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THE DENSITY OF TUPLES RESTRICTED BY RELATIVELY r-PRIME CONDITIONS

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Abstract. In order to consider j-wise relative r-primality conditions that do not necessarily require all j-tuples of elements in a Dedekind domain to be relatively r-prime, we define the notion of j-wise relative r-primality with respect to a fixed j-uniform hypergraph H. This allows us to provide further generalisations to several results on natural densities not only for a ring of algebraic integers O, but also for the ring $\mathbb{F}_q[x]$.

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1. INTRODUCTION

In 1976, Benkoski proved that the natural density of the set of relatively r-prime m-tuples of positive integers (with $rm > 1$) equals $1/\zeta(rm)$, where ζ is the Riemann zeta function [1]. We note that an m-tuple of positive integers is relatively r-prime if their greatest common rth power divisor is equal to 1.

This acted as a culmination of the work of Mertens [8], Lehmer [7], and Gegenbauer [3]. Thereafter, Tóth [14, 15] and Hu [5] found the natural density of the set of j-wise relatively prime m-tuples of positive integers (where $j \leq m$). Extensions of these results have been made to ideals in a ring of algebraic integers O by Sittinger [11, 13] and subsequently to elements in a ring of algebraic integers as well by Micheli [2] and Sittinger [12]. Moreover, Morrison and Dong [9] as well as Guo, Hou, and Liu [4] gave analogous results for elements in $\mathbb{F}_q[x]$.

We can further generalise the notion of j-wise relatively primality by considering relative primality conditions that require some but not all j-tuples to be relatively prime. A first step in this direction was investigated by Hu [6], who used graphs to notate which pairs of integers are to be relatively prime.

Definition 1. Let D be a Dedekind domain. Fix $r, m \in \mathbb{N}$. We say that $\beta_1, \ldots, \beta_m \in D$ are **relatively r-prime** if $\mathfrak{p}^r \nmid \langle \beta_1, \ldots, \beta_m \rangle$ for any prime ideal $\mathfrak{p} \subseteq D$.

In order to properly generalise the notion of G-wise relative primality, we use the concept of a j-uniform hypergraph H, in which any edge connects exactly j vertices.

Definition 2. Let D be a Dedekind domain. Fix $r, j, m \in \mathbb{N}$ where $j \leq m$, and let H be a simple undirected j-uniform hypergraph whose m vertices are $\beta_1, \ldots, \beta_m \in D$. We say that $\beta_1, \ldots, \beta_m \in D$ are **H-wise relatively r-prime** if any j adjacent vertices of H are relatively r-prime.

A few remarks are now in order. First, although we state the definitions in this generality, we are in particular interested in the cases of a ring of algebraic integers as well the polynomial rings $\mathbb{F}_q[x]$. Next, suppose we take $D = \mathbb{Z}$, $j = 2$, and $r = 1$. Then our hypergraph is a graph G, and Definition 2 reduces to m integers are G-wise relatively prime as defined in [6]. Moreover when $D = O$ and $H = K_m^{(j)}$, the complete j-uniform hypergraph on m vertices, this definition reduces to m elements being j-wise relatively r-prime as defined in [12].

Definition 3. Given a j-uniform hypergraph H, we say that a subset S of vertices from H is an **independent vertex set** if S does not contain any hyperedge of H. Moreover for any non-negative integer k, we let $i_k(H)$ denote the number of independent sets of k vertices in H.

We now state the main results of this article, starting with the algebraic integer case.

Theorem 1. *Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$, and let K be an algebraic number field over $\mathbb{Q}$ with ring of integers O. Then, the density of the set of H-wise relatively r-prime ordered m-tuples of elements in O equals*

$$\prod_{\mathfrak{p}} \left[\sum_{k=0}^{m} i_k(H) \left(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\right)^{m-k} \left(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\right)^{k} \right],$$

where the product is over all nonzero prime ideals in O.

After setting up the pertinent notation in Section 2, we prove Theorem 1.

Since the arithmetic in the rings $\mathbb{Z}$ and $\mathbb{F}_q[x]$ have striking similarities (for further details, see [10]), we would expect that we can derive a H-wise relatively r-prime density statement for $\mathbb{F}_q[x]$. In Section 3, we state and prove an analogue of Theorem 1 for the function field case $\mathbb{F}_q[x]$.

Theorem 2. *Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$. Then the density of the set of H-wise relatively r-prime ordered m-tuples of polynomials in $\mathbb{F}_q[x]$ equals*

$$\prod_{f \text{ irred.}} \left[\sum_{k=0}^{m} i_k(H) \left(1 - \frac{1}{q^{r \deg f}}\right)^{m-k} \left(\frac{1}{q^{r \deg f}}\right)^{k} \right],$$

where it is understood that the product is over all monic irreducible polynomials in $\mathbb{F}_q[x]$.

Remark 1. By noting that $\mathfrak{N}(f) = |\mathbb{F}_q[x]/\langle f\rangle| = q^{\deg f}$, the analogy between this latter density statement and the one given in the algebraic number ring case is made clear.

2. DENSITY OF H-WISE RELATIVELY r-PRIME ELEMENTS IN $\mathcal{O}$

Let K be an algebraic number field of degree n over $\mathbb{Q}$ with $\mathcal{O}$ as its ring of integers having integral basis $\mathcal{B} = \{\alpha_1,\dots,\alpha_n\}$. As a way to generalise the notion of all positive integers less than or equal to some positive constant M, we define

$$\mathcal{O}_{\mathcal{B}}[M] = \Big\{ \sum_{i=1}^{n} c_i\alpha_i \,:\, c_i \in [-M,M)\cap\mathbb{Z} \Big\}.$$

The goal of this section is to derive a H-wise relatively prime density statement in $\mathcal{O}$ by using the methods developed by [2] and [12]. First, we define a notion of density for a subset T of $\mathcal{O}^m$ that reduces to the classic notion of density over $\mathbb{Z}$ as follows.

Definition 4. Let $T \subseteq \mathcal{O}^m$ and fix an integral basis $\mathcal{B}$ of $\mathcal{O}$. The **upper and lower densities of** T **with respect to** $\mathcal{B}$ are respectively defined as

$$\overline{\mathbb{D}}_{\mathcal{B}}(T) = \limsup_{M\to\infty} \frac{|T\cap \mathcal{O}_{\mathcal{B}}[M]^m|}{|\mathcal{O}_{\mathcal{B}}[M]^m|} \text{ and } \underline{\mathbb{D}}_{\mathcal{B}}(T) = \liminf_{M\to\infty} \frac{|T\cap \mathcal{O}_{\mathcal{B}}[M]^m|}{|\mathcal{O}_{\mathcal{B}}[M]^m|}.$$

If $\overline{\mathbb{D}}_{\mathcal{B}}(T) = \underline{\mathbb{D}}_{\mathcal{B}}(T)$, we say that its common value is called the **density of** T **with respect to** $\mathcal{B}$ and denote this as $\mathbb{D}_{\mathcal{B}}(T)$. Whenever this density is independent of the chosen integral basis $\mathcal{B}$, we denote this density as $\mathbb{D}(T)$.

Although the manner in which we cover $\mathcal{O}$ could potentially depend on the choice of the given integral basis $\mathcal{B}$, it is a direct corollary to Theorem 1 that the density of the set of H-wise relatively r-prime elements in $\mathcal{O}$ is actually independent of the integral basis used.

For the remainder of this section, let S be a finite set of rational primes, and fix positive integers r, j, m such that $j \le m$. Fix a j-uniform hypergraph H, and define E_S to be the set of m-tuples $z = (z_1,\dots,z_m)$ in $\mathcal{O}^m$ such that any ideal generated by j entries of z is H-wise relatively r-prime with respect to all $\mathfrak{p} \mid \langle p\rangle$ for each $p \in S$. That is, E_S consists of the H-wise relatively r-prime m-tuples of algebraic integers from $\mathcal{O}$ with respect to S.

In order to aid us in analysing E_S, let

$$\pi\colon\, \mathcal{O}^m \to \Big(\prod_{\substack{\mathfrak{p}\mid\langle p\rangle \\ p\in S}} \mathcal{O}/\mathfrak{p}^r \Big)^m$$

be the surjective homomorphism induced by the family of natural projections

$$\pi_{\mathfrak{p}^r}\colon\, \mathcal{O} \to \mathcal{O}/\mathfrak{p}^r \text{ for all } \mathfrak{p} \mid \langle p\rangle \text{ where } p \in S.$$

From the definition of H-wise relative r-primality of algebraic integers, we immediately deduce the following lemma.

Lemma 1. *For a given prime ideal $\mathfrak{p} \mid \langle p \rangle$ where $p \in S$ and $k \in \{1,2,\ldots,m\}$, let $A_k^{(\mathfrak{p})}$ denote the set of elements in $(O/\mathfrak{p}^r)^m$ where exactly k of their m components are 0, and these k components form an independent vertex set in H. Then,*

$$E_S = \pi^{-1}\Big(\prod_{\substack{\mathfrak{p}\mid\langle p\rangle \\ p\in S}} \bigcup_{k=0}^{m} A_k^{(\mathfrak{p})}\Big).$$

Proposition 1. *Suppose that $\mathfrak{p}$ is a prime ideal in O that lies above a fixed rational prime p, and let $D_p = \sum_{\mathfrak{p}\mid\langle p\rangle} f_p$ where f_p denotes the inertial degree of $\mathfrak{p}$. If we fix $q \in \mathbb{N}$ and set $N = \prod_{p\in S} p^r$, then*

$$|E_S \cap O_{\mathcal{B}}[qN]^m| = (2q)^{mn} \prod_{\substack{\mathfrak{p}\mid\langle p\rangle \\ p\in S}} p^{rm(n-D_p)} \Big[\sum_{k=0}^{m} i_k(H)\big(\mathfrak{N}(\mathfrak{p}^r)-1\big)^{m-k}\mathfrak{N}(\mathfrak{p}^r)^k\Big].$$

Proof. We first examine the map π. For brevity, we set $R_p = \prod_{\mathfrak{p}\mid\langle p\rangle} O/\mathfrak{p}^r$. Then we let π_N denote the reduction modulo N homomorphism, and $\psi = (\psi_p)_{p\in S}$ where $\psi_p : (O/\langle p\rangle^r)^m \to R_p^m$ is the homomorphism induced by the projection maps $O/\langle p\rangle^r \to R_p$. Finally, let $\overline{\psi}$ be its extension to $(O/\langle N\rangle)^m$ (by applying the Chinese Remainder Theorem to the primes in S). These maps are related to each other through the following diagram

$$\begin{array}{ccccc} O^m & \xrightarrow{\pi_N} & (O/\langle N\rangle)^m & \xrightarrow{\overline{\psi}} & (\prod_{p\in S} R_p)^m \\ & & \cong\downarrow & & \downarrow = \\ & & (\prod_{p\in S} O/\langle p^r\rangle)^m & \xrightarrow{\psi} & (\prod_{p\in S} R_p)^m \end{array}$$

and it follows that $\pi = \overline{\psi}\circ\pi_N$.

To prove this proposition, we start by examining ψ^{-1}. Since for each rational prime p the mapping $\psi_p : (O/\langle p^r\rangle)^m \to R_p^m$ is a surjective free $\mathbb{Z}_{p^r}$-module homomorphism, we have for all $y \in (\prod_{p\in S} R_p)^m$:

$$|\overline{\psi}^{-1}(y)| = \prod_{p\in S} |\psi_p^{-1}(y_p)| = \prod_{p\in S} |\ker\psi_p| = \prod_{p\in S} p^{rm(n-D_p)}.$$

Next, we compute $\left|\pi_N^{-1}(z) \cap O_{\mathcal{B}}[qN]^m\right|$. Given $\bar{z} = (\overline{z_1},\ldots,\overline{z_m}) \in (O/\langle N\rangle)^m$, observe that since $O/\langle N\rangle$ is a free $\mathbb{Z}_N$-module with basis $\{\pi(\alpha_1),\ldots,\pi(\alpha_n)\}$, there exist unique $c_t^j \in [0,N)\cap\mathbb{Z}$ such that

$$\overline{z_j} = \sum_{t=1}^{n} c_t^j \pi(\alpha_t).$$

Then for $z = (z_1, \dots, z_m) \in \mathcal{O}^m$, it follows that $\pi_N(z) = \bar{z}$ if and only if

$$z_j = \sum_{t=1}^{n} (c_t^j + l_t^j N)\alpha_t$$

for some $l_t^j \in \mathbb{Z}$. Moreover, since we need $l_t^j \in [-q, q) \cap \mathbb{Z}$ for each pair of indices j and t, we deduce that

$$\left| \pi_N^{-1}(z) \cap \mathcal{O}_{\mathcal{B}}[qN]^m \right| = (2q)^{mn}.$$

We are ready to compute $\left| E_S \cap \mathcal{O}_{\mathcal{B}}[qN]^m \right|$. By the definition of $A_k^{(\mathfrak{p})}$, we have for any fixed k and $\mathfrak{p}$:

$$|A_k^{(\mathfrak{p})}| = i_k(H) \big(\mathfrak{N}(\mathfrak{p}^r) - 1 \big)^{m-k} \mathfrak{N}(\mathfrak{p}^r)^k.$$

Since we know from the last lemma that $E_S = \pi^{-1}(J)$, where

$$J = \psi^{-1} \Big(\prod_{\substack{\mathfrak{p} \mid \langle p \rangle \\ p \in S}} \bigcup_{k=0}^{m} A_k^{(\mathfrak{p})} \Big),$$

it immediately follows that

$$|J| = \prod_{\substack{\mathfrak{p} \mid \langle p \rangle \\ p \in S}} p^{rm(n-D_p)} \sum_{k=0}^{m} i_k(H) \big(\mathfrak{N}(\mathfrak{p}^r) - 1 \big)^{m-k} \mathfrak{N}(\mathfrak{p}^r)^k.$$

Therefore, we conclude that

$$\begin{aligned} \left| E_S \cap \mathcal{O}_{\mathcal{B}}[qN]^m \right| &= (2q)^{mn} |J| \\ &= (2q)^{mn} \prod_{\substack{\mathfrak{p} \mid \langle p \rangle \\ p \in S}} p^{rm(n-D_p)} \Big[\sum_{k=0}^{m} i_k(H) \big(\mathfrak{N}(\mathfrak{p}^r) - 1 \big)^{m-k} \mathfrak{N}(\mathfrak{p}^r)^k \Big], \end{aligned}$$

as desired. □

We now compute the density of E_S.

Lemma 2. *Using the previous notation, we have for any integral basis $\mathcal{B}$ of $\mathcal{O}$,*

$$\mathbb{D}(E_S) = \mathbb{D}_{\mathcal{B}}(E_S) = \prod_{\substack{\mathfrak{p} \mid \langle p \rangle \\ p \in S}} \Big[\sum_{k=0}^{m} i_k(H) \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)} \Big)^{m-k} \Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)} \Big)^k \Big].$$

Proof. Define the sequence $\{a_j\}$ by $a_j = \dfrac{|E_S \cap \mathcal{O}_{\mathcal{B}}[j]^m|}{|\mathcal{O}_{\mathcal{B}}[j]^m|}$, and let D denote the value of the density in question.

First, we consider the subsequence $\{a_{qN}\}_{q\in\mathbb{N}}$, where $N=\prod_{p\in S}p^r$. We claim that this subsequence is constant. By the previous proposition along with the definitions for N and D_p,

$$a_{qN}=\frac{1}{(2qN)^{mn}}\Big[(2q)^{mn}\cdot\prod_{\substack{\mathfrak{p}\mid\langle p\rangle\\ p\in S}}p^{rm(n-D_p)}\sum_{k=0}^{m}i_k(H)\Big(\mathfrak{N}(\mathfrak{p}^r)-1\Big)^{m-k}\mathfrak{N}(\mathfrak{p}^r)^k\Big]$$

$$=\prod_{\substack{\mathfrak{p}\mid\langle p\rangle\\ p\in S}}\Big[\sum_{k=0}^{m}i_k(H)\Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{m-k}\Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^k\Big].$$

Hence, $\{a_{qN}\}$ is a constant subsequence and converges to D.

Next, we show that $\{a_{c+qN}\}$ also converges to D for any $c\in\{1,2,\ldots,N-1\}$, we first find bounds for a_{c+qN}. To this end, note that

$$a_{qN}\Big(\frac{2qN}{2c+2qN}\Big)^{mn}\le a_{c+qN}\le a_{(q+1)N}\Big(\frac{2(q+1)N}{2c+2qN}\Big)^{mn}.$$

By letting $q\to\infty$ and applying the Squeeze Theorem, we conclude that $\{a_{c+qN}\}$ converges to D for any $c\in\{1,2,\ldots,N-1\}$. Finally, since $\{a_{c+qN}\}$ converges to D for any $c\in\{0,1,\ldots,N-1\}$, we conclude that $\{a_j\}$ converges to D. □

Note that the density in Lemma 2 is independent of the integral basis $\mathcal{B}$ used. Now we are ready to establish to the main theorem of this section. For convenience, we restate it here before proving it.

Theorem 3. *Fix $r,j,m\in\mathbb{N}$ such that $j\le m$ and $rm\ge 2$, and let K be an algebraic number field over $\mathbb{Q}$ with ring of integers $\mathcal{O}$. Then, the density of the set E consisting of H-wise relatively r-prime ordered m-tuples of elements in $\mathcal{O}$ equals*

$$\prod_{\mathfrak{p}}\Big[\sum_{k=0}^{m}i_k(H)\Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{m-k}\Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^k\Big],$$

where the product is over all nonzero prime ideals in $\mathcal{O}$.

Proof. Fix $t\in\mathbb{N}$ and let S_t denote the set of the first t rational primes. For brevity, we write $E_t=E_{S_t}$. Since $E_t\supseteq E$,

$$\overline{\mathbb{D}}_{\mathcal{B}}(E)\le\overline{\mathbb{D}}_{\mathcal{B}}(E_t)=\mathbb{D}(E).$$

Observe that the last equality is due to the existence of $\mathbb{D}(E)$. Letting $t\to\infty$,

$$\mathbb{D}_{\mathcal{B}}(E)\le\prod_{\mathfrak{p}}\Big[\sum_{k=0}^{m}i_k(H)\Big(1-\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{m-k}\Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^k\Big].$$

It remains to show the opposite inequality. Noting that $\mathbb{D}_{\mathcal{B}}(E_t)-\overline{\mathbb{D}}_{\mathcal{B}}(E_t\backslash E)\le\underline{\mathbb{D}}_{\mathcal{B}}(E)$, it suffices to show that $\lim_{t\to\infty}\overline{\mathbb{D}}_{\mathcal{B}}(E_t\backslash E)=0$.

To this end, we introduce the following notation. Let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$, p_t be the t^{th} rational prime, and M be a positive integer.

(1) We write $\mathfrak{p} \succ M$ iff $\mathfrak{p}$ lies over a rational prime greater than M.
(2) We write $M \succ \mathfrak{p}$ iff the rational prime lying under $\mathfrak{p}$ is less than M.

Using this notation, we can write

$$E_t \backslash E \subseteq \bigcup_{\mathfrak{p} \succ p_t} \Big(\prod_{j=1}^{m} \mathfrak{p}^r \Big) \subseteq O^m,$$

where it is understood that $\prod_{j=1}^m \mathfrak{p}^r$ is the subset of O^m such that each entry of the m-tuple is an element of $\mathfrak{p}^r$. Then, we see that

$$(E_t \backslash E) \cap O_{\mathcal{B}}[M]^m \subseteq \bigcup_{CM^n \succ \mathfrak{p} \succ p_t} \prod_{j=1}^{m} \Big(\mathfrak{p}^r \cap O_{\mathcal{B}}[M] \Big)$$

for some constant $C > 0$ dependent only on $\mathcal{B}$, and thus

$$\overline{\mathbb{D}}_{\mathcal{B}}(E_t \backslash E) \leq \limsup_{M \to \infty} \sum_{CM^n \succ \mathfrak{p} \succ p_t} |(\mathfrak{p}^r \cap O_{\mathcal{B}}[M])^m| \cdot (2M)^{-mn}.$$

By [2, Proposition 13], there exist constants $c, d > 0$ independent of M and $\mathfrak{p}$ such that

$$|(\mathfrak{p}^r \cap O_{\mathcal{B}}[M])^m| \leq \frac{(2M)^{mn}}{\mathfrak{N}(\mathfrak{p}^r)^m} + c\Big(\frac{2M}{d\, \mathfrak{N}(\mathfrak{p}^r)^{1/n}} + 1 \Big)^{mn-1}.$$

Using this bound along with the facts that $\mathfrak{N}(\mathfrak{p}) \geq p$ for every $\mathfrak{p}$ lying above a fixed rational prime p, and at most n prime ideals lie above a fixed rational prime, we obtain

$$\begin{aligned} \overline{\mathbb{D}}_{\mathcal{B}}(E_t \backslash E) &\leq \limsup_{M \to \infty} \sum_{CM^n \succ \mathfrak{p} \succ p_t} \Big[\frac{1}{\mathfrak{N}(\mathfrak{p}^r)^m} + c\Big(\frac{2M}{d\, \mathfrak{N}(\mathfrak{p}^r)^{1/n}} + 1 \Big)^{mn-1} (2M)^{-mn} \Big] \\ &\leq \limsup_{M \to \infty} \sum_{CM^n > p > p_t} \Big[\frac{n}{p^{rm}} + cn\Big(\frac{2M}{d p^{r/n}} + 1 \Big)^{mn-1} (2M)^{-mn} \Big]. \end{aligned}$$

It remains to show that the right side goes to 0 as $t \to \infty$. First, observe that for all sufficiently large M, we have $2M/dp^{r/n} > 1$ and thus

$$\Big(\frac{2M}{d p^{r/n}} + 1 \Big)^{mn-1} (2M)^{-mn} < \Big(\frac{2}{d} \Big)^{mn} \cdot \frac{1}{p^{rm}}.$$

Then, by writing $A = n + cn(2/d)^{mn}$ which is a constant independent of M and p, we deduce that

$$\overline{\mathbb{D}}_{\mathcal{B}}(E_t \backslash E) \leq \limsup_{M \to \infty} \sum_{CM^n > p > p_t} \frac{A}{p^{rm}} \leq \sum_{k=p_t}^{\infty} \frac{A}{k^{rm}}$$

for all sufficiently large M.

Finally since $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{rm}}$ is convergent, we conclude that $\overline{\mathbb{D}}_{\mathcal{B}}(E_t \backslash E) = 0$. □

To conclude this section, we now state a corollary that indicates how this main result provides a generalisation of the work from [12].

Corollary 1. *Fix $r, j, m \in \mathbb{N}$ such that $j \leq m$ and $rm \geq 2$, and let K be an algebraic number field over $\mathbb{Q}$ with ring of integers O. Then the density of the set of j-wise relatively r-prime ordered m-tuples of elements in O equals*

$$\prod_{\mathfrak{p}} \Big[\sum_{k=0}^{j-1} \binom{m}{k} \Big(1 - \frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{m-k} \Big(\frac{1}{\mathfrak{N}(\mathfrak{p}^r)}\Big)^{k} \Big].$$

Proof. Take $H = K_m^{(j)}$ as the hypergraph, and observe that

$$i_k(H) = \begin{cases} \binom{m}{k} & \text{if } 0 \leq k \leq j-1 \\ 0 & \text{otherwise.} \end{cases}$$

Applying Theorem 3 immediately yields the desired result. □

3. Density of H-wise relatively r-prime elements in $\mathbb{F}_q[x]$

Let $\mathbb{F}_q[x]$ be the ring of polynomials over the finite field $\mathbb{F}_q$ where $q = p^k$ for some prime p and $k \in \mathbb{N}$. The goal of this section is to derive a H-wise density statement in $\mathbb{F}_q[x]$ by using methods developed in [4].

In order to define a suitable definition of density in $\mathbb{F}_q[x]$, we begin by giving an enumeration of the polynomials in $\mathbb{F}_q[x]$. Denoting the elements of $\mathbb{F}_q$ as $a_0 = 0$, $a_1, \dots, a_{q-1}$, let Σ be the set of all $(a_{d_0}, a_{d_1}, a_{d_2}, \dots)$ whose entries are in $\mathbb{F}_q$ and $d_i = 0$ for all sufficiently large i. Then since non-negative integers have a unique expansion base q, where q is a positive integer greater than 1, we have a bijection $\Phi\colon \Sigma \to \mathbb{Z}_{\geq 0}$ defined by

$$\Phi(a_{d_0}, a_{d_1}, \dots) = \sum_{i=0}^{\infty} d_i q^i.$$

Using this bijection, we define for each $j \in \mathbb{Z}_{\geq 0}$

$$f_j(x) = \sum_{i=0}^{\infty} a_{d_i} x^i, \text{ where } j = \phi(a_{d_0}, a_{d_1}, \dots).$$

Note that $\mathbb{F}_q[x] = \{f_j(x) : j \in \mathbb{Z}_{\geq 0}\}$, thereby giving an ordering of the elements in $\mathbb{F}_q[x]$. Now, we are able to define a density in this ring.

Definition 5. Fix a positive integer $m \geq 2$, and let $\mathcal{M}_N$ be the subset of $(\mathbb{F}_q[x])^m$ consisting of m-tuples of elements in $\mathbb{F}_q[x]$ whose entries are taken from $\{f_0, f_1, \dots, f_N\}$. For any subset $T \subseteq (\mathbb{F}_q[x])^m$, we define the **upper and lower densities of** T are respectively defined as

$$\overline{\mathbb{D}}(T) = \limsup_{N\to\infty} \frac{|T \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \text{ and } \underline{\mathbb{D}}(T) = \liminf_{N\to\infty} \frac{|T \cap \mathcal{M}_N|}{|\mathcal{M}_N|}.$$

If $\overline{\mathbb{D}}(T) = \underline{\mathbb{D}}(T)$, we say that its common value is called the **density of** T and denote this as $\mathbb{D}(T)$.

Let S be a finite set of irreducible polynomials in $\mathbb{F}_q[x]$, and fix $r, j, m \in \mathbb{N}$ satisfying $j \leq m$. Fix a j-uniform hypergraph H, and let E_S denote the set of m-tuples of polynomials from $\mathbb{F}_q[x]$ that are H-wise relatively r-prime with respect to all irreducible polynomials in S.

For the following lemma and proposition, let

$$\pi\colon (\mathbb{F}_q[x])^m \to \Big(\prod_{f\in S}\mathbb{F}_q[x]/\langle f^r\rangle\Big)^m$$

be the surjective homomorphism induced by the family of natural projections

$$\pi_{f^r}: \mathbb{F}_q[x] \to \mathbb{F}_q[x]/\langle f^r\rangle \text{ for each } f \in S.$$

As in the algebraic integer case, the following lemma follows immediately from the definition of H-wise relative r-primality of elements in $\mathbb{F}_q[x]$.

Lemma 3. *For a given irreducible polynomial $f \in S$, let $A_k^{(f)}$ denote the set of elements in $(\mathbb{F}_q[x]/\langle f^r\rangle)^m$ where exactly k of their m components are 0, and these k components form an independent vertex set in H. Then,*

$$E_S = \pi^{-1}\Big(\prod_{f\in S}\bigcup_{k=0}^{m} A_k^{(f)}\Big).$$

Proposition 2. *Let $N = bq^{\deg F} - 1$ where $b \in \mathbb{N}$, and $F = \prod_{f\in S} f^r$. Then,*

$$\big|E_S \cap \mathcal{M}_N\big| = (bq^{\deg F})^m \prod_{f\in S} q^{-rm\deg f} \cdot \sum_{k=0}^{m} i_k(H)(q^{r\deg f} - 1)^{m-k}(q^{r\deg f})^k.$$

Proof. Let π_F denote the reduction modulo F homomorphism, and let

$$\psi\colon (\mathbb{F}_q[x]/\langle F\rangle)^m \to \Big(\prod_{f\in S}(\mathbb{F}_q[x]/\langle f^r\rangle\Big)^m \to \prod_{f\in S}(\mathbb{F}_q[x]/\langle f^r\rangle)^m,$$

where the first part of ψ is induced by the Chinese Remainder Theorem and the second part is an obvious isomorphism of free $\mathbb{F}_q[x]$-modules.

Now we compute $|\pi_F^{-1}\big(h(x)\big) \cap \mathcal{M}_N|$. By the Division Algorithm, we have that

$$\{f_l(x)\}_{l=0}^{N} = \{f_s(x)\cdot x^{\deg F} + f_t(x) \mid 0 \leq t \leq q^{\deg F} - 1 \,\text{and}\, 0 \leq s \leq b-1\}.$$

Then for any fixed $s \in \{0, 1, \dots, b-1\}$, the map π_F restricted to

$$\{f_s(x)\cdot x^{\deg F} + f_t(x)\}_{t=0}^{q^{\deg F}-1} \to \mathbb{F}_q[x]/\langle F\rangle$$

is one-to-one. Since $|\ker(\pi_F)| = b^m$, we conclude that $|\pi_F^{-1}\big(h(x)\big) \cap \mathcal{M}_N| = b^m$.

We are now ready to compute $|E_S \cap \mathcal{M}_N|$. We know that $E_S = \pi^{-1}(J)$, where

$$J = \psi^{-1}\Big(\prod_{f\in S}\bigcup_{k=0}^{m} A_k^{(f)}\Big).$$

Since for any fixed $k \in \{0,1,\ldots,m\}$ and $f \in S$ we have

$$|A_k^{(f)}| = i_k(H)(q^{r\deg f}-1)^{m-k}(q^{r\deg f})^k,$$

we deduce that

$$|J| = q^{m\deg F}\prod_{f\in S} q^{-rm\deg f}\cdot\sum_{k=0}^{m} i_k(H)(q^{r\deg f}-1)^{m-k}(q^{r\deg f})^k.$$

Therefore,

$$\begin{aligned}|E_S\cap\mathcal{M}_N| &= b^m\cdot|J|\\ &= (bq^{\deg F})^m\prod_{f\in S} q^{-rm\deg f}\cdot\sum_{k=0}^{m} i_k(H)(q^{r\deg f}-1)^{m-k}(q^{r\deg f})^k.\end{aligned}$$

□

We now find the density of E_S.

Lemma 4. *Using the notation from Proposition 2,*

$$\mathbb{D}(E_S) = \prod_{f\in S}\left[\sum_{k=0}^{m} i_k(H)\Big(1-\frac{1}{q^{r\deg f}}\Big)^{m-k}\Big(\frac{1}{q^{r\deg f}}\Big)^k\right].$$

Proof. Let $a_j = \dfrac{|E_S\cap\mathcal{M}_j|}{|\mathcal{M}_j|}$ and let D be the value of the density in question. For notational brevity, we let $n = q^{\deg F}$.

We first consider the subsequence $\{a_{bn-1}\}_{b\in\mathbb{N}}$. By Proposition 2, we find that

$$\frac{|E_S\cap\mathcal{M}_{bn-1}|}{|\mathcal{M}_{bn-1}|} = \prod_{f\in S}\left[\sum_{k=0}^{m} i_k(H)\Big(1-\frac{1}{q^{r\deg f}}\Big)^{m-k}\Big(\frac{1}{q^{r\deg f}}\Big)^k\right].$$

Hence, $\{a_{bn-1}\}$ trivially converges to D.

Next, we show $\{a_{bn+c}\}$ converges to D as well for each $c\in\{0,1,\ldots,n-2\}$. In a manner reminiscent of the proof to Lemma 4, we find that

$$\Big(\frac{bn}{bn+c+1}\Big)^m a_{bn-1}\le a_{bn+c}\le\Big(\frac{(b+1)n}{(b+1)n+c+1}\Big)^m a_{(b+1)n-1}.$$

Letting $b\to\infty$, the Squeeze Theorem implies that $\{a_{bn+c}\}$ converges to D for each $c\in\{0,1,\ldots,n-2\}$. Finally, since $\{a_{bn+c}\}$ converges to D for each $c\in\{0,1,\ldots,n-1\}$, we conclude that $\{a_j\}$ converges to D, as desired. □

Now we are ready to state and prove the main theorem of this section.

Theorem 4. *Fix $r,j,m\in\mathbb{N}$ such that $j\le m$ and $rm\ge 2$. Then the density of the set of H-wise relatively r-prime ordered m-tuples of polynomials in $\mathbb{F}_q[x]$ equals*

$$\prod_{f \text{ irred.}}\left[\sum_{k=0}^{m} i_k(H)\Big(1-\frac{1}{q^{r\deg f}}\Big)^{m-k}\Big(\frac{1}{q^{r\deg f}}\Big)^k\right],$$

where it is understood that the product is over all monic irreducible polynomials in $\mathbb{F}_q[x]$.

Proof. Fix a monic irreducible polynomial $f \in \mathbb{F}_q[x]$ and let K_f denote the set of ordered m-tuples $(g_1, \ldots, g_m)$ such that f divides the gcd of k of the entries from $(g_1, \ldots, g_m)$ whenever these k entries form an independent vertex set. Then by Lemma 4, we have

$$\mathbb{D}(K_f) = 1 - \sum_{k=0}^{m} i_k(H)\Big(1 - \frac{1}{q^{r \deg f}}\Big)^{m-k}\Big(\frac{1}{q^{r \deg f}}\Big)^{k}.$$

However for any $x \in [0,1]$, Bernoulli's Inequality implies that

$$\begin{aligned}\sum_{k=0}^{m} i_k(H)x^k(1-x)^{m-k} &\geq (1-x)^m + mx(1-x)^{m-1} \\ &= (1-x)^{m-1}(1+(m-1)x) \\ &\geq (1-(m-1)x)(1+(m-1)x) \\ &= 1-(m-1)^2x^2.\end{aligned}$$

Therefore, letting $x = q^{-\deg f}$ yields

$$\mathbb{D}(K_f) \leq \Big(\frac{m-1}{q^{r \deg f}}\Big)^2.$$

Next, let S_t be the set of monic irreducible polynomials of a degree greater or equal to t where $t \in \mathbb{N}$, and set $E_t = E_{S_t}$. Moreover, let $\hat{S}$ be the set of all monic irreducible polynomials in $\mathbb{F}_q[x]$. Then,

$$\begin{aligned}\overline{\mathbb{D}}(E_t \backslash E) &\leq \limsup_{N\to\infty} \frac{|(\bigcup_{f \in \hat{S}\backslash S_t} K_f) \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \\ &\leq \limsup_{N\to\infty} \frac{\sum_{f \in \hat{S}\backslash S_t} |K_f \cap \mathcal{M}_N|}{|\mathcal{M}_N|} \\ &\leq \sum_{f \in \hat{S}\backslash S_t} \overline{\mathbb{D}}(K_f).\end{aligned}$$

Since $\overline{\mathbb{D}}(K_f) = \mathbb{D}(K_f)$, we obtain

$$\begin{aligned}\overline{\mathbb{D}}(E_t \backslash E) &\leq \sum_{f \in \hat{S}\backslash S_t} \mathbb{D}(K_f) \\ &\leq \sum_{f \in \hat{S}\backslash S_t} \Big(\frac{m-1}{q^{r \deg f}}\Big)^2 \\ &= \sum_{j=t+1}^{\infty} \frac{(m-1)^2}{q^{2rj}} \cdot \varphi(j),\end{aligned}$$

where $\varphi(j)$ denotes the number of monic irreducible polynomials of degree j in $\mathbb{F}_q[x]$.

Since any irreducible polynomial over $\mathbb{F}_q[x]$ with degree j divides $x^{q^j} - x$ (which has no multiple roots), we have $j \cdot \varphi(j) \le q^j$. Therefore

$$\overline{\mathbb{D}}(E_t \backslash E) \le \sum_{j=t+1}^{\infty} \frac{(m-1)^2}{jq^{(2r-1)j}} \le \frac{(m-1)^2}{q^t(q-1)},$$

in which the last inequality follows from

$$\begin{aligned}\sum_{j=t+1}^{\infty} \frac{1}{jq^{(2r-1)j}} &= \frac{1}{q^{(2r-1)(t+1)}} \cdot \sum_{j=0}^{\infty} \frac{1}{(j+t+1)q^{(2r-1)j}} \\ &\le \frac{1}{q^{(2r-1)(t+1)}} \cdot \sum_{j=0}^{\infty} \frac{1}{q^{(2r-1)j}} \\ &\le \frac{1}{q^t(q-1)}.\end{aligned}$$

Next, since $E \cap \mathcal{M}_N \subseteq E_t \cap \mathcal{M}_N$, it follows that

$$\overline{\mathbb{D}}(E) \le \overline{\mathbb{D}}(E_t) \le \mathbb{D}(E_t).$$

Similarly, since $E \cap \mathcal{M}_N = (E_t \cap \mathcal{M}_N) - \big((E_t \backslash E) \cap \mathcal{M}_N\big)$, we obtain

$$\begin{aligned}\underline{\mathbb{D}}(E) &\ge \underline{\mathbb{D}}(E) - \overline{\mathbb{D}}(E \backslash E_t) \\ &\ge \mathbb{D}(E_t) - \frac{(m-1)^2}{q^t(q-1)}.\end{aligned}$$

Finally noting that $\mathbb{D}(E_t)$ exists, we conclude by letting $t \to \infty$ that

$$\begin{aligned}\mathbb{D}(E) &= \lim_{t\to\infty} \mathbb{D}(E_t) \\ &= \lim_{t\to\infty} \prod_{f \in S_t} \Big[\sum_{k=0}^{m} i_k(H)\Big(1 - \frac{1}{q^{r \deg f}}\Big)^{m-k}\Big(\frac{1}{q^{r \deg f}}\Big)^{k}\Big] \\ &= \prod_{f \text{ irred.}} \Big[\sum_{k=0}^{m} i_k(H)\Big(1 - \frac{1}{q^{r \deg f}}\Big)^{m-k}\Big(\frac{1}{q^{r \deg f}}\Big)^{k}\Big],\end{aligned}$$

and this concludes the proof. □

In a manner reminiscent of the previous section, we conclude by giving without proof the analogue of Corollary 2 for $\mathbb{F}_q[x]$ as originally given in [4].

Corollary 2. *Fix $r, j, m \in \mathbb{N}$ such that $j \le m$ and $rm \ge 2$. Then the density of the set of j-wise relatively r-prime ordered m-tuples of elements in $\mathbb{F}_q[x]$ equals*

$$\prod_{f \textit{ irred.}} \Big[\sum_{k=0}^{j-1} \binom{m}{k}\Big(1 - \frac{1}{q^{r \deg f}}\Big)^{m-k}\Big(\frac{1}{q^{r \deg f}}\Big)^{k}\Big],$$

where it is understood that the product is over all monic irreducible polynomials in $\mathbb{F}_q[x]$.

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