Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 833–843

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4366

PETROVIĆ-TYPE INEQUALITY VIA FRACTIONAL CALCULUS

PÉTER KÓRUS, JUAN EDUARDO NÁPOLES VALDÉS,
JOSÉ MANUEL RODRÍGUEZ, AND JOSÉ MARÍA SIGARRETA ALMIRA

Received 11 August, 2022

Abstract. Inequalities play an important role in pure and applied mathematics. In particular, Petrović inequality is an important inequality which have several interesting generalizations. In this work we prove a new Petrović-type inequality for measurable functions defined on a space with finite measure, and we apply it to generalized Riemann–Liouville-type integral operators.

2010 *Mathematics Subject Classification:* 26A33; 26A51; 26D15

Keywords: Petrović's inequality, fractional derivatives and integrals, fractional integral inequalities, convex functions

1. INTRODUCTION

Integral inequalities are used in countless mathematical problems such as approximation theory, spectral analysis, statistical analysis, distribution theory, etc. Studies involving integral inequalities play an important role in several areas of science and engineering.

In recent years there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives, since integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics. Some of the inequalities studied are Gronwall, Chebyshev, Hermite–Hadamard-type, Ostrowski-type, Opial-type, Grüss-type, Hardy-type, Gagliardo–Nirenberg-type, reverse Minkowski and reverse Hölder inequalities (see, e.g., [3, 5, 9–11, 14–16, 20, 21]).

In 1905, J. Jensen was the first to define convex functions (see [7] and [19, p. 8]) and to draw attention to their importance. One of the most significant inequalities is the distinguished Petrović's inequality for convex functions (see Theorem 1). There are many generalizations of Petrović's inequality (see, e.g., [1, 6, 13, 17, 18] and the references therein).

In this work we obtain a new Petrović-type inequality for measurable functions defined on a space with finite measure, and we apply it to the generalized Riemann–Liouville-type integral operators defined in [2], which include the well-known Riemann–Liouville-type integral operators.

2. Preliminaries

One of the first operators that can be called fractional is the Riemann–Liouville fractional integrals and derivatives of order $\alpha \in \mathbb{C}$, with $\mathrm{Re}(\alpha) > 0$, defined as follows (see [4]).

Definition 1. Let $a < b$ and $f \in L^1([a,b])$. The *left and right side Riemann–Liouville fractional integrals of order* α, with $\mathrm{Re}(\alpha) > 0$, are defined, respectively, by

$$ {}^{RL}J^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\, ds, \tag{2.1}$$

and

$$ {}^{RL}J^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s)\, ds, \tag{2.2}$$

with $t \in (a,b)$.

When $\alpha \in (0,1)$, their corresponding *Riemann–Liouville fractional derivatives* are given by

$$\begin{aligned} \left({}^{RL}D^{\alpha}_{a^+} f\right)(t) &= \frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{a^+} f(t)\right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^{\alpha}}\, ds, \\ \left({}^{RL}D^{\alpha}_{b^-} f\right)(t) &= -\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{b^-} f(t)\right) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^{\alpha}}\, ds. \end{aligned}$$

Other definitions of fractional operators are the following ones.

Definition 2. Let $a < b$ and $f \in L^1([a,b]))$. The *left and right side Hadamard fractional integrals of order* α, with $\mathrm{Re}(\alpha) > 0$, are defined, respectively, by

$$ H^{\alpha}_{a^+} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \tag{2.3}$$

and

$$ H^{\alpha}_{b^-} f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{s}{t}\right)^{\alpha-1} \frac{f(s)}{s}\, ds, \tag{2.4}$$

with $t \in (a,b)$.

When $\alpha \in (0,1)$, *Hadamard fractional derivatives* are given by the following expressions:

$$\left({}^{H}D^{\alpha}_{a^+} f\right)(t) = t\,\frac{d}{dt}\left(H^{1-\alpha}_{a^+} f(t)\right) = \frac{1}{\Gamma(1-\alpha)}\,t\,\frac{d}{dt}\int_a^t \left(\log\frac{t}{s}\right)^{-\alpha}\frac{f(s)}{s}\,ds,$$

$$\left({}^{H}D^{\alpha}_{b^-} f\right)(t) = -t\,\frac{d}{dt}\left(H^{1-\alpha}_{b^-} f(t)\right) = \frac{-1}{\Gamma(1-\alpha)}\,t\,\frac{d}{dt}\int_t^b \left(\log\frac{s}{t}\right)^{-\alpha}\frac{f(s)}{s}\,ds,$$

with $t \in (a,b)$.

Definition 3. Let $0 < a < b$, $g\colon [a,b] \to \mathbb{R}$ an increasing positive function on $(a,b]$ with continuous derivative on (a,b), $f\colon [a,b] \to \mathbb{R}$ an integrable function, and $\alpha \in (0,1)$ fixed real number. The left and right side fractional integrals in [8] of order α of f with respect to g are defined, respectively, by

$$I^{\alpha}_{g,a^+} f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t \frac{g'(s)f(s)}{\left(g(t)-g(s)\right)^{1-\alpha}}\,ds, \tag{2.5}$$

and

$$I^{\alpha}_{g,b^-} f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b \frac{g'(s)f(s)}{\left(g(s)-g(t)\right)^{1-\alpha}}\,ds, \tag{2.6}$$

with $t \in (a,b)$.

There are other definitions of integral operators in the global case, but they are slight modifications of the previous ones.

3. General fractional integral of Riemann–Liouville type

Now, we give the definition of a general fractional integral introduced in [2].

Definition 4. Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g\colon [a,b] \to \mathbb{R}$ be a positive function on $(a,b]$ with continuous positive derivative on (a,b), and $G\colon [0, g(b)-g(a)] \times (0,\infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b)-g(a)] \times (0,\infty)$. Let us define the function $T\colon [a,b]\times[a,b]\times(0,\infty) \to \mathbb{R}$ by

$$T(t,s,\alpha) = \frac{G\left(|g(t)-g(s)|,\alpha\right)}{g'(s)}.$$

The *left and right integral operators*, denoted respectively by J^{α}_{T,a^+} and J^{α}_{T,b^-}, are defined for each measurable function f on $[a,b]$ as

$$J^{\alpha}_{T,a^+} f(t) = \int_a^t \frac{f(s)}{T(t,s,\alpha)}\,ds,$$

$$J^{\alpha}_{T,b^-} f(t) = \int_t^b \frac{f(s)}{T(t,s,\alpha)}\,ds,$$

with $t \in [a,b]$.

We say that $f \in L^1_T([a,b])$ if $J^{\alpha}_{T,a^+}|f|(t), J^{\alpha}_{T,b^-}|f|(t) < \infty$ for every $t \in [a,b]$.

Note that these operators generalize the integral operators in Definitions 1, 2 and 3:

(A) If we choose

$$g(t)=t, \quad G(x,\alpha)=\Gamma(\alpha)\,x^{1-\alpha}, \quad T(t,s,\alpha)=\Gamma(\alpha)\,|t-s|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} become the left and right Riemann–Liouville fractional integrals ${}^{RL}J^{\alpha}_{a^+}$ and ${}^{RL}J^{\alpha}_{b^-}$ in (2.1) and (2.2), respectively. Its corresponding left and right Riemann–Liouville fractional derivatives are

$$\left({}^{RL}D^{\alpha}_{a^+}f\right)(t)=\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{a^+}f(t)\right), \quad \left({}^{RL}D^{\alpha}_{b^-}f\right)(t)=-\frac{d}{dt}\left({}^{RL}J^{1-\alpha}_{b^-}f(t)\right).$$

(B) If we choose

$$g(t)=\log t, \quad G(x,\alpha)=\Gamma(\alpha)\,x^{1-\alpha}, \quad T(t,s,\alpha)=\Gamma(\alpha)\,t\left|\log\frac{t}{s}\right|^{1-\alpha},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} become the left and right Hadamard fractional integrals $H^{\alpha}_{a^+}$ and $H^{\alpha}_{b^-}$ in (2.3) and (2.4), respectively. Its corresponding left and right Hadamard fractional derivatives are

$$\left({}^{H}D^{\alpha}_{a^+}f\right)(t)=t\,\frac{d}{dt}\left(H^{1-\alpha}_{a^+}f(t)\right), \quad \left({}^{H}D^{\alpha}_{b^-}f\right)(t)=-t\,\frac{d}{dt}\left(H^{1-\alpha}_{b^-}f(t)\right).$$

(C) If we choose a function g with the properties in Definition 4 and

$$G(x,\alpha)=\Gamma(\alpha)\,x^{1-\alpha}, \quad T(t,s,\alpha)=\Gamma(\alpha)\,\frac{|g(t)-g(s)|^{1-\alpha}}{g'(s)},$$

then J^{α}_{T,a^+} and J^{α}_{T,b^-} are the left and right fractional integrals I^{α}_{g,a^+} and I^{α}_{g,b^-} in (2.5) and (2.6), respectively.

Definition 5. Let $a<b$ and $\alpha\in\mathbb{R}^+$. Let $g\colon[a,b]\to\mathbb{R}$ be a positive function on $(a,b]$ with continuous positive derivative on (a,b), and $G\colon[0,g(b)-g(a)]\times(0,\infty)\to\mathbb{R}$ a continuous function which is positive on $(0,g(b)-g(a)]\times(0,\infty)$. For each function $f\in L^1_T[a,b]$, its *left and right generalized derivative of order* α are defined, respectively, by

$$D^{\alpha}_{T,a^+}f(t)=\frac{1}{g'(t)}\,\frac{d}{dt}\left(J^{1-\alpha}_{T,a^+}f(t)\right),$$
$$D^{\alpha}_{T,b^-}f(t)=\frac{-1}{g'(t)}\,\frac{d}{dt}\left(J^{1-\alpha}_{T,b^-}f(t)\right).$$

for each $t\in(a,b)$.

Note that if we choose

$$g(t)=t, \quad G(x,\alpha)=\Gamma(\alpha)\,x^{1-\alpha}, \quad T(t,s,\alpha)=\Gamma(\alpha)\,|t-s|^{1-\alpha},$$

then $D^{\alpha}_{T,a^+}f(t)={}^{RL}D^{\alpha}_{a^+}f(t)$ and $D^{\alpha}_{T,b^-}f(t)={}^{RL}D^{\alpha}_{b^-}f(t)$. Also, we can obtain other fractional derivatives such as Hadamard type as particular cases of this generalized derivative.

4. Petrović-type inequality

The famous Petrović inequality is stated as follows, see [12]:

Theorem 1. *Let* φ *be a convex function on* $[0,a]$*, and* $w_1,\dots,w_n \geq 0$*. If* $t_1,\dots,t_n \in [0,a]$ *satisfy* $\sum_{k=1}^{n} t_k w_k \in (0,a]$*, and*

$$\sum_{k=1}^{n} t_k w_k \geq t_j, \qquad j=1,\dots,n,$$

then

$$\sum_{k=1}^{n} \varphi(t_k) w_k \leq \varphi\Big(\sum_{k=1}^{n} t_k w_k\Big) + \Big(\sum_{k=1}^{n} w_k - 1\Big)\varphi(0).$$

There are many generalizations of Petrović's inequality (see, e.g., [1, 6, 13, 17, 18] and the references therein).

Next, we present a continuous version of the above Petrović inequality.

Theorem 2. *Let* μ *be a finite measure on the space* X*,* φ *be a convex function on* $[0,a]$*, and* $f\colon X \to [0,a]$ *be a measurable function with* $\int_X f\,d\mu \in (0,a]$*. Then* $\varphi \circ f$ *is a* μ*-integrable function. If*

$$f(x) \leq \int_X f\,d\mu$$

for every $x \in X$*, then*

$$\int_X \varphi \circ f\,d\mu \leq \varphi\Big(\int_X f\,d\mu\Big) + \big(\mu(X)-1\big)\varphi(0).$$

Proof. Assume first that f is constant a.e. $f = c$. Thus, Theorem 1 with $n=1$, $t_1 = c$ and $w_1 = \mu(X)$ gives the conclusion.

Assume now that f is not constant a.e. We have

$$0 \leq f(x) \leq A = \int_X f\,d\mu$$

for every $x \in X$ and $0 < A = \int_X f\,d\mu$, since f is not constant a.e. Since φ is a convex function on $[0,a]$, $\varphi \circ f$ is a bounded function. Since μ is a finite measure, $\varphi \circ f$ is a μ-integrable function.

For each $n \geq 1$ and $1 \leq k < 2^n$, consider the sets

$$I_{n,k} = \big((k-1)2^{-n}A,\, k2^{-n}A\big],$$
$$I_{n,2^n} = \big((2^n-1)2^{-n}A,\, A\big),$$
$$I_{n,0} = \{0\}, \qquad I_{n,2^n+1} = \{A\}.$$

Note that $\{I_{n,k}\}_{k=0}^{2^n+1}$ is a partition of $[0,A]$ for each $n \geq 1$.

For each $n \geq 1$ and $0 \leq k \leq 2^n+1$, define the sets

$$S_{n,k} = f^{-1}\big(I_{n,k}\big)$$

and choose constants $a_{n,k} \in I_{n,k}$ satisfying

$$a_{n,k}\,\mu\big(S_{n,k}\big) = \int_{S_{n,k}} f\,d\mu.$$

Thus, $a_{n,0} = 0$ and $a_{n,2^n+1} = A$.

Since f is a measurable function satisfying $0 \le f \le A$, we have that $\{S_{n,k}\}_{k=0}^{2^n+1}$ are pairwise disjoint measurable sets and $X = \cup_{k=0}^{2^n+1} S_{n,k}$ for each n.

Recall that the characteristic function of a set E is defined as $\chi_E(x) = 1$ if $x \in E$ and $\chi_E(x) = 0$ if $x \notin E$. If we define

$$f_n = \sum_{k=0}^{2^n+1} a_{n,k}\,\chi_{S_{n,k}},$$

then

$$\int_X f_n\,d\mu = \sum_{k=0}^{2^n+1} a_{n,k}\,\mu(S_{n,k}) = \sum_{k=0}^{2^n+1} \int_{S_{n,k}} f\,d\mu = \int_X f\,d\mu.$$

It is clear that

$$0 \le a_{n,k} \le \int_X f\,d\mu = \int_X f_n\,d\mu, \qquad 0 \le f_n \le \int_X f\,d\mu, \qquad |f_n - f| \le 2^{-n}A.$$

Hence, f_n uniformly converges to f and

$$0 \le f_n \le \int_X f\,d\mu = \int_X f_n\,d\mu.$$

Since $\{S_{n,k}\}_{k=0}^{2^n+1}$ are pairwise disjoint sets and $X = \cup_{k=0}^{2^n+1} S_{n,k}$, we have

$$\varphi \circ f_n = \sum_{k=0}^{2^n+1} \varphi(a_{n,k})\chi_{S_{n,k}},$$

$$\int_X \varphi \circ f_n\,d\mu = \sum_{k=0}^{2^n+1} \varphi(a_{n,k})\mu(S_{n,k}).$$

Since φ is a convex function on $[0,A]$, it is continuous on $(0,A)$ and it can be written on $[0,A]$ as

$$\varphi = \varphi_0 + \delta_0\chi_{\{0\}} + \delta_A\chi_{\{A\}},$$

where φ_0 is a continuous convex function on $[0,A]$ and $\delta_0, \delta_A \ge 0$. Hence

$$\begin{aligned}
\varphi \circ f_n &= \sum_{k=0}^{2^n+1} \varphi(a_{n,k})\chi_{S_{n,k}} \\
&= \sum_{k=0}^{2^n+1} \varphi_0(a_{n,k})\chi_{S_{n,k}} + \delta_0\chi_{f^{-1}(\{0\})} + \delta_A\chi_{f^{-1}(\{A\})} \\
&= \varphi_0 \circ f_n + \delta_0\chi_{f^{-1}(\{0\})} + \delta_A\chi_{f^{-1}(\{A\})}
\end{aligned}$$

and

$$\int_X \varphi \circ f_n \, d\mu = \sum_{k=0}^{2^n+1} \varphi(a_{n,k})\mu(S_{n,k})$$
$$= \sum_{k=0}^{2^n+1} \varphi_0(a_{n,k})\mu(S_{n,k}) + \delta_0\mu(f^{-1}(\{0\})) + \delta_A\mu(f^{-1}(\{A\}))$$
$$= \delta_0\mu(f^{-1}(\{0\})) + \delta_A\mu(f^{-1}(\{A\})) + \int_X \varphi_0 \circ f_n \, d\mu.$$

Since

$$0 \le a_{n,j} \le \int_X f \, d\mu = \int_X f_n \, d\mu = \sum_{k=0}^{2^n+1} a_{n,k}\,\mu(S_{n,k})$$

for $0 \le j \le 2^n+1$, Petrović's inequality gives

$$\sum_{k=0}^{2^n+1} \varphi(a_{n,k})\mu(S_{n,k}) \le \varphi\Big(\sum_{k=0}^{2^n+1} a_{n,k}\,\mu(S_{n,k})\Big) + \big(\mu(X)-1\big)\varphi(0). \tag{4.1}$$

Then the right hand side of (4.1) is equal to

$$\varphi\Big(\int_X f \, d\mu\Big) + \big(\mu(X)-1\big)\varphi(0).$$

Since $0 \le f_n \le A = \int_X f \, d\mu$ for every n, $\lim_{n\to\infty} f_n = f$ and φ_0 is a continuous function on $[0,A]$, we have $\lim_{n\to\infty} \varphi_0 \circ f_n = \varphi_0 \circ f$.

Since φ_0 is a continuous function on $[0,A]$, there exists a constant M with $|\varphi_0| \le M$ on $[0,A]$, hence $|\varphi_0 \circ f_n| \le M$ for every n.

Since μ is a finite measure, $M \in L^1(X,\mu)$ and the dominated convergence theorem gives

$$\lim_{n\to\infty} \int_X \varphi_0 \circ f_n \, d\mu = \int_X \varphi_0 \circ f \, d\mu.$$

Therefore, the left hand side of (4.1) has limit

$$\lim_{n\to\infty} \sum_{k=0}^{2^n+1} \varphi(a_{n,k})\mu(S_{n,k}) = \lim_{n\to\infty} \int_X \varphi \circ f_n \, d\mu$$
$$= \delta_0\mu(f^{-1}(\{0\})) + \delta_A\mu(f^{-1}(\{A\})) + \lim_{n\to\infty} \int_X \varphi_0 \circ f_n \, d\mu$$
$$= \delta_0\mu(f^{-1}(\{0\})) + \delta_A\mu(f^{-1}(\{A\})) + \int_X \varphi_0 \circ f \, d\mu$$
$$= \int_X \varphi \circ f \, d\mu.$$

This fact finishes the proof. □

Jensen's inequality and Theorem 2 provide the following result.

Corollary 1. *Let μ be a finite measure on the space X, φ be a convex function on $[0,a]$, and $f\colon X \to [0,a]$ be a measurable function with $\int_X f\,d\mu \in (0,a]$. Then $\varphi \circ f$ is a μ-integrable function. If*

$$f(x) \le \int_X f\,d\mu$$

for every $x \in X$, then

$$\mu(X)\varphi\Big(\frac{1}{\mu(X)}\int_X f\,d\mu\Big) \le \int_X \varphi\circ f\,d\mu \le \varphi\Big(\int_X f\,d\mu\Big) + \big(\mu(X)-1\big)\varphi(0).$$

5. Application to fractional integrals

Corollary 1 has the following direct consequence for general fractional integrals of Riemann–Liouville type.

Proposition 1. *Let $a<b$, $\alpha, B>0$ be real constants and T be a function as in Definition 4. If $f\colon [a,b]\to[0,B]$ is a measurable function,*

$$\mathbb{T}(\alpha) = \int_a^b \frac{1}{T(b,s,\alpha)}\,ds = \int_0^{g(b)-g(a)} \frac{dx}{G(x,\alpha)} < \infty, \qquad \int_a^b \frac{f(s)}{T(b,s,\alpha)}\,ds \in (0,B],$$

φ is a convex function on $[0,B]$, and

$$f(x) \le \int_a^b \frac{f(s)}{T(b,s,\alpha)}\,ds$$

for every $x\in[a,b]$, then $\varphi(f(s))/T\big(b,s,\alpha\big) \in L^1([a,b])$ and

$$\mathbb{T}(\alpha)\varphi\Big(\frac{1}{\mathbb{T}(\alpha)}\int_a^b \frac{f(s)}{T\big(b,s,\alpha\big)}\,ds\Big) \le \int_a^b \frac{\varphi(f(s))}{T\big(b,s,\alpha\big)}\,ds$$
$$\le \varphi\Big(\int_a^b \frac{f(s)}{T\big(b,s,\alpha\big)}\,ds\Big) + \big(\mathbb{T}(\alpha)-1\big)\varphi(0).$$

Proposition 1 for the classical Riemann–Liouville fractional integrals ${}^{RL}J^{\alpha}_{a^+}$, ${}^{RL}J^{\alpha}_{b^-}$ and for the Hadamard fractional integrals $H^{\alpha}_{a^+}$, $H^{\alpha}_{b^-}$ reads as follows.

Corollary 2. *Let $a<b$, $\alpha,B>0$ be real constants. If $f\colon[a,b]\to[0,B]$ is a measurable function,*

$${}^{RL}J^{\alpha}_{a^+}f(b) = \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s)\,ds \in (0,B],$$

φ is a convex function on $[0,B]$, and

$$f(x) \le \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s)\,ds$$

for every $x \in [a,b]$*, then* $(b-s)^{\alpha-1}\varphi(f(s)) \in L^1([a,b])$ *and*

$$\begin{aligned}\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}\varphi\Big(\frac{\alpha}{(b-a)^{\alpha}}\int_a^b (b-s)^{\alpha-1} f(s)\,ds\Big) \\ \le \frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1}\varphi(f(s))\,ds \\ \le \varphi\Big(\frac{1}{\Gamma(\alpha)}\int_a^b (b-s)^{\alpha-1} f(s)\,ds\Big) + \Big(\frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)} - 1\Big)\varphi(0).\end{aligned}$$

Similarly for the right Riemann–Liouville fractional integral ${}^{RL}J^{\alpha}_{b^-}f(a)$.

Corollary 3. *Let* $a < b$, $\alpha, B > 0$ *be real constants. If* $f\colon [a,b] \to [0,B]$ *is a measurable function,*

$$H^{\alpha}_{a^+}f(b) = \frac{1}{\Gamma(\alpha)}\int_a^b \Big(\log\frac{b}{s}\Big)^{\alpha-1}\frac{f(s)}{s}\,ds \in (0,B],$$

φ *is a convex function on* $[0,B]$*, and*

$$f(x) \le \frac{1}{\Gamma(\alpha)}\int_a^b \Big(\log\frac{b}{s}\Big)^{\alpha-1}\frac{f(s)}{s}\,ds$$

for every $x \in [a,b]$*, then* $\big(\log\frac{b}{s}\big)^{\alpha-1}\frac{\varphi(f(s))}{s} \in L^1([a,b])$ *and*

$$\begin{aligned}\frac{\big(\log\frac{b}{a}\big)^{\alpha}}{\Gamma(\alpha+1)}\varphi\Big(\frac{\alpha}{\big(\log\frac{b}{a}\big)^{\alpha}}\int_a^b \Big(\log\frac{b}{s}\Big)^{\alpha-1}\frac{f(s)}{s}\,ds\Big) \\ \le \frac{1}{\Gamma(\alpha)}\int_a^b \Big(\log\frac{b}{s}\Big)^{\alpha-1}\frac{\varphi(f(s))}{s}\,ds \\ \le \varphi\Big(\frac{1}{\Gamma(\alpha)}\int_a^b \Big(\log\frac{b}{s}\Big)^{\alpha-1}\frac{f(s)}{s}\,ds\Big) + \Big(\frac{\big(\log\frac{b}{a}\big)^{\alpha}}{\Gamma(\alpha+1)} - 1\Big)\varphi(0).\end{aligned}$$

Similarly for the right Hadamard fractional integral $H^{\alpha}_{b^-}f(a)$.

ACKNOWLEDGEMENTS

The research of José Manuel Rodríguez and José María Sigarreta Almira is supported by a grant from Agencia Estatal de Investigación (PID2019-106433GB-I00 / AEI / 10.13039 / 501100011033), Spain. The research of José Manuel Rodríguez is supported by the Madrid Government (Comunidad de Madrid-Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M23), and in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

REFERENCES

[1] I. A. Baloch and Y.-M. Chu, "Petrović-type inequalities for harmonic h-convex functions," *J. Funct. Spaces*, vol. 2020, p. 3075390, 2020, doi: 10.1155/2020/3075390.

[2] P. Bosch, H. J. Carmenate, J. M. Rodríguez, and J. M. Sigarreta, "Generalized inequalities involving fractional operators of Riemann-Liouville type," *AIMS Math.*, vol. 7, no. 1, pp. 1470–1485, 2022, doi: 10.3934/math.2022087.

[3] Z. Dahmani, "On Minkowski and Hermite-Hadamard integral inequalities via fractional integral," *Ann. Funct. Anal.*, vol. 1, pp. 51–58, 2010, doi: 10.15352/afa/1399900993.

[4] R. Gorenflo and F. Mainardi, *Fractals and fractional calculus in continuum mechanics*, 1st ed. Vienna: Springer, 1997.

[5] J. Han, P. O. Mohammed, and H. Zeng, "Generalized fractional integral inequalities of Hermite-Hadamard-type for a convex function," *Open Math.*, vol. 18, pp. 794–806, 2020, doi: 10.1515/math-2020-0038.

[6] W. Iqbal, K. M. Awan, A. U. Rehman, and G. Farid, "An extension of Petrović's inequality for h-convex (h-concave) functions in plane," *Open J. Math. Sci.*, vol. 2019, no. 3, pp. 398–403, 2019, doi: 10.30538/oms2019.0082.

[7] J. L. W. V. Jensen, "Om konvekse funktioner og uligheder mellem middelvaerdier [About convex functions and inequalities between mean values]," *Nyt Tidsskr. Math.*, vol. 16 B, pp. 49–69, 1905.

[8] A. A. Kilbas, O. I. Marichev, and S. G. Samko, *Fractional integrals and derivatives. Theory and applications*, 1st ed. Pennsylvania: Gordon & Breach, 1993.

[9] P. Kórus and J. E. Nápoles Valdés, "q-hermite-hadamard inequalities for functions with convex or h-convex q-derivative," *Math. Ineq. Appl.*, vol. 25, no. 2, pp. 601–610, 2022, doi: 10.7153/mia-2022-25-36.

[10] S. Mubeen, S. Habib, and M. N. Naeem, "The Minkowski inequality involving generalized k-fractional conformable integral," *J. Inequal. Appl.*, vol. 2019, p. 81, 2019, doi: 10.1186/s13660-019-2040-8.

[11] K. S. Nisar, F. Qi, G. Rahman, S. Mubeen, and M. Arshad, "Some inequalities involving the extended gamma function and the Kummer confluent hypergeometric k-function," *J. Inequal. Appl.*, vol. 2018, p. 135, 2018, doi: 10.1186/s13660-018-1717-8.

[12] M. Petrović, "Sur une fonctionnelle," *Publ. Math. Univ. Belgrade*, vol. 1, pp. 146–149, 1932.

[13] J. E. Pečarić, "On the Petrović inequality for convex functions," *Glasnik Matem.*, vol. 18, no. 38, pp. 77–85, 1983.

[14] G. Rahman, T. Abdeljawad, F. Jarad, A. Khan, and K. S. Nisar, "Certain inequalities via generalized proportional Hadamard fractional integral operators," *Adv. Difference Eq.*, vol. 2019, p. 454, 2019, doi: 10.1186/s13662-019-2381-0.

[15] G. Rahman, K. S. Nisar, B. Ghanbari, and T. Abdeljawad, "On generalized fractional integral inequalities for the monotone weighted Chebyshev functionals," *Adv. Difference Eq.*, vol. 2020, p. 368, 2020, doi: 10.1186/s13662-020-02830-7.

[16] S. Rashid, M. A. Noor, K. I. Noor, and Y.-M. Chu, "Ostrowski type inequalities in the sense of generalized $\mathcal{K}$-fractional integral operator for exponentially convex functions," *AIMS Math.*, vol. 5, no. 3, pp. 2629–2645, 2020, doi: 10.3934/math.2020171.

[17] A. U. Rehman, G. Farid, and V. N. Mishra, "Generalized convex function and associated Petrović's inequality," *Int. J. Anal. Appl.*, vol. 17, no. 1, pp. 122–131, 2019, doi: 10.28924/2291-8639-17-2019-122.

[18] A. U. Rehman, M. Mudessir, H. T. Fazal, and G. Farid, "Petrović's inequality on coordinates and related results," *Cogent Math.*, vol. 3, p. 1227298, 2016, doi: 10.1080/23311835.2016.1227298.

[19] A. W. Roberts and D. E. Varberg, *Convex functions*. New York: Academic Press, 1973.

[20] Y. Sawano and H. Wadade, “On the Gagliardo-Nirenberg type inequality in the critical Sobolev-Morrey space,” *J. Fourier Anal. Appl.*, vol. 19, no. 1, pp. 20–47, 2013, doi: 10.1007/s00041-012-9223-8.

[21] E. Set, M. Tomar, and M. Z. Sarikaya, “On generalized Grüss type inequalities for *k*-fractional integrals,” *Appl. Math. Comput.*, vol. 269, pp. 29–34, 2015, doi: 10.1016/j.amc.2015.07.026.

Authors’ addresses

Péter Kórus
(**Corresponding author**) Institute of Applied Pedagogy, Juhász Gyula Faculty of Education, University of Szeged, Hattyas utca 10, H-6725 Szeged, Hungary
E-mail address: korus.peter@szte.hu

Juan Eduardo Nápoles Valdés
UNNE, FaCENA, Av. Libertad 5450, Corrientes 3400, Argentina
E-mail address: jnapoles@exa.unne.edu.ar

José Manuel Rodríguez
Universidad Carlos III de Madrid, Departamento de Matemáticas, Avenida de la Universidad 30, 28911 Leganés, Madrid, Spain
E-mail address: jomaro@math.uc3m.es

José María Sigarreta Almira
Universidad Autónoma de Guerrero, Centro Acapulco, CP 39610, Acapulco de Juárez, Guerrero, Mexico
E-mail address: josemariasigarretaalmira@hotmail.com