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STATISTICALLY p_τ-CONVERGENCE IN LATTICE-NORMED LOCALLY SOLID RIESZ SPACES

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Abstract. In this paper, we introduce the concept of the statistical convergence with respect to solid topology and Riesz valued norms on lattice-normed locally solid Riesz spaces. Moreover, we give the notions of statistically p_τ-bounded and statistically p_τ-dense sequence, and we introduce statistically p_τ-continuous and statistically p_τ-bounded operators. We also investigate some properties and examples of these concepts.

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1. INTRODUCTORY FACTS

Lattice-valued norms on Riesz spaces and statistical convergence of sequences provide natural and efficient tools in the theory of functional analysis. A Riesz space that was introduced by Riesz [20] is an ordered vector space having many applications in measure theory, operator theory, and economics [2, 3, 19, 23]. On the other hand, a lattice-normed space is a lattice-valued norm, and it is enough to mention the theory of lattice-normed vector lattices [11, 17, 18]. As an active area of research, statistical convergence is a generalization of the ordinary convergence of a real sequence, and the idea of statistical convergence was firstly introduced by Zygmund [16]. After then, Fast [13] and Steinhaus [21] independently improved that idea. Several applications and generalizations of the statistical convergence of sequences have been investigated by several authors [1, 5, 6, 8, 9, 14, 21, 22]. The main aim of the present paper is to introduce the concept of *statistical p_τ-convergence* on lattice-normed locally solid Riesz spaces, which attracted the attention of several authors in a series of recent papers [1, 5, 6, 9].

The concept of the statistical convergence in Riesz spaces was introduced by Ercan [12], where the notion of the statistically u-uniformly convergent sequence was introduced in Riesz spaces. Then Albayrak and Pehlivan extended the statistical convergence to locally solid Riesz spaces with respect to solid topology [1]. Recently, Aydın et al., have investigated some studies about the statistical convergence on Riesz spaces and locally solid Riesz spaces [5,6,9].

We now turn our attention to some basic notions which will be used in this paper. A real-valued vector space E with a partial order relation "$\leq$" on E (i.e. it is an antisymmetric, reflexive and transitive relation) is called an *ordered vector space* whenever, for every $x,y \in E$, we have

(a) $x \leq y$ implies $x+z \leq y+z$ for all $z \in E$,
(b) $x \leq y$ implies $\lambda x \leq \lambda y$ for every $0 \leq \lambda \in \mathbb{R}$.

An ordered vector space E is called a *Riesz space* or a *vector lattice* if, for any two vectors $x,y \in E$, the infimum $x \wedge y = \inf\{x,y\}$ and the supremum $x \vee y = \sup\{x,y\}$ exist in E. For an element x in a Riesz space E, the *positive part*, the *negative part*, and the *module* of x are

$$x^+ := x \vee 0, \quad x^- := (-x) \vee 0 \quad \text{and} \quad |x| := x \vee (-x),$$

respectively. In the present paper, the vertical bar $|\cdot|$ of elements of Riesz spaces will stand for the module of the given elements.

By a *linear topology* τ on a vector space X, we mean a topology τ on X that makes the addition and the scalar multiplication continuous. A *topological vector space* (X,τ) is a vector space X equipped with a linear topology τ. A linear topology τ on a vector space E has a base $\mathcal{N}$ for the zero neighborhoods satisfying the following:

(i) Each $V \in \mathcal{N}$ is *balanced*, i.e., $\lambda V \subseteq V$ for all scalars $|\lambda| \leq 1$;
(ii) Every $V \in \mathcal{N}$ is *absorbing*, i.e., for every element x, there exists a positive real $\lambda > 0$ such that $x \in \lambda V$;
(iii) For each $V_1, V_2 \in \mathcal{N}$, there is $V \in \mathcal{N}$ such that $V \subseteq V_1 \cap V_2$;
(iv) For every $V \in \mathcal{N}$, there exists $U \in \mathcal{N}$ with $U + U \subseteq V$;
(v) For any scalar λ and each $V \in \mathcal{N}$, the set λV is also in $\mathcal{N}$.

In this article, unless otherwise stated, when we mention a zero neighborhood, it means that it always belongs to a base that holds the above properties. In this paper, neighborhoods of zero will often be referred to as *zero neighborhoods*.

A subset A of a vector lattice E is called *solid* if, for each $x \in A$ and $y \in E$ with $|y| \leq |x|$ it holds $y \in A$. Let E be a Riesz space and τ be a linear topology on it. Then the pair (E,τ) is said to be a *locally solid Riesz space* if τ has a base that consists of solid sets; for much more details on these notions, see [2,3,23].

Definition 1. Let X be a vector space and E be a Riesz space. Then $p\colon X \to E_+$ is called an *E-valued vector norm* whenever it satisfies the following conditions:

(1) $p(x) = 0 \Leftrightarrow x = 0$;
(2) $p(\lambda x) = |\lambda| p(x)$ for all $\lambda \in \mathbb{R}$;
(3) $p(x+y) \leq p(x) + p(y)$ for all $x,y \in X$.

Then the triple (X,p,E) is called a *lattice-normed space*, abbreviated as *LNS*. If, in addition, X is a Riesz space and the vector norm p is monotone (i.e., $|x|\leq|y|\Rightarrow p(x)\leq p(y)$ holds for all $x,y\in X$) then the triple (X,p,E) is called a *lattice-normed Riesz space*. We abbreviate it as *LNRS*. A subset $Y\subseteq X$ is called *p-bounded* if there exists $e\in E$ such that $p(y)\leq e$ for all $y\in Y$. A sequence (x_n) in X is called *p-convergent* to $x\in X$ (or, shortly, $x_n\xrightarrow{\mathrm{p}}x$) whenever $p(x_n-x)\xrightarrow{\mathrm{o}}0$ holds in E. We refer the reader for more information on *LNS*s [11, 17, 18]. We shall keep in mind also the following examples.

Example 1. Let X be a normed space with a norm $\|\cdot\|$. Then $(X,\|\cdot\|,\mathbb{R})$ is an *LNS*.

Example 2. Let X be a Riesz space. Then $(X,|\cdot|,X)$ is an *LNRS*.

Now, we remind some basic properties of the concept related to the statistical convergence. Consider a set K of positive integers. Then the *natural density* of K is defined by

$$\delta(K):=\lim_{n\to\infty}\frac{1}{n}\left|\{k\leq n:k\in K\}\right|,$$

where the vertical bar of sets will stand for the cardinality of the given sets. We refer the reader to an exposition on the natural density of sets [13, 14]. In the same way, a real sequence (x_n) is called *statistically convergent* to L provided that

$$\lim_{m\to\infty}\frac{1}{m}\left|\{n\leq m:|x_n-L|\geq\varepsilon\}\right|=0$$

for each $\varepsilon>0$.

Let X be a topological space and (x_n) be a sequence in X. Then (x_n) is said to be *statistically convergent* to $x\in X$ whenever, for each neighborhood U of x, we have $\delta\big(\{n\in\mathbb{N}:x_n\notin U\}\big)=0$. On the other hand, a sequence (x_n) in a locally solid Riesz space (E,τ) is *statistically τ-convergent* to $x\in E$ whenever we have $\delta\big(\{n\in\mathbb{N}:(x_n-x)\notin U\}\big)=0$ for every zero neighborhood U [1, 5–10].

2. STATISTICALLY p_τ-CONVERGENCE

In this section, we introduce the statistically topological convergence on lattice-normed spaces. Also, we give some basic results about this concept.

Definition 2. Let (X,p,E) be an *LNRS*. If (E,τ) is a locally solid Riesz space then (X,p,E_τ) is called a *lattice-normed locally solid Riesz space*. We abbreviate it as *LNLS*.

Definition 3. Let (X,p,E_τ) be an *LNLS* and (x_n) be a sequence in X. Then (x_n) is said to be *statistically p_τ-convergent* (*st*-p_τ-convergent, for short) to x if it is provided that

$$\lim_{m\to\infty}\frac{1}{m}\Big|\{n\leq m:p(x_n-x)\notin U\}\Big|=0$$

holds for every zero neighborhood U. In this case, we write $x_n \xrightarrow{\text{st-p}_\tau} x$.

Briefly, a sequence (x_n) is statistically p_τ-convergent to $x \in X$ if $\delta(K_U) = 0$ for each zero neighborhood U, where $K_U = \{n \in \mathbb{N} : p(x_n - x) \notin U\}$. Note that, in order to simplify the presentation, we take zero neighborhoods from a solid base because, for every zero neighborhood V, there exists a zero neighborhood solid set U such that $U \subseteq V$.

Example 3. Consider the *LNLS* $(E, |\cdot|, E_\tau)$ for an arbitrary locally solid Riesz space (E, τ). Then statistically τ- and p_τ-convergence coincide.

Example 4. Let $\|\cdot\|$ be a lattice norm (i.e., $|x| \leq |y|$ implies $\|x\| \leq \|y\|$) on a Riesz space X. Then, it follows from [2, Theorem 2.28] that the topology τ on X generated by the norm is a solid topology. Moreover, the statistically norm convergence and p_τ-convergence agree on the *LNLS* $(X, |\cdot|, X_\tau)$.

Example 5. Let's consider the *LNLS* $(c, |\cdot|, c_\tau)$, where c is the set of all convergent real sequences and τ is the topology generated by the supremum norm on c. Take the sequence (x_n) in c such that $x_n = (0, .., \frac{1}{n}, 0, ...)$ for each n. Now, consider the set of all zero neighborhoods $\mathcal{N} := \{U_r : r \in \mathbb{R}_+\}$, where U_r is the set $\{x \in c : \|x\|_\infty \leq r\}$. Thus, it follows that $x_n \xrightarrow{\text{st-p}_\tau} 0$ in c. Indeed, fix an arbitrary zero neighborhood set U in c. Then, there exist some $r > 0$ such that $U_r \subseteq U$. Consider the set

$$K_r = \{n \in \mathbb{N} : p(x_n - 0) \in U_r\} = \left\{n \in \mathbb{N} : \|x_n\|_\infty = \frac{1}{n} < r\right\}.$$

It can be seen that $\delta(K_r) = 1$. Thus, we have $x_n \xrightarrow{\text{st-p}_\tau} 0$.

Remark 1. Let (X, p, E_τ) be an *LNLS*. If $|y| \leq |x|$ for any $x, y \in X$, then we have that $p(y) \notin U$ implies $p(x) \notin U$ for an arbitrary zero neighborhood U. Indeed, assume that $p(y) \notin U$ and $p(x) \in U$ hold. It follows from the solidness of U that we have $p(y) \in U$ because $|y| \leq |x|$ implies $p(y) \leq p(x) \in U$. So, there is a contradiction, and so, we have the desired result, $p(x) \notin U$.

Proposition 1. *Let (X, p, E_τ) be an LNLS. Then, the following statements hold:*

(i) *every p-convergent sequence is st-p_τ-convergent;*
(ii) *every order convergent sequence is st-p_τ-convergent if order convergence implies p-convergence in X.*

Proof.

(i) Let $x_n \xrightarrow{\text{p}} x$ in X. Then, there exists a sequence $q_n \downarrow 0$ in E such that $p(x_n - x) \leq q_n$ for every $n \in \mathbb{N}$. On the other hand, take an arbitrary zero neighborhood U and arbitrary $m \in \mathbb{N}$. Then, there exists a positive integer $k > 0$ such that $\frac{1}{k} q_m \in U$, and so, we have $\frac{1}{k} q_n \in U$ for all $n \geq m$ because U is solid and absorbing set. Now, take an arbitrary index n_k such that $q_{n_k} \leq \frac{1}{k} q_m$. Hence, we have $q_n \in U$ for all $n \geq n_k$. So, we observe that

$$\delta(\{n \in \mathbb{N} : q_n \notin U\}) = \delta(\{1, 2, \dots, n_k - 1\}) = 0.$$

It follows from the inequality $p(x_n - x) \leq q_n$ for all n that we have $\delta(\{n \in \mathbb{N} : p(x_n - x) \notin U\}) = 0$, i.e., $x_n \xrightarrow{\text{st-p}_\tau} x$.

(ii) The proof is similar to the first part.

□

Theorem 1. *The st-p_τ-limit is linear in LNLSs.*

Proof. Suppose that $x_n \xrightarrow{\text{st-p}_\tau} x$ and $y_n \xrightarrow{\text{st-p}_\tau} y$ in an *LNLS* (X, p, E_τ). For any zero neighborhood U, there exists another zero neighborhood V such that $V + V \subseteq U$. Then, we have $\delta(\{n \in \mathbb{N} : p(x_n - x) \notin V\}) = 0$ and $\delta(\{n \in \mathbb{N} : p(y_n - y) \notin V\}) = 0$. It follows from the following inequality

$$p(x_n + y_n - x - y) \leq p(x_n - x) + p(y_n - y) \in V + V \subseteq U$$

that if $p(x_k - x) \in V$ and $p(y_k - y) \in V$ for some indexes k, then $p(x_k + y_k - x - y) \in U$. Thus, we observe that $\{n \in \mathbb{N} : p(x_n + y_n - x - y) \notin U\} \subseteq \{n \in \mathbb{N} : p(x_n - x) \notin V\} \cup \{n \in \mathbb{N} : p(y_n - y) \notin V\}$, and so, we obtain $\delta(\{n \in \mathbb{N} : p(x_n + y_n - x - y) \notin U\}) = 0$. Therefore, we get the desired result, i.e., $x_n + y_n \xrightarrow{\text{st-p}_\tau} x + y$.

Now, take a scalar $\alpha \in \mathbb{R}$ such that $|\alpha| \leq 1$ and an arbitrary zero neighborhood U. Then, it follows that $\delta(K) = 1$, where $K = \{n \in \mathbb{N} : p(x_n - x) \in U\}$. Since U is a balanced set, by the equality

$$p(\alpha x_n - \alpha x) = |\alpha| p(x_n - x) \in |\alpha| U \subseteq U,$$

we get $p(\alpha x_n - \alpha x) \in U$ for all $n \in K$. Thus, we obtain

$$\{n \in \mathbb{N} : p(x_n - x) \in U\} \subseteq \{n \in \mathbb{N} : p(\alpha x_n - \alpha x) \in U\},$$

and so, we have $\delta(\{n \in \mathbb{N} : p(\alpha x_n - \alpha x) \in U\}) = 1$, i.e., $\alpha x_n \xrightarrow{\text{st-p}_\tau} \alpha x$.

Consider the case $|\alpha| > 1$. Then, for a given zero neighborhood U, we have $\frac{1}{|\alpha|} U \subseteq U$ because of the balanced property. Fix $\gamma = \frac{1}{|\alpha|}$. Since γU is a solid zero neighborhood, there exists another zero neighborhood W such that $W \subseteq \gamma U$, and so, we have $|\alpha| W \subseteq U$. By the same consideration of the above part, it follows from

$$p(\alpha x_n - \alpha x) = |\alpha| p(x_n - x) \in |\alpha| W \subseteq U$$

that $\delta(\{n \in \mathbb{N} : p(\alpha x_n - \alpha x) \in U\}) = 1$. So, we get $\alpha x_n \xrightarrow{\text{st-p}_\tau} \alpha x$ for every $\alpha \in \mathbb{R}$. □

Theorem 2. *Let (X, p, E_τ) be an LNLS. Then,*

(i) *if τ is a Hausdorff solid topology, then the st-p_τ-limit is uniquely determined,*
(ii) *the statistically p_τ-version of the squeeze law holds.*

Proof.

(i) Assume that $x_n \xrightarrow{\text{st-p}_\tau} x_1$ and $x_n \xrightarrow{\text{st-p}_\tau} x_2$ and τ has the Hausdorff property. Let U be a zero neighborhood in E. Then, there exists a zero neighborhood V satisfying $V+V \subseteq U$. Thus, we have $\delta(K_1) = \delta(K_2) = 1$ for the sets $K_1 = \{n \in \mathbb{N} : p(x_n - x_1) \in V\}$ and $K_2 = \{n \in \mathbb{N} : p(x_n - x_2) \in V\}$. It follows that
$$p(x_1 - x_2) \leq p(x_1 - x_n) + p(x_n - x_2) \in V + V \subseteq U$$
for every $n \in K_1 \cap K_2$. Then, by the solidness of U, $p(x_1 - x_2) \in U$ holds. So, we have $p(x_1 - x_2) \in U$ for every zero neighborhood. Since the intersection of all zero neighborhoods in a Hausdorff space is zero, we obtain $p(x_1 - x_2) = 0$, i.e., $x_1 = x_2$.

(ii) Suppose that $x_n \leq y_n \leq z_n$ holds for all $n \in \mathbb{N}$, and $x_n \xrightarrow{\text{st-p}_\tau} w$ and $z_n \xrightarrow{\text{st-p}_\tau} w$. Take an arbitrary zero neighborhood U, and so, there exists another zero neighborhood V such that $V+V \subseteq U$. Hence, we have $\delta(K_1) = \delta(K_2) = 1$, where $K_1 = \{n \in \mathbb{N} : p(x_n - w) \in V\}$ and $K_2 = \{n \in \mathbb{N} : p(z_n - w) \in V\}$. By the inequality $x_n \leq y_n \leq z_n$, we have $x_n - w \leq y_n - w \leq z_n - w$, and so, $|y_n - w| \leq |x_n - w| + |z_n - w|$ holds for all n. Therefore, we obtain
$$p(y_n - w) \leq p(x_n - w) + p(z_n - w) \in V + V \subseteq U$$
for every $n \in K_1 \cap K_2$. Thus, we obtain $\delta(\{n \in \mathbb{N} : p(y_n - w) \in U\}) = 1$. So, we get the desired result, $y_n \xrightarrow{\text{st-p}_\tau} w$.

□

Definition 4. Let (X, p, E_τ) be an *LNLS* and (x_n) be a sequence in X. Then, (x_n) is said to be *statistically* p_τ*-bounded* if, for every zero neighborhood U, there exist some $\lambda > 0$ such that
$$\delta(\{n \in \mathbb{N} : p(x_n) \notin \lambda U\}) = 0.$$

Remark 2.

(i) Every p-bounded sequence is statistically p_τ-bounded. Indeed, let (x_n) be a p-bounded sequence in an *LNLS* (X, p, E_τ). It follows from [2, Theorem 2.19 (1)] that $p(x_n)$ is topologically bounded in E because it is order bounded in E. So, for each zero neighborhood U, there exists $\lambda > 0$ such that $p(x_n) \in \lambda U$ for all $n \in \mathbb{N}$. Thus, (x_n) is statistically p_τ-bounded.

(ii) Let (X, p, E_τ) be an *LNLS* and (x_n) be an order bounded sequence in X. Then, (x_n) is statistically p_τ-bounded. Indeed, since (x_n) is order bounded, there exists $x \in X_+$ such that $|x_n| \leq x$ for all $n \in \mathbb{N}$. It follows from the monotonicity of p that $p(x_n) \leq p(x)$ for all n, i.e., $p(x_n)$ is order bounded in E. Thus, by applying [2, Thmeorem 2.19 (1)], $p(x_n)$ is topologically bounded in E, and so, it is statistically p_τ-bounded.

Proposition 2. *Every st-*p_τ*-convergent sequence is statistically* p_τ*-bounded.*

Proof. Let (x_n) be an *st*-p_τ-convergent sequence to x in an *LNLS* (X,p,E_τ). Take an arbitrary zero neighborhood U with another zero neighborhood V such that $V+V\subseteq U$. Then, we have $\delta(\{n\in\mathbb{N}: p(x_n-x)\notin V\})=0$. On the other hand, by using the absorbing property of V, there exists a positive scalar λ such that $\lambda p(x)\in V$. Choose $r:=\min\{1,\lambda\}$. Then, we have $rp(x)\in V$ because of $rp(x)\leq\lambda p(x)$. Therefore, if $p(x_n-x)\in V$, then $rp(x_n-x)\in V$ by the balancing property of V. It follows from the inequality $rp(x_n)\leq rp(x_n-x)+rp(x)\in V+V\subseteq U$ that we obtain $\delta(\{n\in\mathbb{N}: rp(x_n)\notin U\})=0$, i.e., (x_n) is statistically p_τ-bounded. □

3. STATISTICAL p_τ-DENSITY AND p_τ-LIMIT POINTS

Recall that a sublattice Y of a Riesz space X is called

(-) *dense with respect to order convergence* if every vector in X is the order limit of a net in Y,

(-) *order dense* whenever for each $0<x\in X$ there exists some $y\in Y$ with $0<y\leq x$.

Motivated by these definitions, we give the following notions.

Definition 5. Let (X,p,E_τ) be an *LNLS*. A sublattice $Y\subseteq X$ is called

(i) *statistically p_τ-dense* in X with respect to *st*-p_τ-convergence whenever for all $x\in X$ there exists a non zero sequence (y_n) in Y such that $y_n\xrightarrow{\text{st-p}_\tau}x$;

(ii) *p_τ-dense* in X with respect to *st*-p_τ-boundedness if, for any $x\in X$, there exists a non zero sequence (y_n) in Y such that the sequence $(x-y_n)$ is statistically p_τ-bounded.

It is clear from Proposition 2 that statistical p_τ-density implies p_τ-density. But, the converse need not hold in general. To see this, consider the following example.

Example 6. Let X be the set of real-valued bounded functions on $[0,1]$ denoted in the form $f:=g+h$, where g is continuous and h vanishes except at finitely many points. Then, $(X,|\cdot|,X_\tau)$ is an *LNLS*, where τ is the topology generated by supremum norm on X. Then, the sublattice $Y:=C[0,1]$, all continuous functions on $[0,1]$, is statistically p_τ-dense in X. Indeed, it is clear that, for any $f\in X$, there exists a sequence (g_n) in Y such that $g_n\xrightarrow{o}f$. It follows from [4, Remark 2.2 (2)] and Proposition 1(*ii*) that we obtain $g_n\xrightarrow{\text{st-p}_\tau}f$. On the other hand, we observe that Y is not p_τ-dense in X because, for the characteristic function $\chi_{\{\frac{1}{3}\}}$ in X, there is not any sequence (g_n) in Y such that $(f-g_n)$ is statistically p_τ-bounded.

Proposition 3. *Let Y be a sublattice of a locally solid Riesz space (X,τ). If Y is p_τ-dense in LNLS $(X,|\cdot|,X_\tau)$, then Y is order dense.*

Proof. Take a positive nonzero element $0\neq x\in X_+$. Then, there is a sequence (y_n) in Y such that $(y_n-\frac{1}{2}x)$ is statistically p_τ-bounded. Thus, for an arbitrary zero neighborhood U, there exist some $\lambda>0$ such that $\delta(\{n\in\mathbb{N}:\lambda|y_n-\frac{1}{2}x|\in U\})=1$.

Take $u \in U$ such that $|y_m - \frac{1}{2}x| = \frac{1}{\lambda}u \leq \frac{1}{3}x$ for some $m \in \mathbb{N}$. Then, it follows that $0 < y_m \leq x$, and so, Y is order dense in X. □

Example 7. Let Y be the Riesz space c_0 of all convergent to zero real sequences and X be the Riesz space ℓ_∞ of all bounded reel sequences. Then, for the *LNLS* $(\ell_\infty, |\cdot|, \ell_\infty)$, Y is order bounded in X. But, it is not p_τ-dense in X.

Now, we turn our attention to statistical p_τ-limit and p_τ-cluster points. The following notions are p_τ-versions of classical statistical points [10, 14].

Definition 6. Let (X, p, E_τ) be an *LNLS*. Then, a point $x \in X$ is called

(1) an *st-p_τ-limit point* of a sequence (x_n) in X whenever there is an index set $K = \{k_1 < k_2 < \dots\} \subset \mathbb{N}$ such that $\delta(K) > 0$ and $p(x_{k_n} - x) \xrightarrow{\tau} 0$.
(2) an *st-p_τ-cluster point* of a sequence (x_n) in X if, for each zero neighborhood U, we have $\delta(\{n \in \mathbb{N} : p(x_n - x) \in U\}) > 0$.

For a sequence (x_n) in a space X, let $\Lambda_p(x_n)$ denote the set of all *st*-p_τ-limit points of (x_n), and let $\Theta_p(x_n)$ denote the set of all *st*-p_τ-cluster points of (x_n).

Example 8. Let $(E, |\cdot|, E_\tau)$ be an *LNLS* and $x, y \in E$. Then, take a sequence (x_n) denoted by $x_n := x$ if n is a square and $x_n := y$ otherwise. Thus, it is clear that $\Lambda_p(x_n) = \Theta_p(x_n) = \{y\}$.

Theorem 3. *For an LNLS (X, p, E_τ) and sequence (x_n) in X, $\Lambda_p(x_n) \subseteq \Theta_p(x_n)$.*

Proof. Let $x \in \Lambda_p(x_n)$ and $p(x_{n_k} - x) \xrightarrow{\tau} 0$ holds on an index set K such that $\delta(K) = \lambda > 0$. Fix a zero neighborhood U in E. Then, it follows from $p(x_{n_k} - x) \xrightarrow{\tau} 0$ that there exists an $n_j \in K$ such that $p(x_{n_k} - x) \in U$ for $n_j \leq n_k \in K$. Also, we can easily observe the following subset

$$\{n_k : k \in \mathbb{N}\} \setminus \{n_1, n_2, \dots, n_{j-1}\} \subseteq \{n \in \mathbb{N} : p(x_n - x) \in U\}.$$

Therefore, we have

$$\delta(\{n \in \mathbb{N} : p(x_n - x) \in U\}) \geq \lambda > 0$$

Then, we get $x \in \Theta_p(x_n)$ because U is arbitrary. □

Theorem 4. *Let (x_n) and (y_n) be two sequences in an LNLS (X, p, E_τ). If the natural density of $\{n \in \mathbb{N} : x_n \neq y_n\}$ is zero, then $\Theta_p(x_n) = \Theta_p(y_n)$ and $\Lambda_p(x_n) = \Lambda_p(y_n)$.*

Proof. Assume that $w \in \Theta_p(x_n)$ and U is an arbitrary zero neighborhood. Thus, we have $\delta(\{n \in \mathbb{N} : p(x_n - w) \in U\}) > 0$. It follows from the subset

$$\{n \in \mathbb{N} : p(x_n - w) \in U\} \setminus \{n \in \mathbb{N} : x_n \neq y_n\} \subseteq \{n \in \mathbb{N} : p(y_n - w) \in U\}$$

that we obtain $\delta(\{n \in \mathbb{N} : p(y_n - w) \in U\}) > 0$. Therefore, we get $w \in \Theta_p(y_n)$. In a similar way, one proves $\Theta_p(y_n) \subseteq \Theta_p(x_n)$. Hence, we get the desired result, $\Theta_p(x_n) = \Theta_p(y_n)$.

The equality $\Lambda_p(x_n) = \Lambda_p(y_n)$ can be obtained similarly. $\square$

Theorem 5. *Let (x_n) be a sequence in an LNLS (X,p,E_τ) and F be a compact subset of E satisfying $\delta(\{n \in \mathbb{N} : p(x_n) \in F\}) > 0$. Then, we have $p(\Theta_p(x_n)) \cap F \neq \varnothing$.*

Proof. Assume that $p(\Theta_p(x_n)) \cap F = \varnothing$. Then, for every $w \in X$ such that $p(w) \in F$, we have $w \notin \Theta_p(x_n)$. Thus, there is a zero neighborhood U_w such that $\delta(K_w) = 0$, where $K_w := \{n \in \mathbb{N} : p(x_n - w) \in U_w\}$. On the other hand, consider an open cover $\{U_w : p(w) \in F\}$ of F. Then, we have a finite subcover $U_{w_1}, U_{w_2}, \dots, U_{w_j}$ of F. So, it follows from

$$\{n \in \mathbb{N} : p(x_n - w) \in F\} \subseteq K_{w_1} \cup K_{w_2} \cup \dots \cup K_{w_j}$$

that $\delta(\{n \in \mathbb{N} : p(x_n - w) \in F\}) = 0$, a contradiction. $\square$

4. STATISTICALLY p_τ-CONTINUOUS OPERATOR

In this section, we introduce continuous and bounded operators with respect to the statistical p_τ-convergence. Recall that an operator T between *LNS* (X,p,E) and (Y,q,F) is called

- *p-continuous* whenever $x_n \xrightarrow{p} 0$ in X implies $T(x_n) \xrightarrow{p} 0$ in Y,
- *p-bounded* if it maps p-bounded sets in X to p-bounded sets in Y.

Motivated by these definitions, we introduce the following notions.

Definition 7. Let $T : (X,p,E_\tau) \to (Y,q,F_{\tau'})$ be an operator between *LNLS*s. Then, T is said to be

(1) *statistically p_τ-continuous* if $x_n \xrightarrow{\text{st-p}_\tau} x$ in X implies $T(x_n) \xrightarrow{\text{st-p}_\tau} T(x)$ in Y,
(2) *statistically p_τ-bounded* if it sends statistically p_τ-bounded sequences to statistically p_τ-bounded sequences.

It is clear that the collection of all statistically p_τ-continuous operators between *LNLS*s is a vector space.

Example 9. Consider the *LNLS* $(c_{00}, |\cdot|, \ell_\infty)$, where the solid topology on ℓ_∞ is generated by the supremum norm $\|\cdot\|_\infty$. Define an operator $S : (c_{00}, |\cdot|, \ell_\infty) \to (c_{00}, |\cdot|, c_{00})$ denoted by

$$S(x) = \left(\sum_{n=1}^{\infty} |x_n| \right) x$$

for all $x := (x_n) \in c_{00}$. Consider a sequence $x = (x_n^k) = (x_1^k, x_2^k, \dots)$ in c_{00} denoted by $x_n^k = (x_n^1, x_n^2, \dots) = (1, \dots, 1, 0, 0, \dots)$ in c_{00}. Thus, it is order bounded by the element $(1) = (1, 1, \dots)$ in ℓ_∞, and so, it is topological bounded in ℓ_∞ by applying [2, Theorem 2.19]. Hence, it is statistically p_τ-bounded in c_{00}. Then, it follows that

$$S(x) = (1x_1^k, 2x_2^k, 3x_3^k, \dots, nx_n^k, \dots)$$

is not bounded in c_{00}, and so, S is not a statistically p_τ-bounded operator.

Theorem 6. *If an operator $(E,|\cdot|,E_\tau) \to (F,|\cdot|,F_{\tau'})$ between LNLSs is uniformly continuous, then it is statistically p_τ-continuous.*

Proof. Assume that $T\colon E \to F$ is a uniformly continuous operator. Then, we show that $T\colon (E,|\cdot|,E_\tau) \to (F,|\cdot|,F_{\tau'})$ is statistically p_τ-continuous. Let $x_n \xrightarrow{\text{st-p}_\tau} x$ in E. Take any fixed zero neighborhood V in F. Now, by applying uniform continuity of T, we have some zero neighborhood U in E so that $(a-b) \in U$ implies $T(a-b) \in V$. On the other hand, by $x_n \xrightarrow{\text{st-p}_\tau} x$, we have $\delta(K) = 1$, where $K = \{n \in \mathbb{N} : (x_n - x) \in U\}$. Also, we have $T(x_n - x) \in V$ for every $n \in K$. Then, it follows that $K \subseteq M := \{n \in \mathbb{N} : T(x_n - x) \in V\}$, and so, we obtain $\delta(M) = 1$. Therefore, $T(x_n) \xrightarrow{\text{st-p}_\tau} T(x)$. □

It is well know that every order continuous operator is order bounded; see [3, Lemma 1.54]. But, a statistically p_τ-continuity as an operator between two Riesz spaces need not be order bounded. To see this, we consider Lozanovsky's example; see [3, Exercise 10. p.289].

Example 10. Take an operator $T\colon L_1[0,1] \to c_0$ defined by

$$T(f) = \left(\int_0^1 f(x)\sin x \, dx, \int_0^1 f(x)\sin 2x \, dx, \dots \right).$$

Then, T is not order bounded. But, it can be shown that T is norm continuous. Therefore, by applying Theorem 6, the operator $T\colon (L_1[0,1],|\cdot|,L_1[0,1]) \to (c_0,|\cdot|,c_0)$ is statistically p_τ-continuous because the continuity and the uniform continuity are equivalent for operators between normed spaces.

Proposition 4. *Every p-bounded operator from an LNLS to another LNLS with an order bounded zero neighborhood is statistically p_τ-bounded.*

Proof. Suppose that $T\colon X \to Y$ is a p-bounded operator. We show that

$$T\colon (X,p,E_\tau) \to (Y,q,F_{\tau'})$$

is statistically p_τ-bounded. Let (x_n) be a statistically p_τ-bounded sequence in X. Fix an arbitrary zero neighborhood U in E. Then, there exists $\lambda > 0$ such that $\delta(K) = 1$, where $K := \{n \in \mathbb{N} : p(x_n) \in \lambda U\}$. Then, it follows from [15, Thmeorem 2.2] that the set $\{p(x_n) : n \in K\}$ is order bounded in E, and so, $\{x_n : n \in K\}$ is p-bounded in X. By applying p-boundedness of T, $\{T(x_n) : n \in K\}$ is p-bounded in Y, i.e., $\{q(T(x_n)) : n \in K\}$ is order bounded in F. Thus, it follows from [2, Thmeorem 2.19] that $\{q(T(x_n)) : n \in K\}$ is topologically bounded in F, and so, $T(x_n)$ is statistically p_τ bounded because of $\delta(K) = 1$. Therefore, we get the desired result. □

Theorem 7. *A statistically p_τ-continuous operator between LNLSs is statistically p_τ-bounded.*

Proof. Let $T\colon (X,p,E_\tau) \to (Y,q,F_{\tau'})$ be a statistically p_τ-continuous operator and (x_n) be a statistically p_τ-bounded sequence in X. Take a zero neighborhood U in

E. Then, there exists $\lambda > 0$ such that $\delta(K) = 1$ for the set $K = \{n \in \mathbb{N} : p(x_n) \in \lambda U\}$. Let's consider an index set $M := \mathbb{N} \times K$ with the lexicographic order. That is, $(m,k') \leq (n,k)$ iff $m < n$ or else $m = n$ and $k' \leq k$. Take the sequence $x_{(n,k)} = \frac{1}{\lambda n} x_k$. Thus, we have $p(x_{(n,k)}) = \frac{1}{\lambda n} p(x_k)$, and so, we get $x_{(n,k)} \xrightarrow{\text{st-p}_\tau} 0$ because U is arbitrary and $\frac{1}{\lambda n} p(x_k) \leq \frac{1}{\lambda} p(x_k)$ for every n. By using the statistically p_τ-continuity of T, we obtain $T(x_{(n,k)}) \xrightarrow{\text{st-p}_\tau} 0$. It follows from Proposition 2 that $T(x_{(n,k)})$ is statistically p_τ-bounded sequence in Y. Now, take an arbitrary zero neighborhood W in F. Then, for a fixed $n_0 \in \mathbb{N}$, there is $\alpha > 0$ such that $\delta(J) = 0$, where

$$J = \left\{ k \in K : q(T(x_{n_0,k})) = \frac{1}{\lambda n_0} q(T(x_k)) \notin \alpha U \right\} = \{k \in K : q(T(x_k)) \in \alpha\lambda n_0 \notin U\}.$$

Therefore, we get $T(x_n)$ is statistically p_τ-bounded. □

The lattice operations in an *LNLS* are statistically p_τ-continuous in the following sense.

Theorem 8. *If $x_n \xrightarrow{\text{st-p}_\tau} x$ and $y_n \xrightarrow{\text{st-p}_\tau} y$ in an LNLS, then $x_n \vee y_n \xrightarrow{\text{st-p}_\tau} x \vee y$.*

Proof. Suppose that $x_n \xrightarrow{\text{st-p}_\tau} x$ and $y_n \xrightarrow{\text{st-p}_\tau} y$ in an *LNLS* (X, p, E_τ) and U be an arbitrary zero neighborhood in E with another zero neighborhood V such that $V + V \subseteq U$. Then, we have $\delta(K) = \delta(M) = 1$ for the sets

$$K = \{n \in \mathbb{N} : p(x_n - x) \in V\} \quad \text{and} \quad M = \{n \in \mathbb{N} : p(y_n - y) \in V\}.$$

Take an index set $J := K \cap M$. Then, it follows from [19, Theorem 12.4] that we observe

$$p(x_n \vee y_n - x \vee y) \leq p(x_n - x) + p(y_n - y) \in V + V \subseteq U$$

for all $n \in J$. Then, by using the solidness of U, we have $p(x_n \vee y_n - x \vee y) \in U$ for all $n \in J$. Thus, we get $\delta(\{n \in \mathbb{N} : p(x_n \vee y_n - x \vee y) \in U\}) = 1$. As a result, we have $x_n \vee y_n \xrightarrow{\text{st-p}_\tau} x \vee y$. □

Corollary 1. *Let $x_n \xrightarrow{\text{st-p}_\tau} x$ and $y_n \xrightarrow{\text{st-p}_\tau} y$ hold in an LNLS. Then, we have the following statements:*

(i) $x_n^+ \xrightarrow{\text{st-p}_\tau} x^+$;
(ii) $x_n^- \xrightarrow{\text{st-p}_\tau} x^-$;
(iii) $|x_n| \xrightarrow{\text{st-p}_\tau} |x|$;
(iv) $x_n \wedge y_n \xrightarrow{\text{st-p}_\tau} x \wedge y$.

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