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ON THE LOCAL EXISTENCE OF SOLUTIONS TO p-LAPLACIAN EQUATION WITH LOGARITHMIC NONLINEARITY AND NONLINEAR DAMPING TERM

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Abstract. This paper is concerned with the interaction between logarithmic source term and p-Laplacian term for nonlinear damped semilinear wave equation. We established the local existence and uniquenes under appropriate conditions.

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1. INTRODUCTION

In this work, we investigate the following p-Laplacian hyperbolic type equation with logarithmic nonlinearity and nonlinear damping

$$\begin{cases} u_{tt} - \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) + |u_t|^{k-2}u_t = |u|^{q-2}u\ln|u|, & x\in\Omega,\ t>0,\\ u(x,0)=u_0(x),\ \ u_t(x,0)=u_1(x), & x\in\Omega,\\ u(x,t)=0, & x\in\partial\Omega,\ t\geq 0,\end{cases} \tag{1.1}$$

where $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ and $u_1 \in H_0^1(\Omega)$ are given initial data and $\Omega \subset R^n$ $(n \geq 1)$ is a bounded domain with smooth boundary $\partial\Omega$. The parameter $k \geq 2$ and the exponents p, q satisfy

$$2 < p < q < p\left(1+\frac{2}{n}\right). \tag{1.2}$$

The logarithmic nonlinearity occured naturally in quantum mechanics, inflation cosmolog, supersymmetric field theories, and a lot of different areas of physics such as, optics, geophysics and nuclear physics It was a classical field equation whose popularity increased especially when it was shown in [1, 3]. The qualitative behavior of solutions for problems with logarithmic nonlinearity in the absence of the

p-Laplacian term

$$u_{tt} - \Delta u + f(u_t) = |u|^{q-2} u \ln |u|$$

have attracted the attention of several mathematicians. Some of the based work in this subject are [2, 5–7, 9, 10, 12, 16, 18, 19]. In [14], Nhan and Truong investigated

$$u_t - \operatorname{div}\left(|\nabla u|^{p-2} \nabla u\right) - \Delta u_t = |u|^{p-2} u \ln |u|, \tag{1.3}$$

and they established the global existence, blow up and decay of the solutions for $p > 2$. The problem (1.3) was studied by Cao and Liu[4] and they proved global boundedness and blowing-up at ∞ for $1 < p < 2$. Ding and Zhou [8] studied the problem (1.3) replaced $|u|^{p-2} u \ln |u|$ with $|u|^{q-2} u \ln |u|$. They established global existence, blow up in finite time and blow up at infinite time. He et al. [11] studied the decay of solutions the same problem. Our aim in this study will be existence of solution hyperbolic type equation with logarithmic source term and p-Laplacian term.

2. Preliminaries

In order to state the main results to problem 1.1 more clearly, we start to our work by introducing some notations, lemmas and definitions which will be used in this paper. Throughout this paper, we denote

$$\|u\|_m = \|u\|_{L^m(\Omega)}, \qquad \|u\|_{1,m} = \|u\|_{W_0^{1,m}(\Omega)} = \left(\|u\|_m^m + \|\nabla u\|_m^m\right)^{\frac{1}{m}}$$

for $1 < m < \infty$. We consider $W_0^{-1,m'}(\Omega)$ to denote the dual space of $W_0^{1,m'}(\Omega)$ where m' is Hölder conjugate exponent for $m > 1$.

We define energy function as follows

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \ln |u|\, dx + \frac{1}{q^2}\|u\|_q^q.$$

Let us define some useful funcionals as follows

$$J(u) = \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q \ln |u|\, dx + \frac{1}{q^2}\|u\|_q^q, \tag{2.1}$$

and

$$I(u) = \|\nabla u\|_p^p - \int_\Omega |u|^q \ln |u|\, dx. \tag{2.2}$$

By the Gagliardo-Nirenberg multiplicative embedding inequality that $J(u)$ and $I(u)$ are continuous. Then, by (2.1) and (2.2), it tells us that

$$J(u) = \frac{1}{q} I(u) + \left(\frac{1}{p} - \frac{1}{q}\right)\|\nabla u\|_p^p + \frac{1}{q^2}\|u\|_q^q \tag{2.3}$$

and

$$E(t) = \frac{1}{2}\|u_t\|^2 + J(u).$$

We can define the mountain-pass level

$$d = \inf_{u \in \aleph} J(t), \tag{2.4}$$

where $\aleph$ is the Nehari manifold, which is defined by

$$\aleph = \left\{u \in W_0^{1,p}(\Omega) \setminus \{0\} : I(u) = 0\right\}.$$

As in [17], we put the potential well depth

$$0 < d = \inf_u \left\{ \sup_{\lambda \geq 0} J(\lambda u) : u \in W_0^{1,p}(\Omega), \|u\|_p^p \neq 0 \right\}.$$

Now, we introduce the potential well U and its corresponding set K

$$U = \left\{u \in W_0^{1,p}(\Omega) : I(u) > 0,\ J(u) < d\right\} \cup \{0\},$$
$$K = \left\{u \in W_0^{1,p}(\Omega) : I(u) < 0,\ J(u) < d\right\}.$$

Lemma 1. *For any $u \in W_0^{1,p}(\Omega)$, we get*

$$\|u\|_s \leq C_p \|\nabla u\|_p, \text{ for } u \in H_0^1(\Omega)$$

for all $1 \leq p \leq \frac{pn}{n-p}$ if $n > p$; $1 \leq p < \infty$ if $n \leq p$, where C_p is the best embedding constant.

Lemma 2 ([13, Chapter II, Lemma 1.1])**.**

(i) For any function $u \in W_0^{1,p}(\Omega)$, we have

$$\|u\|_q \leq B_{q,p} \|\nabla u\|_p,$$

for all $q \in [1,\infty]$ if $n \leq p$, and $1 \leq q \leq \frac{np}{n-p}$ if $n > p$. The best constant $B_{q,p}$ depends only on Ω, n, p and q. We will denote the constant $B_{p,p}$ by B_p.

(ii) For any $u \in W_0^{1,p}(\Omega)$, $p \geq 1$ and $r \geq 1$, we have

$$\|u\|_q \leq C \|\nabla u\|_p^{\mu} \|u\|_r^{1-\mu},$$

where C is a positive constant

$$\mu = \left(\frac{1}{r} - \frac{1}{q}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{r}\right)^{-1},$$

and

- *for $p \geq n = 1$, $r \leq q \leq \infty$*
- *for $p \geq n = 1$, $r \leq q \leq \infty$*
- *for $n > 1$ and $p < n$, $q \in \left[r, \frac{np}{n-p}\right]$ if $r \leq \frac{np}{n-p}$ and $q \in \left[r, \frac{np}{n-p}\right]$ if $r \leq \frac{np}{n-p}$*

- *for* $p = n > 1$, $r \leq q < \infty$
- *for* $p > n > 1$, $r \leq q \leq \infty$.

Lemma 3. *$E(t)$ is a nonincreasing function, for $t \geq 0$*

$$E'(t) = -\|u_t\|_k^k \leq 0.$$

Proof. Multiplying the equation (1.1) by u_t and integrating on Ω, we have

$$\int_\Omega u_{tt}u_t\,dx - \int_\Omega \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)u_t\,dx + \int_\Omega |u_t|^{k-1}u_t\,dx = \int_\Omega u^{q-2}u\ln|u|\,u_t\,dx,$$

$$\frac{d}{dt}\left(\frac{1}{2}\|u_t\|^2 + \frac{1}{p}\|\nabla u\|_p^p - \frac{1}{q}\int_\Omega |u|^q\ln|u|\,dx + \frac{1}{q^2}\|u\|_q^q\right) = -\|u_t\|_k^k,$$

$$E'(t) = -\|u_t\|_k^k.$$

□

Lemma 4. *Suppose that $\lambda > 0$, $u \in W_0^{1,p}(\Omega)\setminus\{0\}$ and $\|u\|_q \neq 0$. Then we get*

(i) $\lim\limits_{\lambda\to 0^+} J(\lambda u) = 0,\ \lim\limits_{\lambda\to\infty} J(\lambda u) = -\infty$;

(ii) there exists a unique λ^* such that

$$\frac{d}{d\lambda}J(\lambda u)\,|_{\lambda=\lambda^*} = 0;$$

(iii) $J(\lambda u)$ is strictly decreasig on $\lambda^* < \lambda < \infty$, strictly increasing on $0 \leq \lambda \leq \lambda^*$, and takes maximum at $\lambda = \lambda^*$;

(iv) For any $\lambda \geq 0$, we get

$$I(\lambda u) = \lambda\frac{d}{d\lambda}J(\lambda u) = \begin{cases} > 0, & 0 < \lambda < \lambda^*, \\ = 0, & \lambda = \lambda^*, \\ < 0, & \lambda^* < \lambda < \infty. \end{cases} \tag{2.5}$$

Proof.

(i) It is obvious that by the definition of $J(u)$,

$$\begin{aligned} J(\lambda u) &= \frac{1}{p}\|\lambda\nabla u\|_p^p + \frac{1}{q^2}\|\lambda u\|_q^q - \frac{1}{q}\int_\Omega (\lambda u)^q\ln|\lambda u|\,dx \\ &= \frac{\lambda^p}{p}\|\nabla u\|_p^p + \frac{\lambda^q}{q^2}\|u\|_q^q - \frac{\lambda^q}{q}\ln|\lambda|\,\|u\|_q^q - \frac{\lambda^q}{q}\int_\Omega \ln|u|\,|u|^q\,dx. \end{aligned}$$

By virtue of $\|u\|_p^p \neq 0$, we obtain $\lim\limits_{\lambda\to 0} g(\lambda) = 0,\ \lim\limits_{\lambda\to\infty} g(\lambda) = -\infty$.

(ii) Now, differentiating $J(\lambda u)$ with respect to λ, we obtain

$$\begin{aligned}\frac{d}{d\lambda}J(\lambda u) &= \lambda^{p-1}\|\nabla u\|_p^p - \lambda^{q-1}\ln|\lambda|\,\|u\|_q^q - \lambda^{q-1}\int_\Omega |u|^q \ln|u|\,dx\\ &= \lambda\left(\lambda^{p-2}\|\nabla u\|_p^p - \lambda^{q-2}\ln|\lambda|\,\|u\|_q^q - \lambda^{q-2}\int_\Omega |u|^q \ln|u|\,dx\right)\\ &= \lambda\varphi(\lambda)\end{aligned}$$

where

$$\varphi(\lambda) = \lambda^{p-2}\|\nabla u\|_p^p - \lambda^{q-2}\ln|\lambda|\,\|u\|_q^q - \lambda^{q-2}\int_\Omega |u|^q \ln|u|\,dx$$

We observe from $2 < p < q$ that

$$\begin{aligned}\varphi(\lambda) &= \lambda^{p-2}\|\nabla u\|_p^p - \lambda^{q-2}\ln|\lambda|\,\|u\|_q^q - \lambda^{q-2}\int_\Omega |u|^q \ln|u|\,dx\\ &= \lambda^{q-2}\left(\lambda^{p-q}\|\nabla u\|_p^p - \ln|\lambda|\,\|u\|_q^q - \int_\Omega |u|^q \ln|u|\,dx\right)\\ &= \lambda^{q-2}\left(x\lambda^{p-q} - y\ln|\lambda| - z\right)\end{aligned}$$

where $x = \|\nabla u\|_p^p \geq 0$, $y = \|u\|_q^q \geq 0$ and $z = \int_\Omega |u|^q \ln|u|\,dx$. Also we obtain

$$\begin{aligned}\varphi'(\lambda) &= (q-2)\lambda^{q-3}\left(x\lambda^{p-q} - y\ln|\lambda| - z\right) + \lambda^{q-3}\left(x(p-q)\lambda^{p-q} - y\right)\\ &= \lambda^{q-3}\left[(p-2)x\lambda^{p-q} - y((q-2)\ln|\lambda| + 1) - (q-2)z\right].\end{aligned}$$

Let

$$g(\lambda) = (p-2)x\lambda^{p-q} - y((q-2)\ln|\lambda| + 1) - (q-2)z$$

which together with $2 < p < q$ satisfies that

$$\lim_{\lambda\to 0} g(\lambda) = \infty, \qquad \lim_{\lambda\to\infty} g(\lambda) = -\infty$$

and

$$g'(\lambda) = \frac{(p-q)(p-1)\lambda^{p-q} - (q-1)z}{\lambda} < 0.$$

Now, we deduce that there exist a unique λ_0 such that $g(\lambda)\,|_{\lambda=\lambda_0} = 0$, which satisfies

$$\begin{cases}\varphi'(\lambda) > 0, & \text{for } 0 < \lambda < \lambda_0,\\ \varphi'(\lambda) = 0, & \text{for } \lambda = \lambda_0,\\ \varphi'(\lambda) < 0, & \text{for } \lambda > \lambda_0.\end{cases}$$

Therefore, we conclude that there exists a unique $\lambda_1 > \lambda_0$ such that $\varphi(\lambda)\,|_{\lambda=\lambda_1}=$ 0 and $\varphi(\lambda)$ is monotone decreasing $\lambda > \lambda_1$. Hence, there exists $\lambda^* > \lambda_1$ such that $\left(\|\nabla u\|^2+\varphi(\lambda)\right)=0$,which means $\frac{d}{d\lambda}J(\lambda u)\,|_{\lambda=\lambda^*}$.

(iii) From (ii), we can see clearly

$$\frac{d}{d\lambda}J(\lambda u)>0 \qquad \text{for } 0\le\lambda\le\lambda^*,$$

$$\frac{d}{d\lambda}J(\lambda u)<0 \qquad \text{for } \lambda^*<\lambda<\infty,$$

which gives (iii).

(iv) Thus, by definition of $I(u)$ we have the desired results such that

$$I(\lambda u)=\lambda^p\,\|\nabla u\|_p^p-\lambda^q\ln|\lambda|\,\|u\|_q^q-\lambda^q\int_\Omega |u|^q\ln|u|\,dx=\lambda\frac{d}{d\lambda}J(\lambda u) \tag{2.6}$$

We obtain (2.5) from the proof of the (ii) and (2.6).

□

Lemma 5.

(i) d is positive and there exists a positive function $u\in\aleph$ such that $J(u)=d$

(ii) The depth of potential well d is defined as

$$d=\left(\frac{q-p}{pq}\right)\left(\frac{e\alpha}{C}\right)^{\frac{p}{q+\alpha-p}}$$

Proof.

(i) By (2.3), our aim is to show that there is a positive function $u\in\aleph$ such that $J(u)=d$. Let $\{u_m\}_{m=1}^\infty\subset\aleph$ be a minimum sequence of $J(u)$, i.e.

$$\lim_{m\to\infty}J(u_m)=d.$$

Hence, we have $\{|u_m|\}_{m=1}^\infty\subset\aleph$ is a minimum sequence of $J(u)$ from $|u_m|\subset u_m\in\aleph$ and $J(|u_m|)=J(|u_m|)$ .Morever, we can assume that $u_m>0$ a.e. for all $m\in\mathbb{N}$.

Otherwise, we have already observed that, $J(u)$ is coercive on $\aleph$ which satisfies that $\{u_m\}_{m=1}^\infty\subset\aleph$ is bounded in $u\in W_0^{1,p}(\Omega)$. Let $\alpha>0$ is a sufficiently small such that $q+\alpha<\frac{np}{n-p}$, so the embedding $W_0^{1,p}\hookrightarrow L^{q+\alpha}$ is compact, and there is a function u and subsequence $\{u_m\}_{m=1}^\infty$, still denoted by $\{u_m\}_{m=1}^\infty$, such that

$$u_m\to u,\text{weakly in } W_0^{1,p}(\Omega),$$
$$u_m\to u,\text{strongly in } L^{q+\alpha}(\Omega),$$
$$u_m\to u,\text{a.e. in } \Omega.$$

Thus, we get $u \geq 0$ a.e. in Ω. By Lebesgue dominated convergence theorem, we see that

$$\int_{\Omega} |u|^{q} \ln|u| \, dx = \lim_{m\to\infty} \int_{\Omega} |u_m|^{q} \ln|u_m| \, dx, \tag{2.7}$$

$$\int_{\Omega} |u|^{q} \, dx = \lim_{m\to\infty} \int_{\Omega} |u_m|^{q} \, dx. \tag{2.8}$$

The weak lower semicontinuity of $\|.\|_{W_0^{1,p}}$ implies

$$\|\nabla u\|_p \leq \lim_{m\to\infty} \inf \|\nabla u_m\|_p \,. \tag{2.9}$$

Combining definition of the $J(u)$ and $I(u)$, (2.7) - (2.9), we conclude that

$$J(u) \leq \lim_{m\to\infty} \inf J(u_m) = d, \tag{2.10}$$

$$I(u) \leq \lim_{m\to\infty} \inf I(u_m) = 0. \tag{2.11}$$

Thanks to $u_m \in \aleph$ one has $u_m \in W_0^{1,p}(\Omega)$ and $I(u_m) = 0$. Therefore, by using the fact

$$\ln x \leq \frac{1}{e\alpha} x^{\alpha} \text{ for } x \geq 1 \tag{2.12}$$

and the Sobolev embedding inequality, we have

$$\begin{aligned}
\|\nabla u_m\|_p^p &= \int_{\Omega} |u_m|^{q} \ln|u_m| \, dx \\
&= \int_{\{x\in\Omega:|u_m(x)|\geq 1\}} |u_m|^{q} \ln|u_m| \, dx + \int_{\{x\in\Omega:|u_m(x)|<1\}} |u_m|^{q} \ln|u_m| \, dx \\
&\leq \int_{\{x\in\Omega:|u_m(x)|\geq 1\}} |u_m|^{q} \ln|u_m| \, dx \\
&\leq \frac{1}{e\alpha} \int_{\{x\in\Omega:|u_m(x)|\geq 1\}} |u_m|^{q+\alpha} \, dx \leq C \|\nabla u_m\|_{p+\alpha}^{p+\alpha},
\end{aligned}$$

for some positive constant C, which implies

$$\int_{\Omega} |u_m|^{q} \ln|u_m| \, dx = \|\nabla u_m\|_p^p \geq C. \tag{2.13}$$

From (2.13) and (2.7), we reproduce

$$\int_{\Omega} |u|^{q} \ln|u| \, dx \geq C.$$

Therefore, we obtain $u \in W_0^{1,p}(\Omega)$. By (2.11), we easily have $I(u) \le 0$. Now, we show that $I(u) = 0$. Indeed, if it false, we get $I(u) < 0$, then by Lemma 2.4, there exists a λ^* such that $0 < \lambda^* < 1$ and $I(\lambda^* u) = 0$. Thus, we conclude that

$$\begin{aligned} d &\le J(\lambda^* u) \\ &= \frac{1}{q} I(\lambda^* u) + \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla(\lambda^* u)\|_p^p + \frac{1}{q^2} \|\lambda^* u\|_q^q \\ &= \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla(\lambda^* u)\|_p^p + \frac{1}{q^2} \|\lambda^* u\|_q^q \\ &\le (\lambda^*)^p \left(\left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p + \frac{1}{q^2} \|u\|_q^q\right) \\ &\le (\lambda^*)^p \liminf_{m\to\infty} \left(\left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u_m\|_p^p + \frac{1}{q^2} \|u_m\|_q^q\right) \\ &\le (\lambda^*)^p \liminf_{m\to\infty} J(u_m) = (\lambda^*)^p d < d. \end{aligned}$$

This is impossible, so we derive $I(u) = 0$ and $u_m \in \aleph$. From (2.10) and (2.4), we obtain $J(u) = d$, and the proof of (i) is complete.

(ii) By $I(u) = 0$ and the definition of $I(u)$, we obtain

$$\|\nabla u\|_p^p = \int_\Omega |u|^q \ln |u| \, dx. \tag{2.14}$$

Then, by using tha fact (2.12) and Sobolev embedding theorem, (2.14) becomes

$$\|\nabla u\|_p^p < \frac{1}{e\alpha} \|u\|_{q+\alpha}^{q+\alpha} \le \frac{C}{e\alpha} \|\nabla u\|_p^{q+\alpha}$$

where $C > 0$, which means that

$$\left(\frac{e\alpha}{C}\right)^{\frac{1}{q+\alpha-p}} \le \|\nabla u\|_p. \tag{2.15}$$

From the (i) we know that, $u \in \aleph$. By $I(u) = 0$, (2.3) and (2.15), we note that

$$\begin{aligned} J(u) &= \frac{1}{q} I(u) + \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p + \frac{1}{q^2} \|u\|_q^q \ge \left(\frac{1}{p} - \frac{1}{q}\right) \|\nabla u\|_p^p \\ &\ge \left(\frac{q-p}{pq}\right) \left(\frac{e\alpha}{C}\right)^{\frac{p}{q+\alpha-p}} \end{aligned}$$

where $q > p$, which implies that

$$d = \left(\frac{q-p}{pq}\right) \left(\frac{e\alpha}{C}\right)^{\frac{p}{q+\alpha-p}}.$$

This completes the proof.

□

3. Local existence of solution for $E(0) < d$

In this part, we established the global existence of the problem (1.1). Firstly, we start the definition of the weak solution to the problem (1.1).

Definition 1. *A function $u(t)$ is called a weak solution to problem (1.1) on $\Omega \times [0,T)$, if*

$$u \in L^{\infty}\left(0,T;W_0^{1,p}(\Omega)\right)$$

and

$$u_t \in L^{\infty}\left(0,T;L^k(\Omega)\right)$$

satisfy for $t \in [0,T)$ and llw $\in W_0^{1,p}(\Omega)$

$$\begin{cases} \int\limits_{\Omega} u_{tt}(x,t)\,w(x)\,dx + \int\limits_{\Omega} |u_t(x,t)|^{k-2}\,u_t(x,t)\,w(x)\,dx \\ + \int\limits_{\Omega} |\nabla u(x,t)|^{p-2}\,\nabla u(x,t)\,\nabla w(x)\,dx \\ = \int\limits_{\Omega} \ln|u(x,t)|\,u^{q-2}(x,t)\,w(x)\,dx, \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \end{cases}$$

Theorem 1.

Rational case: *Let $(u_0,u_1) \in W_0^{1,p}(\Omega) \times L^k(\Omega)$ and $2 < p < q < p\left(1+\frac{2}{n}\right)$ for every $T > 0$. Then problem (1.1) has a unique weak solution*

$$u \in C\left([0,T);W_0^{1,p}(\Omega)(\Omega)\right), \qquad u_t \in C\left([0,T);L^k(\Omega)\right).$$

Irrational case: *Moreover, u satisfies the following energy inequality*

$$E(t) + \int\limits_0^t \|u_t(s)\|_k^k\,ds \le E(0) \qquad \text{for } 0 \le t \le T.$$

Proof. To consider the well-posedness of problem (1.1), we employ the standard Faedo–Galerkin method. The proof will consist of three steps.

Step 1: Approximate Problem: Let $\{w_j\}_{j=1}^{\infty}$ be the orthogonal basis of $W_0^{1,p}(\Omega)$ space. We take the finite dimensional space

$$V_m = \operatorname{span}\{w_1,w_2,\dots,w_m\}.$$

Let the projections of the initial data on the finite dimensional subspace V_m be given by

$$\begin{aligned} u_m(0) = u_{m0}(x) &= \sum_{j=1}^{m} a_{jm} w_j(x) \to u_0 && \text{in } W_0^{1,p}(\Omega), \\ u_{mt}(0) = u_{m1}(x) &= \sum_{j=1}^{m} b_{jm} w_j(x) \to u_1 && \text{in } L^k(\Omega), \end{aligned} \tag{3.1}$$

for $j = 1, 2, \ldots, m$.

We construct the approximate solutions $u_m(x,t)$ for problem (1.1) in the form

$$u_m(x,t) = \sum_{j=1}^{m} h_{jm}(t) w_j(x) \tag{3.2}$$

which satisfy the approximate problem in V_m

$$\begin{aligned}(u_{mtt}, w_s) &+ \left(|\nabla u_m|^{p-2} \nabla u_m, \nabla w_s\right) \\ &= \left(|u_m|^{q-2} u_m \log|u_m|, w_s\right) ds - \left(|u_{mt}|^{k-2} u_{mt}, w_s\right)\end{aligned} \tag{3.3}$$

for conditions

$$\begin{cases} u_0^m(x) = \sum_{j=1}^m a_j w_j(x) \to u_0 & \text{in } W_0^{1,p}(\Omega), \\ u_1^m(x) = \sum_{j=1}^m b_j w_j(x) \to u_1 & \text{in } L^k(\Omega), \end{cases}$$

$s = 1, 2, \ldots m$, where $w \in V_m$ as $m \to \infty$.

This leads to a system of ordinary differantial equations for unknown functions $h_j^m(t)$. Based on standard existence theory for ordinary differantial equation, one can obtain functions

$$h_j : [0, t_m) \to R, \qquad j = 1, 2, \ldots, m,$$

which satisfy (3.3) in a maximal interval $[0, t_m)$, $0 < t_m \le T$ and therefore $u_m \in C\left([0, t_m); W_0^{1,p}(\Omega)\right)$, $u_{mt} \in C\left([0, t_m); H^1(\Omega)\right)$.

Step 2: A priori estimates: Our purpose is to show that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. Now, taking the derivative of (3.3) with respect to t, multiplying the obtained equation by $h'_{mj}(t)$ and summing for j=1,2,...,m, we obtain

$$(u_{mtt}, w) + \left(|u_{mt}|^{k-2} u_{mt}, w\right) + \left(|\nabla u_m|^{p-2} \nabla u_m, \nabla w\right) = \left(|u_m|^{q-2} \ln|u_m|, w\right)$$

for $\forall w \in H_0^1(\Omega)$. Let us replace w by u_{mt} in and integrate by parts we obtain

$$\frac{d}{dt} E_m(t) = -\int_0^t \|u_{mt}(s)\|_k^k \, ds$$

where

$$E_m(t) = \frac{1}{2}\|u_{mt}\|^2 + \frac{1}{p}\|\nabla u_m\|_p^p - \frac{1}{q}\int_\Omega |u_m|^q \ln|u_m| \, dx + \frac{1}{q^2}\|u_m\|_q^q.$$

Then Integrating (1.2) with respect to t from 0 to t, we have

$$E_m(t) + \int_0^t \|u_{mt}(s)\|_k^k \, ds = E_m(0). \tag{3.4}$$

Otherwise, for $\alpha > 0$, we obtain

$$\int_\Omega |u_m|^q \ln|u_m|\,dx \le \frac{1}{e\alpha}\|u_m\|_{q+\alpha}^{q+\alpha},$$

where α is taken such that $0 < \alpha < p\left(1+\frac{2}{n}\right) - q$. Then by using Lemma 2.2 and Young's inequality

$$ab \le \delta a^k + C(\delta)\,b^l$$

with $k = \frac{p}{q+\alpha}$ and $l = \frac{p(1-\mu)(q+\alpha)}{p-\mu(q+\alpha)}$ for $\delta \in (0,1)$, we have

$$\begin{aligned}\int_\Omega |u_m|^q \ln|u_m|\,dx &\le B\|\nabla u_m\|_p^{\mu(q+\alpha)}\|u_m\|_2^{(1-\mu)(q+\alpha)}\\ &\le \delta\|\nabla u_m\|_p^p + C(\delta)\|u_m\|_2^{\frac{p(1-\mu)(q+\alpha)}{p-\mu(q+\alpha)}},\end{aligned} \tag{3.5}$$

where

$$\mu = \left(\frac{1}{2} - \frac{1}{q+\alpha}\right)\left(\frac{1}{n} - \frac{1}{p} + \frac{1}{2}\right)^{-1}.$$

Here, we take $\alpha > 0$ such that $p - \mu(q+\alpha)$ and $0 < \alpha < p\left(1+\frac{2}{n}\right) - q$ hold. Let

$$h = \frac{p(1-\mu)(q+\alpha)}{p-\mu(q+\alpha)} = \frac{p(n+q+\alpha) - n(q+\alpha)}{p(2+n) - n(q+\alpha)}$$

then $h > 1$ because $2 < p < q < p\left(1+\frac{2}{n}\right)$. Morever, by the combination of (3.2), (3.4) and (3.5), we obtain

$$E_m(t) \le C_1 + C_2\int_0^t E_m^h(s)ds, \tag{3.6}$$

where C_1, C_2 are positive constants independent of m. By using of the Gronwall inequality, we have a positive consatant

$$T < \frac{C_1^{1-h}}{C_2(h-1)}$$

such that

$$E_m(t) \le C_T \tag{3.7}$$

for any $t \in [0,T]$.

Subsequently, there exists the solution of (3.3) on $[0,T]$, for any m.

On the other hand, multiplying (3.3) by $h'_{mj}(t)$ and summing for s, we derive

$$\frac{1}{2}\|u_{mt}\|^2 + J(u_m) = E_m(0) \tag{3.8}$$

for $\forall t \in [0,T]$.

By the continuity of J and (3.1), we consider

$$E_m(0) \leq C \tag{3.9}$$

where C is the positive constant for any m.

Therefore, it follows from the definion of $E(t)$, (3.5), (3.7)-(3.9) and using Hölder's inequality, we have

$$\begin{aligned} C \geq E_m(t) &= \frac{1}{2}\|u_{mt}\|^2 + \frac{1}{p}\|\nabla u_m\|_p^p - \frac{1}{q}\int_\Omega |u_m|^q \ln|u_m|\, dx + \frac{1}{q^2}\|u_m\|_q^q \\ &\geq \frac{1}{p}\|\nabla u_m\|_p^p - \frac{1}{q}\int_\Omega |u_m|^q \ln|u_m|\, dx + \frac{1}{q^2}\|u_m\|_q^q \\ &\geq \frac{1}{p}\|\nabla u_m\|_p^p - \frac{\delta}{q}\|\nabla u_m\|_p^p - \frac{C(\delta)}{q}\|u_m\|_2^{2h} + \frac{1}{q^2}\|u_m\|_q^q \\ &\geq \left(\frac{1}{p} - \frac{\delta}{q}\right)\|\nabla u_m\|_p^p - \frac{C(\delta)}{q} p^{ph} E_m^{ph}(t) + \frac{1}{q^2}\|u_m\|_q^q \\ &\geq \left(\frac{1}{p} - \frac{\delta}{q}\right)\|\nabla u_m\|_p^p + \frac{1}{q^2}\|u_m\|_q^q - C_3. \end{aligned} \tag{3.10}$$

Combining (3.10) and (3.8), we have

$$\begin{aligned} \|u_m\|_{L^\infty(0,T;W^{1,p}(\Omega))} &\leq C, \\ \|u_{mt}\|_{L^\infty(0,T;H^1(\Omega))} &\leq C. \end{aligned} \tag{3.11}$$

It follows from (3.4) and (3.7) that

$$\left\| |\nabla u_m|^{p-2} \nabla u_m \right\|_{L^\infty(0,T;W^{-1,p'}(\Omega))} \leq C. \tag{3.12}$$

Step 3: Passage to the limit: Combining **(3.11)-(3.12),** there are functions u and χ and a subsequence of $\{u_m\}_{m=1}^\infty$ which we still denoted by $\{u_m\}_{m=1}^\infty$ such that

$$\begin{cases} u_m \to u, \ \text{ weakly}^*\text{in } L^\infty\left(0,T;W_0^{1,p}(\Omega)\right), \\ u_{mt} \to u_t, \text{weakly in } L^\infty\left(0,T;L^k(\Omega)\right), \\ |\nabla u_m|^{p-2}\nabla u_m \to \chi \text{ weakly}^*\text{in } L^\infty\left(0,T;W_0^{-1,p'}(\Omega)\right). \end{cases}$$

By Aubin–Lions–Simon Lemma we obtain

$$\begin{aligned} &u_m \to u \text{ strongly in } C([0,T];W_0^{1,p}(\Omega)), \\ &u_m \to u, \text{ a.e. } (x,t) \in \Omega \times (0,T), \end{aligned}$$

which implies that

$$|u_m|^{q-2} u_m \ln|u_m| \to |u|^{q-2} u \ln|u|, \text{a.e.} \qquad (x,t) \in \Omega \times (0,T). \tag{3.13}$$

On the other side, since $2 < p < q < p\left(1+\frac{2}{n}\right) < \frac{np}{n-p}$, we can choose $\alpha > 0$ such that $(q-1+\mu)\,q' < \frac{np}{n-p}$. So, by direct calculation and Sobolev inequality, we note that

$$\begin{aligned}\int_\Omega |\psi_m(x,t)|^{q'}\,dx &= \int_{\{x\in\Omega:|u_m(x,t)|\le 1\}} |\psi_m(x,t)|^{q'}\,dx + \int_{\{x\in\Omega:|u_m(x,t)|>1\}} |\psi_m(x,t)|^{q'}\,dx \\ &\le (e(q-1))^{-q'}|\Omega| + (e\alpha)^{-q'} \int_{\{x\in\Omega:|u_m(x,t)|>1\}} |\psi_m(x,t)|^{(q-1+\alpha)q'}\,dx \\ &\le C_4 + C_5 \|\nabla u_m(t)\|_p^{(q-1+\alpha)q'} \le C \qquad (3.14)\end{aligned}$$

where $\psi_m(x,t) = |u_m|^{q-2}u_m \ln|u_m|$. And we have used

$$\left|x^{p-1}\log x\right| \le (e(p-1))^{-1} \qquad \text{for } 0<x<1,$$

while $x^{-\alpha}\log x \le \frac{1}{e\alpha}$ for $x \ge 1$, $\alpha > 0$, where $\psi_m(x,t) = |u_m|^{q-2}u_m \ln|u_m|$. And we have used $\left|x^{p-1}\log x\right| \le (e(p-1))^{-1}$ for $0<x<1$, while $x^{-\alpha}\log x \le \frac{1}{e\alpha}$ for $x \ge 1$, $\alpha > 0$.

Hence, from (3.13), (3.14) and Lions Lemma [15], we get

$$|u_m|^{q-2}u_m\ln|u_m| \to |u|^{q-2}u\ln|u| \;\text{ weakly}^*\text{in } L^\infty\left(0,T;L^{q'}(\Omega)\right).$$

Now, taking the limit in (3.1) as $m\to\infty$, it follows that u satisfies the initial conditions $u(x,0)=u_0$ in $W_0^{1,p}(\Omega)$ and $u_t(x,0)=u_1$ in $H^1(\Omega)$. Additionally, passing to the limit in (3.3), it follows that $t\in[0,T]$

$$\begin{aligned}&(u_t,w_s) + \int_0^t\int_\Omega |\nabla u|^{p-2}\nabla u\nabla w_s ds + \int_0^t\int_\Omega |u_{mt}|^{k-2}u_{mt}w_s ds \\ &= \int_0^t \left(|u|^{q-2}u\log|u|, w_s\right)ds + (u_1,w_s)\end{aligned}$$

for all $w\in W_0^{1,p}(\Omega)$.

Step 3: Uniqueness: : Firstly, we consider linear problem

$$\begin{cases} v_{tt} + |v_t|^{k-2}v_t - \operatorname{div}\left(|\nabla v|^{p-2}\nabla v\right) & (x,t)\in\Omega\times(0,T), \\ = f(u_1) - f(u_2), & \\ v(x,0) = v_0(x), \quad v_t(x,0) = v_1(x), & x\in\Omega, \\ v = \frac{\partial v}{\partial n} = 0, & x\in\partial\Omega\times R^+. \end{cases} \qquad (3.15)$$

where $f(s) = |s|^{q-2}s\ln|s|$. Suppose there are two solutions u_1 and u_2 to problem (1.1). Then, $v = u_1 - u_2$ solves the problem (3.15).

Multiplying both sides of the first equation for above problem (3.15) by v_t and integrating the obtained result over $\Omega\times(0,T)$, then we obtain

$$\int_0^t\int_\Omega v_{tt}v_t\,dx\,ds+\int_0^t\int_\Omega |v_t|^k\,dx\,ds+\int_0^t\int_\Omega |\nabla v|^{p-2}\nabla v\nabla v_t dxds$$
$$=\int_0^t\int_\Omega\left(|u_1|^{q-2}u_1\ln|u_1|-|u_2|^{q-2}u_2\ln|u_2|\right)v_t\,dx\,ds. \tag{3.16}$$

Making use of mean value theorem, we get

$$\begin{aligned}&|f(u_1)-f(u_2)|\times\left|f'(\vartheta u_1+(1-\vartheta)u_2)(u_1-u_2)\right|\\&\le[1+(q-1)\ln|(u_1+\vartheta u_2)|]\,|(u_1+\vartheta u_2)|^{q-2}|u_1-u_2|\end{aligned} \tag{3.17}$$

where $0<\vartheta<1$. Inserting (3.17) into (3.16), we denote

$$\begin{aligned}&\int_0^t\int_\Omega v_{tt}v_t\,dx\,ds+\int_0^t\int_\Omega |v_t|^k\,dx\,ds+\int_0^t\int_\Omega |\nabla v|^{p-2}\nabla v\nabla v_t\,dx\,ds\\
&\le\int_0^t\int_\Omega\left([1+(q-1)\ln|(u_1+\vartheta u_2)|]\,|(u_1+\vartheta u_2)|^{q-2}\right)vv_t\,dx\,ds\\
&\le\int_0^t\int_\Omega |(u_1+\vartheta u_2)|^{q-2}vv_t\,dx\,ds\\
&\quad+(q-1)\int_0^t\int_\Omega \ln|(u_1+\vartheta u_2)|\,|(u_1+\vartheta u_2)|^{q-2}vv_t\,dx\,ds.\end{aligned} \tag{3.18}$$

Morever, from the Lebesgue and Sobolev inequality and Hölder inequality, we obtain

$$\begin{aligned}\int_0^t\int_\Omega |(u_1+\vartheta u_2)|^{q-2}vv_t\,dx\,ds&\le\int_0^t\|u_1+\vartheta u_2\|_{n(q-2)}^{q-2}\|v\|_{\frac{2n}{n-2}}\|v_t\|_2\,ds\\
&\le C_5^{q-2}C_6\int_0^t\|\nabla u_1+\vartheta\nabla u_2\|_2^{q-2}\|\nabla v\|_2\|v_t\|_2\,ds\\
&\le C_7\int_0^t\|\nabla v\|_2\|v_t\|_2\,ds,\end{aligned}$$

$$\leq C_7 \int_0^t \|\nabla v\|_p \|v_t\|_2 \, ds, \tag{3.19}$$

where C_5, C_6, C_7 are the best constants satisfying Sobolev inequality. We used the condition $n(q-2) < p\left(1+\frac{2}{n}\right)$.

Now, our purpose is to estimate the second term of the (3.18). Furthermore, taking $\alpha > 0$ such that $(q-2+\alpha)n < p\left(1+\frac{2}{n}\right)$, and by using the calculation similar to (3.14), it follows that

$$\begin{aligned}
&\int_0^t \int_\Omega \left|\ln|(u_1+\vartheta u_2)|\,|(u_1+\vartheta u_2)|^{q-2}\right|^n v v_t \, dx\, ds \\
&\leq (e(q-1))^{-n}|\Omega| + (e\alpha)^{-n} C_8^{(q-1+\alpha)n} \|(\nabla u_1+\vartheta \nabla u_2)\|^{(q-1+\alpha)n} \\
&\leq (e(q-1))^{-n}|\Omega| + (e\alpha)^{-n} C_8^{(q-1+\alpha)n} \|(\nabla u_1+\vartheta \nabla u_2)\|_p^{(q-1+\alpha)n}
\end{aligned} \tag{3.20}$$

where C_8 is the optimal constant satisfying

$$\|(u_1+\vartheta u_2)\|_{(q-1+\alpha)n} \leq \|(\nabla u_1+\vartheta \nabla u_2)\|^{(q-1+\alpha)n}.$$

Inserting (3.20) into (3.18), we obtain

$$\begin{aligned}
&(q-1)\int_0^t \int_\Omega \ln|(u_1+\vartheta u_2)|\,|(u_1+\vartheta u_2)|^{q-2} v v_t \, dx\, ds \\
&\leq (q-1)\int_0^t \left(\int_\Omega \ln|(u_1+\vartheta u_2)|\,|(u_1+\vartheta u_2)|^{q-2}\right)^{\frac{1}{n}} \|v\|_{\frac{2n}{n-2}} \|v_t\|_2 \, ds \\
&\leq C_9 \int_0^t \|\nabla v\|_2 \|v_t\|_2 \, ds \leq C_{10} \int_0^t \|\nabla v\|_p \|v_t\|_2 \, ds.
\end{aligned} \tag{3.21}$$

Inserting (3.19) and (3.21) into (3.18) and using $v(x,0)=0, v_t(x,0)=0$, we have

$$\|v_t\|^2 + \|\nabla v\|_p^p \leq C\int_0^t \|\nabla v\|_p \|v_t\|_2 \, ds \leq \int_0^t \left(\|v_t\|^2 + \left(\|\nabla v\|_p^p\right)^{\frac{2}{p}}\right).$$

Using the algebraic inequality

$$z^\nu \leq z+1 \leq \left(1+\frac{1}{\alpha}\right)(z+\alpha), \qquad \forall z \geq 0,\ 0<\nu\leq 1,\ \alpha \geq 0,$$

we obtain

$$\left(\|\nabla v\|_p^p\right)^{\frac{2}{p}} \leq 1 + \|\nabla v\|_p^p$$

where $p > 2$. The uniqueness is derived from the Gronwall's inequality.

Step 3: Energy inequality : We will show that the solutions u satisfy (3.4). First, we prove that

$$\int_{\Omega} |u|^q \ln |u| \, dx = \lim_{m\to\infty} \int_{\Omega} |u_m|^q \ln |u_m| \, dx, \tag{3.22}$$

$$\int_{\Omega} |u|^q \, dx = \lim_{m\to\infty} \int_{\Omega} |u_m|^q \, dx. \tag{3.23}$$

Additionally, for each fixed $t > 0$, by similar calculation to (3.18) and Hölder inequality, we obtain

$$\begin{aligned}
&\left| \int_{\Omega} |u_m|^q \ln |u_m| \, dx - \int_{\Omega} |u|^q \ln |u| \, dx \right| \\
&\leq \int_{\Omega} \left| q\,|\sigma_{1m}|^{q-1} \ln |\sigma_m| + |\sigma_{1m}|^{q-1} \right| |u - u_m| \, dx \\
&\leq q \int_{\Omega} \left(\left| |\sigma_{1m}|^{q-1} \ln |\sigma_{1m}| \right|^{q'} dx \right)^{\frac{1}{q'}} \|u - u_m\|_q + \|\sigma_{1m}\|_q^{q-1} \|u - u_m\|_q \\
&\leq C \|u - u_m\|_q \to 0
\end{aligned}$$

and

$$\begin{aligned}
\left| \int_{\Omega} |u_m|^q \, dx - \int_{\Omega} |u|^q \, dx \right| &\leq \int_{\Omega} \left| |u_m|^q - |u|^q \right| dx \\
&\leq q \int_{\Omega} |\sigma_{2m}|^{q-1} |u - u_m| \, dx \\
&\leq q \|\sigma_{2m}\|_q^{q-1} \|u - u_m\|_q \leq C \|u - u_m\|_q \to 0,
\end{aligned}$$

as $m \to \infty$, where $\sigma_i = u + \vartheta_i u_m$, $0 < \vartheta_i < 1$ $(i = 1, 2)$. Morever, (3.22) and (3.23) hold.

On the other hand, from initial and boundary condition of the (3.3), it follows that $E(u_{0m}, u_{1m}) \to E(u_0, u_1) = E(0)$ as $m \to \infty$. Therefore, making use of Fatou Lemma and (3.4), we note that

$$\begin{aligned}
&\frac{1}{2} \|u_t\|^2 + \frac{1}{p} \|\nabla u\|_p^p + \int_0^t \|u_t(s)\|_k^k ds \\
&\leq \frac{1}{2} \liminf_{m\to\infty} \|u_{mt}\|^2 + \frac{1}{p} \liminf_{m\to\infty} \|\nabla u_m\|_p^p + \int_0^t \liminf_{m\to\infty} \|u_{mt}(s)\|_k^k ds
\end{aligned}$$

$$\leq \liminf_{m\to\infty}\left[\frac{1}{2}\|u_{mt}\|^2+\frac{1}{p}\|\nabla u_m\|_p^p+\int_0^t\|u_t(s)\|_k^k\,ds\right]$$
$$=\liminf_{m\to\infty}\left[E_m(0)+\frac{1}{q}\int_\Omega|u_m|^q\ln|u_m|\,dx-\frac{1}{q^2}\|u_m\|_q^q\right]$$
$$=E(0)+\frac{1}{q}\int_\Omega|u|^q\ln|u|\,dx-\frac{1}{q^2}\|u\|_q^q.$$

So that the proof is completed. □

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