD_q f(x) = \frac{f(x)-f(qx+(1-q)a)}{(1-q)(x-a)},\ x\neq a. \tag{2.1}$$

Definition 2 ([4])**.** The right or q^b-derivative of $f\colon [a,b]\to\mathbb{R}$ at $x\in[a,b]$ is expressed as:

$$ {}^bD_q f(x) = \frac{f(qx+(1-q)b)-f(x)}{(1-q)(b-x)},\ x\neq b.$$

Definition 3 ([16])**.** The left or q_a-integral of $f\colon [a,b]\to\mathbb{R}$ at $x\in[a,b]$ is defined as:

$$I_{a+}^q f(x)=\int_a^x f(t)\ {}_ad_qt = (1-q)(x-a)\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)a).$$

Definition 4 ([3])**.** The right or q^b-integral of $f\colon [a,b]\to\mathbb{R}$ at $x\in[a,b]$ is defined as:

$$I_{b-}^q f(x)=\int_x^b f(t)\ {}^bd_qt = (1-q)(b-x)\sum_{n=0}^{\infty} q^n f(q^n x+(1-q^n)b).$$

Lemma 1 ([12])**.** *For continuous functions $f,g\colon [a,b]\to\mathbb{R}$, the following equality true:*

$$\begin{aligned}&\int_0^c g(t)\ {}^bD_q f(ta+(1-t)b)\,d_qt\\&=\frac{1}{b-a}\int_0^c D_q g(t) f(qta+(1-qt)b)\ {}_0d_qt-\left.\frac{g(t)f(ta+(1-t)b)}{b-a}\right|_0^c.\end{aligned}$$

Lemma 2 ([14])**.** *For continuous functions $f,g\colon [a,b]\to\mathbb{R}$, the following equality true:*

$$\begin{aligned}&\int_0^c g(t)\ {}_aD_q f(tb+(1-t)a)\,d_qt\\&=\left.\frac{g(t)f(tb+(1-t)a)}{b-a}\right|_0^c-\frac{1}{b-a}\int_0^c D_q g(t) f(qtb+(1-qt)a)\ {}_0d_qt.\end{aligned}$$

3. q-HERMITE-HADAMARD INEQUALITY

In this section, we establish a general version of Hermite-Hadamard inequality by using convexity.

Theorem 3. *Let $f\colon [a,b]\to\mathbb{R}$ be a convex mapping, then we have the following inequality for $\lambda\in(0,1]$:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2\lambda(b-a)}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]\leq\frac{f(a)+f(b)}{2}. \tag{3.1}$$

Proof. From convexity, we have

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\left[f(x)+f(y)\right]. \tag{3.2}$$

By setting $x = tb+(1-t)(\lambda a+(1-\lambda)b)$ and $y = ta+(1-t)(\lambda b+(1-\lambda)a)$, we have

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2}\left[f(tb+(1-t)(\lambda a+(1-\lambda)b))+f(ta+(1-t)(\lambda b+(1-\lambda)a))\right]. \tag{3.3}$$

By applying q-integration with respect to t over $[0,1]$, also from Definitions 3 and 4, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leq \frac{1}{2}\left[\begin{array}{c} \int_0^1 f(tb+(1-t)(\lambda a+(1-\lambda)b))\,dt \\ +\int_0^1 f(ta+(1-t)(\lambda b+(1-\lambda)a))\,dt \end{array}\right] \\ &= \frac{1}{2\lambda(b-a)}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]. \end{aligned}$$

Hence the first inequality is proved.

We again use convexity to prove the second inequality as follows:

$$\begin{aligned} &f(tb+(1-t)(\lambda a+(1-\lambda)b))+f(ta+(1-t)(\lambda b+(1-\lambda)a)) \\ &\leq tf(b)+(1-t)f(\lambda a+(1-\lambda)b)+tf(a)+(1-t)f(\lambda b+(1-\lambda)a) \\ &\leq tf(b)+(1-t)(\lambda f(a)+(1-\lambda)f(b))+tf(a)+(1-t)(\lambda f(b)+(1-\lambda)f(a)) \\ &\leq f(a)+f(b). \end{aligned}$$

Thus, we obtain the required inequality by applying q-integration concerning t over $[0,1]$, also from Definitions 3 and 4. □

Remark 3. By setting $\lambda = 1$ in Theorem 3, then we recapture inequality (1.4).

Remark 4. If we set $\lambda = \frac{1}{2}$ in Theorem 3, then we recapture inequality (1.5).

Corollary 1. *If we set $\lambda = \frac{1}{3}$ in Theorem 3, then we have the following new q-Hermite-Hadamard type inequality:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{3}{2(b-a)}\left[I_{\frac{a+2b}{3}+}f(b)+I_{\frac{2a+b}{3}-}f(a)\right] \leq \frac{f(a)+f(b)}{2}.$$

Remark 5. It is worth mentioning here that for different choices of $\lambda \in (0,1]$ in Theorem 3, one can obtain several different and new q-Hermite-Hadamard type inequalities.

4. Parameterized inequalities

In this section, we establish some new q-midpoint and trapezoidal type inequalities for q-differentiable functions.

Let us begin with the following lemma.

Lemma 3. *Let $f\colon [a,b]\subset\mathbb{R}\to\mathbb{R}$ be a left and right q-differentiable function. If the functions ${}_{\lambda a+(1-\lambda)b}D_q f$ and ${}^{\lambda b+(1-\lambda)a}D_q f$ are continuous and integrable over $[a,b]$, then we have the following new equality for $\lambda\in(0,1]$:*

$$\begin{aligned}&\frac{1}{b-a}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]\\&\quad-\lambda\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]\\&=\lambda^2(b-a)\left[\int_0^1(1-qt)\ {}_{\lambda a+(1-\lambda)b}D_q f(tb+(1-t)(\lambda a+(1-\lambda)b))\ {}_0d_qt\right.\\&\quad\left.-\int_0^1(1-qt)\ {}^{\lambda b+(1-\lambda)a}Df(ta+(1-t)(\lambda b+(1-\lambda)a))\ {}_0d_qt\right].\end{aligned}\tag{4.1}$$

Proof. From Lemma 2, we have

$$\begin{aligned}&\int_0^1(1-qt)\ {}_{\lambda a+(1-\lambda)b}D_q f(tb+(1-t)(\lambda a+(1-\lambda)b))\ {}_0d_qt\\&=\frac{(1-q)f(b)}{\lambda(b-a)}-\frac{f(\lambda a+(1-\lambda)b)}{\lambda(b-a)}\\&\quad+\frac{q}{\lambda(b-a)}\int_0^1 f(qtb+(1-qt)(\lambda a+(1-\lambda)b))\\&=\frac{(1-q)f(b)}{\lambda(b-a)}-\frac{f(\lambda a+(1-\lambda)b)}{\lambda(b-a)}\\&\quad+\frac{1}{\lambda(b-a)}\left[(1-q)\sum_{n=0}^{\infty}q^n f(q^nb+(1-q^n)(\lambda a+(1-\lambda)b))-(1-q)f(b)\right]\\&=\frac{1}{\lambda^2(b-a)^2}I_{(\lambda a+(1-\lambda)b)+}f(b)-\frac{f(\lambda a+(1-\lambda)b)}{\lambda(b-a)}.\end{aligned}\tag{4.2}$$

Similarly, from Lemma 1, we have

$$\begin{aligned}&\int_0^1(1-qt)\ {}^{\lambda b+(1-\lambda)a}Df(ta+(1-t)(\lambda b+(1-\lambda)a))\ {}_0d_qt\\&=\frac{f(\lambda b+(1-\lambda)a)}{\lambda(b-a)}-\frac{1}{\lambda^2(b-a)^2}I_{(\lambda b+(1-\lambda)a)-}f(a).\end{aligned}\tag{4.3}$$

Thus, we obtain the required equality (4.1) by subtracting (4.3) from (4.2) and multiplying the resultant one with $\lambda^2(b-a)$. □

Corollary 2. *In Lemma 3, if we set $\lambda=1$, then we obtain the following identity:*

$$\begin{aligned}&\frac{1}{2(b-a)}\left[I_{a+}f(b)+I_{b-}f(a)\right]-\frac{f(a)+f(b)}{2}\\&=\frac{b-a}{2}\left[\int_0^1(1-qt)\ {}_aD_q f(tb+(1-t)a)\ {}_0d_qt\right.\end{aligned}$$

$$-\int_0^1 (1-qt)\ {}^{b}Df\left(ta+(1-t)b\right)\ {}_0d_qt\Bigg].$$

This equality is new and can be used for finding the right estimates of the inequality (1.4) that have not been established earlier.

Remark 6. In Lemma 3, if we set $\lambda = \frac{1}{2}$, then we obtain the following identity:

$$\begin{aligned}&\frac{1}{b-a}\left[I_{\frac{a+b}{2}+}f(b)+I_{\frac{a+b}{2}}f(a)\right]-f\left(\frac{a+b}{2}\right)\\&=\frac{(b-a)}{4}\left[\int_0^1(1-qt)\ {}_{\frac{a+b}{2}}D_qf\left(tb+(1-t)\left(\frac{a+b}{2}\right)\right)\ {}_0d_qt\right.\\&\left.-\int_0^1(1-qt)\ {}^{\frac{a+b}{2}}Df\left(ta+(1-t)\left(\frac{a+b}{2}\right)\right)\ {}_0d_qt\right].\end{aligned}$$

This equality is established by Sitthiwirattham et al. in [13].

Theorem 4. *If the functions* $\left|{}_{\lambda a+(1-\lambda)b}D_qf\right|$ *and* $\left|{}^{\lambda b+(1-\lambda)a}D_qf\right|$ *are convex and Lemma 3 holds, then the following inequality holds:*

$$\begin{aligned}&\Bigg|\frac{1}{b-a}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]\\&\quad-\lambda\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]\Bigg|\\&\le\lambda^2(b-a)\left[\left(\frac{(1-\lambda)q}{[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}_{\lambda a+(1-\lambda)b}D_qf(b)\right|\right.\\&\quad+\frac{\lambda q}{[3]_q}\left|{}_{\lambda a+(1-\lambda)b}D_qf(a)\right|+\left(\frac{(1-\lambda)q}{[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}^{\lambda b+(1-\lambda)a}D_qf(a)\right|\\&\quad\left.+\frac{\lambda q}{[3]_q}\left|{}^{\lambda b+(1-\lambda)a}D_qf(b)\right|\right].\end{aligned}$$

Proof. By taking modulus in (4.1) and from convexity, we have

$$\begin{aligned}&\Bigg|\frac{1}{b-a}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]\\&\quad-\lambda\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]\Bigg|\\&\le\lambda^2(b-a)\left[\int_0^1(1-qt)\left|{}_{\lambda a+(1-\lambda)b}D_qf\left(tb+(1-t)(\lambda a+(1-\lambda)b)\right)\right|\ {}_0d_qt\right.\\&\quad\left.+\int_0^1(1-qt)\left|{}^{\lambda b+(1-\lambda)a}Df\left(ta+(1-t)(\lambda b+(1-\lambda)a)\right)\right|\ {}_0d_qt\right]\\&\le\lambda^2(b-a)\left[\int_0^1(1-qt)\left(t\left|{}_{\lambda a+(1-\lambda)b}D_qf(b)\right|+(1-t)\lambda\left|{}_{\lambda a+(1-\lambda)b}D_qf(a)\right|\right)\ {}_0d_qt\right.\end{aligned}$$

$$+\int_0^1 (1-qt)(1-\lambda)(1-t)\left|{}_{\lambda a+(1-\lambda)b}D_q f(b)\right| {}_0d_qt$$
$$+\int_0^1 (1-qt)\left(t\left|{}^{\lambda b+(1-\lambda)a}Df(a)\right|+(1-t)\lambda\left|{}^{\lambda b+(1-\lambda)a}Df(b)\right|\right) {}_0d_qt$$
$$+\int_0^1 (1-qt)(1-\lambda)(1-t)\left|{}^{\lambda b+(1-\lambda)a}Df(a)\right| {}_0d_qt\Bigg]$$
$$=\lambda^2(b-a)\Bigg[\left(\frac{(1-\lambda)q}{[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}_{\lambda a+(1-\lambda)b}D_q f(b)\right|$$
$$+\frac{\lambda q}{[3]_q}\left|{}_{\lambda a+(1-\lambda)b}D_q f(a)\right|+\left(\frac{(1-\lambda)q}{[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}^{\lambda b+(1-\lambda)a}D_q f(a)\right|$$
$$+\frac{\lambda q}{[3]_q}\left|{}^{\lambda b+(1-\lambda)a}D_q f(b)\right|\Bigg].$$

Thus, the proof is completed. □

Remark 7. If we set $\lambda=\frac{1}{2}$ in Theorem 4, then we have the following inequality:

$$\left|\frac{1}{b-a}\left[I_{\frac{a+b}{2}+}f(b)+I_{\frac{a+b}{2}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)}{4}\Bigg[\left(\frac{q}{2[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}_{\frac{a+b}{2}}D_q f(b)\right|+\frac{q}{2[3]_q}\left|{}_{\frac{a+b}{2}}D_q f(a)\right|$$
$$+\left(\frac{q}{2[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}^{\frac{a+b}{2}}D_q f(a)\right|+\frac{q}{2[3]_q}\left|{}^{\frac{a+b}{2}}D_q f(b)\right|\Bigg].$$

This inequality is established by Sitthiwirattham et al. in [13].

Corollary 3. *In Theorem 4, if we set $\lambda=1$, then we have the following new trapezoidal type inequality linked with the inequality (1.4):*

$$\left|\frac{1}{2(b-a)}\left[I_{a+}f(b)+I_{b-}f(a)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leq\frac{(b-a)}{2}\Bigg[\frac{1}{[4]_q+q[2]_q}\left|{}_aD_q f(b)\right|+\frac{q}{[3]_q}\left|{}_aD_q f(a)\right|$$
$$+\frac{1}{[4]_q+q[2]_q}\left|{}^bD_q f(a)\right|+\frac{q}{[3]_q}\left|{}^bD_q f(b)\right|\Bigg].$$

Theorem 5. *If the functions* $\left|{}_{\lambda a+(1-\lambda)b}D_q f\right|^s$ *and* $\left|{}^{\lambda b+(1-\lambda)a}D_q f\right|^s$*, for* $s \geq 1$ *are convex and Lemma 3 holds, then the following inequality holds:*

$$\begin{aligned}&\Big|\frac{1}{b-a}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]\\&\quad-\lambda\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]\Big|\\&\leq\lambda^2(b-a)\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}\left[\left(\left(\frac{(1-\lambda)q}{[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}_{\lambda a+(1-\lambda)b}D_q f(b)\right|^s\right.\right.\\&\quad\left.+\frac{\lambda q}{[3]_q}\left|{}_{\lambda a+(1-\lambda)b}D_q f(a)\right|^s\right)^{\frac{1}{s}}\\&\quad+\left(\left(\frac{(1-\lambda)q}{[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}^{\lambda b+(1-\lambda)a}D_q f(a)\right|^s\right.\\&\quad\left.\left.+\frac{\lambda q}{[3]_q}\left|{}^{\lambda b+(1-\lambda)a}D_q f(b)\right|^s\right)^{\frac{1}{s}}\right].\end{aligned}$$

Proof. By taking modulus in (4.1) and power mean inequality, we have

$$\begin{aligned}&\Big|\frac{1}{b-a}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]\\&\quad-\lambda\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]\Big|\\&\leq\lambda^2(b-a)\left(\int_0^1(1-qt)\ {}_0d_qt\right)^{1-\frac{1}{s}}\\&\quad\times\left[\left(\int_0^1(1-qt)\left|{}_{\lambda a+(1-\lambda)b}D_q f(tb+(1-t)(\lambda a+(1-\lambda)b))\right|^s\ {}_0d_qt\right)^{\frac{1}{s}}\right.\\&\quad\left.+\left(\int_0^1(1-qt)\left|{}^{\lambda b+(1-\lambda)a}Df(ta+(1-t)(\lambda b+(1-\lambda)a))\right|^s\ {}_0d_qt\right)^{\frac{1}{s}}\right].\end{aligned}$$

By using the convexity of the functions $\left|{}_{\lambda a+(1-\lambda)b}D_q f\right|^s$ and $\left|{}^{\lambda b+(1-\lambda)a}D_q f\right|^s$, we have

$$\begin{aligned}&\Big|\frac{1}{b-a}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]\\&\quad-\lambda\left[f(\lambda a+(1-\lambda)b)+f(\lambda b+(1-\lambda)a)\right]\Big|\\&\leq\lambda^2(b-a)\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}\times\left[\left(\left(\frac{(1-\lambda)q}{[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}_{\lambda a+(1-\lambda)b}D_q f(b)\right|^s\right.\right.\end{aligned}$$

$$+\frac{\lambda q}{[3]_q}\left|{}_{\lambda a+(1-\lambda)b}D_q f(a)\right|^s\Bigg)^{\frac{1}{s}}$$
$$+\left(\left(\frac{(1-\lambda)q}{[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}^{\lambda b+(1-\lambda)a}D_q f(a)\right|^s\right.$$
$$\left.+\frac{\lambda q}{[3]_q}\left|{}^{\lambda b+(1-\lambda)a}D_q f(b)\right|^s\right)^{\frac{1}{s}}\Bigg].$$

Thus, the proof is completed. □

Remark 8. If we set $\lambda=\frac{1}{2}$ in Theorem 4, then we have the following inequality:

$$\left|\frac{1}{b-a}\left[I_{\frac{a+b}{2}+}f(b)+I_{\frac{a+b}{2}}f(a)\right]-f\left(\frac{a+b}{2}\right)\right|$$
$$\leq\frac{(b-a)}{4}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{s}}$$
$$\times\left[\left(\left(\frac{q}{2[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}_{\frac{a+b}{2}}D_q f(b)\right|^s+\frac{q}{2[3]_q}\left|{}_{\frac{a+b}{2}}D_q f(a)\right|^s\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\left(\frac{q}{2[3]_q}+\frac{1}{[4]_q+q[2]_q}\right)\left|{}^{\frac{a+b}{2}}D_q f(a)\right|^s+\frac{q}{2[3]_q}\left|{}^{\frac{a+b}{2}}D_q f(b)\right|^s\right)^{\frac{1}{s}}\right]$$

This inequality is established by Sitthiwirattham et al. in [13].

Corollary 4. *In Theorem 4, if we set $\lambda=1$, then we have the following new trapezoidal type inequality linked with the inequality (1.4):*

$$\left|\frac{1}{2(b-a)}\left[I_{a+}f(b)+I_{b-}f(a)\right]-\frac{f(a)+f(b)}{2}\right|$$
$$\leq\frac{(b-a)}{2}\left[\frac{1}{[4]_q+q[2]_q}\left|{}_aD_q f(b)\right|+\frac{q}{[3]_q}\left|{}_aD_q f(a)\right|\right.$$
$$\left.+\frac{1}{[4]_q+q[2]_q}\left|{}^bD_q f(a)\right|+\frac{q}{[3]_q}\left|{}^bD_q f(b)\right|\right].$$

Theorem 6. *If the functions $\left|{}_{\lambda a+(1-\lambda)b}D_q f\right|^s$ and $\left|{}^{\lambda b+(1-\lambda)a}D_q f\right|^s$, for $s>1$ are convex and Lemma 3 holds, then the following inequality holds:*

$$\left|\frac{1}{b-a}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]\right.$$

$$-\lambda\left[f\left(\lambda a+(1-\lambda)b\right)+f\left(\lambda b+(1-\lambda)a\right)\right]\Big|$$
$$\leq \lambda^2(b-a)\left(\frac{1-(1-q)^{r+1}}{q[r+1]_q}\right)^{\frac{1}{r}}\times\left[\left(\left(\frac{1}{[2]_q}(1+q(1-\lambda))\right)\left|{}_{\lambda a+(1-\lambda)b}D_q f(b)\right|^s\right.\right.$$
$$\left.+\frac{\lambda q}{[2]_q}\left|{}_{\lambda a+(1-\lambda)b}D_q f(a)\right|^s\right)^{\frac{1}{s}}+\left(\left(\frac{1}{[2]_q}(1+q(1-\lambda))\right)\left|{}^{\lambda b+(1-\lambda)a}D_q f(a)\right|^s\right.$$
$$\left.\left.+\frac{\lambda q}{[2]_q}\left|{}^{\lambda b+(1-\lambda)a}D_q f(b)\right|^s\right)^{\frac{1}{s}}\right].$$

where $s^{-1}+r^{-1}=1$.

Proof. By taking modulus in (4.1) and Hölder inequality, we have

$$\Big|\frac{1}{b-a}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]$$
$$-\lambda\left[f\left(\lambda a+(1-\lambda)b\right)+f\left(\lambda b+(1-\lambda)a\right)\right]\Big|$$
$$\leq\frac{(b-a)}{4}\left(\int_0^1(1-qt)^r\ {}_0d_qt\right)^{\frac{1}{r}}$$
$$\times\left[\left(\int_0^1\left|{}_{\lambda a+(1-\lambda)b}D_q f\left(tb+(1-t)\left(\lambda a+(1-\lambda)b\right)\right)\right|^s\ {}_0d_qt\right)^{\frac{1}{s}}\right.$$
$$\left.+\left(\int_0^1\left|{}^{\lambda b+(1-\lambda)a}Df\left(ta+(1-t)\left(\lambda b+(1-\lambda)a\right)\right)\right|^s\ {}_0d_qt\right)^{\frac{1}{s}}\right].$$

Since the functions $\left|{}_{\lambda a+(1-\lambda)b}D_q f\right|^s$ and $\left|{}^{\lambda b+(1-\lambda)a}D_q f\right|^s$ are convex, we have

$$\Big|\frac{1}{b-a}\left[I_{(\lambda a+(1-\lambda)b)+}f(b)+I_{(\lambda b+(1-\lambda)a)-}f(a)\right]$$
$$-\lambda\left[f\left(\lambda a+(1-\lambda)b\right)+f\left(\lambda b+(1-\lambda)a\right)\right]\Big|$$
$$\leq \lambda^2(b-a)\left(\frac{1-(1-q)^{r+1}}{q[r+1]_q}\right)^{\frac{1}{r}}\times\left[\left(\left(\frac{1}{[2]_q}(1+q(1-\lambda))\right)\left|{}_{\lambda a+(1-\lambda)b}D_q f(b)\right|^s\right.\right.$$
$$\left.+\frac{\lambda q}{[2]_q}\left|{}_{\lambda a+(1-\lambda)b}D_q f(a)\right|^s\right)^{\frac{1}{s}}+\left(\left(\frac{1}{[2]_q}(1+q(1-\lambda))\right)\left|{}^{\lambda b+(1-\lambda)a}D_q f(a)\right|^s\right.$$
$$\left.\left.+\frac{\lambda q}{[2]_q}\left|{}^{\lambda b+(1-\lambda)a}D_q f(b)\right|^s\right)^{\frac{1}{s}}\right].$$

Hence, the proof is completed. □

Remark 9. In Theorem 6, if we set $\lambda = \frac{1}{2}$, then we have the following inequality:

$$\left| \frac{1}{b-a}\left[I_{\frac{a+b}{2}+} f(b) + I_{\frac{a+b}{2}} f(a) \right] - f\left(\frac{a+b}{2}\right) \right|$$

$$\leq \frac{(b-a)}{4}\left(\frac{1-(1-q)^{r+1}}{q[r+1]_q}\right)^{\frac{1}{r}} \times \left[\left(\frac{2+q}{2[2]_q} \left| {}_{\frac{a+b}{2}}D_q f(b) \right|^s + \frac{q}{2[2]_q} \left| {}_{\frac{a+b}{2}}D_q f(a) \right|^s \right)^{\frac{1}{s}} \right.$$

$$\left. + \left(\frac{2+q}{2[2]_q} \left| {}^{\frac{a+b}{2}}D_q f(a) \right|^s + \frac{q}{2[2]_q} \left| {}^{\frac{a+b}{2}}D_q f(b) \right|^s \right)^{\frac{1}{s}} \right].$$

This inequality is established by Sitthiwirattham et al. in [13].

Corollary 5. *In Theorem 6, if we set $\lambda = 1$, then we have the following new trapezoidal type inequality linked with the inequality (1.4):*

$$\left| \frac{1}{2(b-a)}\left[I_{a+} f(b) + I_{b-} f(a) \right] \frac{f(a)+f(b)}{2} \right|$$

$$\leq \frac{(b-a)}{2}\left(\frac{1-(1-q)^{r+1}}{q[r+1]_q}\right)^{\frac{1}{r}} \left[\left(\frac{1}{[2]_q} \left| {}_aD_q f(b) \right|^s + \frac{q}{[2]_q} \left| {}_aD_q f(a) \right|^s \right)^{\frac{1}{s}} \right.$$

$$\left. + \left(\frac{1}{[2]_q} \left| {}^bD_q f(a) \right|^s + \frac{q}{[2]_q} \left| {}^bD_q f(b) \right|^s \right)^{\frac{1}{s}} \right].$$

5. Examples

In this part of the paper, we give some examples to show the validity of newly established inequalities.

Example 1. We consider a convex function $f(x) = x^2$. From Theorem 3 with $a = 1$, $b = 2$, $q = \frac{1}{2}$ and $\lambda = \frac{1}{3}$, we have

$$f\left(\frac{a+b}{2}\right) = \left(\frac{3}{2}\right)^2 = \frac{9}{4},$$

$$\frac{3}{2(b-a)}\left[I_{\frac{a+2b}{3}+} f(b) + I_{\frac{2a+b}{3}-} f(a) \right]$$

$$= \frac{3}{2}\left(\left(1-\frac{1}{2}\right)\left(2-\frac{5}{3}\right)\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left(\left(\frac{1}{2}\right)^n \times 2 + \left(1-\left(\frac{1}{2}\right)^n\right)\frac{5}{3}\right)^2\right.$$

$$+\left(1-\frac{1}{2}\right)\left(\frac{4}{3}-1\right)\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{n}\left(\left(\frac{1}{2}\right)^{n}+\left(1-\left(\frac{1}{2}\right)^{n}\right)\frac{4}{3}\right)^{2}\right)=\frac{913}{378}$$

and

$$\frac{f(a)+f(b)}{2}=\frac{1+2^2}{2}=\frac{5}{2}.$$

Thus,

$$\frac{9}{4}<\frac{913}{378}<\frac{5}{2}$$

and Theorem 3 is valid.

Example 2. We consider a convex function $f(x)=x^2$. From Theorem 3 with $a=1$, $b=2$, $q=\frac{1}{2}$ and $\lambda=\frac{1}{4}$, we have

$$f\left(\frac{a+b}{2}\right)=\left(\frac{3}{2}\right)^2=\frac{9}{4},$$

$$\frac{2}{(b-a)}\left[I_{\frac{a+3b}{4}+}f(b)+I_{\frac{3a+b}{4}-}f(a)\right]=\frac{817}{336}$$

and

$$\frac{f(a)+f(b)}{2}=\frac{1+2^2}{2}=\frac{5}{2}.$$

Thus,

$$\frac{9}{4}<\frac{817}{336}<\frac{5}{2}$$

and Theorem 3 is valid.

Example 3. We consider a convex function $f(x)=x^2$. From Theorem 4 with $a=1$, $b=2$, $q=\frac{1}{2}$ and $\lambda=1$, we have

$$\left|\frac{1}{2(b-a)}\left[I_{a+}f(b)+I_{b-}f(a)\right]-\frac{f(a)+f(b)}{2}\right|=\frac{2}{21}$$

and

$$\begin{aligned}&\frac{(b-a)}{2}\left[\frac{1}{[4]_q+q[2]_q}\left|{}_aD_qf(b)\right|+\frac{q}{[3]_q}\left|{}_aD_qf(a)\right|\right.\\&\left.+\frac{1}{[4]_q+q[2]_q}\left|{}^bD_qf(a)\right|+\frac{q}{[3]_q}\left|{}^bD_qf(b)\right|\right]\\&=\frac{1}{2}\left(\frac{1}{1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\frac{1}{2}\left(1+\frac{1}{2}\right)}\times\frac{7}{2}+\frac{\frac{1}{2}}{1+\frac{1}{2}+\left(\frac{1}{2}\right)^2}\times 2\right.\\&\left.+\frac{1}{1+\frac{1}{2}+\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^3+\frac{1}{2}\left(1+\frac{1}{2}\right)}\times\frac{5}{2}+\frac{\frac{1}{2}}{1+\frac{1}{2}+\left(\frac{1}{2}\right)^2}\times 4\right)=2.\end{aligned}$$

Thus,

$$\frac{2}{21} < 2$$

and Theorem 4 is valid.

Remark 10. Similarly, one can check the validity of all the results.

6. Concluding Remarks

In this paper, we established a family of q-Hermite–Hadamard type inequalities for convex functions. Moreover, we proved some new integral inequalities for differentiable convex functions in the setting of q-calculus. The new inequalities can help us to find the error bounds for the midpoint, trapezoidal, and many other numerical integration formulas in q-calculus. It is also shown with the examples that the newly established inequalities are valid for convex functions. It is an interesting problem that the upcoming researchers can obtain similar inequalities for the functions of two variables.

References

[1] M. A. Ali, H. Budak, M. Fečkan, and S. Khan, "A new version of q-Hermite-Hadamard's midpoint and trapezoid type inequalities for convex functions," *Mathematica Slovaca*, vol. 73, no. 2, pp. 369–386, 2023, doi: https://doi.org/10.1515/ms-2023-0029.

[2] M. A. Ali, H. Budak, Z. Zhang, and H. Yildirim, "Some new Simpson's type inequalities for coordinated convex functions in quantum calculus," *Math. Methods Appl. Sci.*, vol. 44, no. 6, pp. 4515–4540, 2021, doi: https://doi.org/10.1002/mma.7048.

[3] N. Alp, M. Z. Sarıkaya, M. Kunt, and İ. İşcan, "q-Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions," *Journal of King Saud University-Science*, vol. 30, no. 2, pp. 193–203, 2018, doi: 10.1016/j.jksus.2016.09.007.

[4] S. Bermudo, P. Kórus, and J. Nápoles Valdés, "On q-Hermite–Hadamard inequalities for general convex functions," *Acta Math. Hungar.*, vol. 162, no. 1, pp. 364–374, 2020, doi: https://doi.org/10.1007/s10474-020-01025-6.

[5] H. Budak, "Some trapezoid and midpoint type inequalities for newly defined quantum integrals," *Proyecciones (Antofagasta)*, vol. 40, no. 1, pp. 199–215, 2021, doi: http://dx.doi.org/10.22199/issn.0717-6279-2021-01-0013.

[6] H. Budak, S. Erden, and M. A. Ali, "Simpson and Newton type inequalities for convex functions via newly defined quantum integrals," *Math. Methods Appl. Sci.*, vol. 44, no. 1, pp. 378–390, 2021, doi: https://doi.org/10.1002/mma.6742.

[7] S. Dragomir and R. Agarwal, "Two inequalities for differentiable mappings and applications to special means of real numbers and to trapezoidal formula," *Appl. Math. Lett.*, vol. 11, no. 5, pp. 91–95, 1998, doi: https://doi.org/10.1016/S0893-9659(98)00086-X.

[8] H. Gauchman, "Integral inequalities in q-calculus," *Comput. Math. Appl.,*, vol. 47, no. 2-3, pp. 281–300, 2004, doi: https://doi.org/10.1016/S0898-1221(04)90025-9.

[9] V. G. Kac and P. Cheung, *Quantum calculus*. Springer, 2002, vol. 113.

[10] U. S. Kirmaci, "Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula," *Appl. Math. Comput.*, vol. 147, no. 1, pp. 137–146, 2004, doi: https://doi.org/10.1016/S0096-3003(02)00657-4.

[11] M. A. Noor, K. I. Noor, and M. U. Awan, "Some quantum estimates for Hermite–Hadamard inequalities," *Appl. Math. Comput.*, vol. 251, pp. 675–679, 2015, doi: https://doi.org/10.1016/j.amc.2014.11.090.

[12] I. B. Sial, S. Mei, M. A. Ali, and K. Nonlaopon, "On some generalized Simpson's and Newton's inequalities for (α, m)-convex functions in q-calculus," *Mathematics*, vol. 9, no. 24, p. 3266, 2021, doi: https://doi.org/10.3390/math9243266.

[13] T. Sitthiwirattham, M. A. Ali, A. Ali, and H. Budak, "A new q-hermite-hadamard's inequality and estimates for midpoint type inequalities for convex functions," *Miskolc Math. Notes,*, 2023, in press.

[14] J. Soontharanon, M. A. Ali, H. Budak, K. Nonlaopon, and Z. Abdullah, "Simpson's and Newton's inequalities for (α, m)-convex functions via quantum calculus," *Symmetry*, vol. 2022, no. 14, p. 736, 2022, doi: https://doi.org/10.3390/sym14040736.

[15] W. Sudsutad, S. K. Ntouyas, and J. Tariboon, "Quantum integral inequalities for convex functions," *J. Math. Inequal*, vol. 9, no. 3, pp. 781–793, 2015, doi: dx.doi.org/10.7153/jmi-09-64.

[16] J. Tariboon and S. K. Ntouyas, "Quantum calculus on finite intervals and applications to impulsive difference equations," *Adv. Difference Equ.*, vol. 2013, no. 1, pp. 1–19, 2013, doi: https://doi.org/10.1186/1687-1847-2013-282.

[17] H. Zhuang, W. Liu, and J. Park, "Some quantum estimates of Hermite-Hadamard inequalities for quasi-convex functions," *Mathematics*, vol. 7, no. 2, p. 152, 2019, doi: https://doi.org/10.3390/math7020152.

Authors' addresses

Zoya Abdullah
Department of Mathematics, COMSATS University Islamabad Sahiwal Campus, Pakistan
E-mail address: zoya@cuisahiwal.edu.pk

Awais Yousaf
(**Corresponding author**) Department of Mathematics, The Islamia University of Bahawalpur, Bahawalpur, Pakistan
E-mail address: awais.yousaf@iub.edu.pk

Chanon Promsakon
Department of Mathematics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok 10800, Thailand
E-mail address: chanon.p@sci.kmutnb.ac.th

Thanin Sitthiwirattham
Mathematics Department, Faculty of Science and Technology, Suan Dusit University, Bangkok, Thailand
E-mail address: thanin_sit@dusit.ac.th