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OPTIMAL CASH HOLDING MODEL BASED ON MINIMIZATION OF RUIN PROBABILITY

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Abstract. The management of optimal cash holding is particularly important for the sustainability of enterprises' survival and business decision-making. It is not good for enterprises to have more or less cash holding. In the meantime, the probability of enterprise bankruptcy is also one of the most important research topics in the field of economic finance and management science, such as the early warning of investment risk and financial decision support. In this paper, an optimal cash holding model is established. It is assumed that the objective of optimal cash holding is to minimize the probability of bankruptcy and the security area of cash holding is the constraint condition. By using the dynamic stochastic programming method, the optimal conversion strategy and the analytic expressions of the value function are obtained, and the relevant economic explanations and numerical examples are given. The effects of capital market parameters and consumption function parameters on the optimal conversion strategy and optimal cash holding are discussed.

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1. INTRODUCTION

Capital is the blood of enterprises. The health of enterprises is closely related to the successful operation of funds. Since the financial crisis took place in 2008, Chinese enterprises, especially small and medium-sized enterprises have paid more and more attention to their cash holding decisions due to China's macroeconomic policies. This is especially true after the breakout of COVID-19 in 2020. This paper holds that enterprises will put survival in the first place and then consider income in some circumstances. Take financial institutions, such as insurance companies as an example, due to the particularity of their business, the enterprises usually pay more

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attention to the problem of ruin probability. Another example is that for enterprises in the process of recession avoiding bankrupty is the first thing to do. Also, for enterprises under some special background, no bankruptcy means victory, for example, companies in manufacture, tourism, and catering industries have to struggle to survival after being hit by COVID-19. At this time, the decision of cash holding is particularly important.

In the field of economic finance and management science, the ruin probability of enterprises is also one of the most important research topics. It plays a significant role in the early warning of investment risk and financial decision support. In finace, when a company's net assets is negative, that is, when the value of assets is less than the value of liabilities, stock bankruptcy will happen. When its operating cash flow is not enough to meet the existing debts due, flow bankruptcy will happen, which means that an enterprise cannot pay its debts. The possibility of enterprises' bankruptcy is an indisputable fact. The possibility of bankruptcy can be measured by ruin probability. Different companies and industries should have different bankruptcy probability. [14] defined the moment of investors' bankruptcy is when remaining wealth reaches zero for the first time. In order to fully consider the safety of the company's operations, this article defines the moment of bankruptcy as the first time that company's residual wealth falls below a threshold m instead of zero, ie $\tau = \inf\{t > 0; 0 < X_t \leq m\}$, the ruin probability is $P_{x_t}\{\tau < +\infty\}$, X_t is the residual assets of an enterprise. When the company's remaining wealth is less than m, it will be considered bankrupt. However, in practice, it does not really mean that the company is going bankrupt. It may just indicate that the company needs to adjust its cash holding strategy and maintain the liquidity of the company's normal operations, or the company's financial situation is not very bad after having considered many other aspects of the company. But studying the ruin probability of companies is still of great theoretical and practical significance. The ruin probability can be used as an indicator of the stability and development of a company as a comprehensive residual wealth management process. Taking insurance companies as an example, it is supposed that bankruptcy occurs when its surplus is zero for the first time in most cases. Although this does not mean it will happen in reality.

The ruin probability was first proposed by [6]. Under the discrete-time and discrete-state model, the optimal investment strategy problem that minimizes the ruin probability is considered for general investors. Long afterwards, [4] first considered the problem of minimizing the ruin probability for insurance companies. Different from previous studies on bankruptcy, [5] adopted the dynamic programming method provided by [7] and [11] to obtain the explicit expression of the minimum ruin probability and the corresponding optimal strategy. This is the first application of dynamic programming principle and HJB equation theory in bankruptcy problems. Since then, they have become the main tools to study the minimization of bankruptcy probability. Then [5] studied the maximization survival strategy of companies with fixed debt.

[9] considered the problem of minimizing the ruin probability of allowed investment under the classical Cramer-Lundberg model. And after that, [17] introduced linear reinsurance control based on their model. For more models, see the results of [[16], [8], [18], [20], [2], [1], [13], [3], [12], [10], [22]].

Assuming that the objective function pursued by an enterprise is the pursuit of minimizing ruin probability, the main problem to be studied in this paper can be summarized as follows: for an enterprise, how to determine the optimal cash holding in a safe area to achieve the minimum ruin probability.

2. MODELING

In this paper, we assume that the residual assets of an enterprise is X_t at time t, and the ratio of the residual assets in the form of cash is $0 \leq \beta_t \leq 1$. As a result, the cash holding amount is $C_t = \beta_t X_t$, and the holding amount of risky assets is $R_t = (1-\beta_t)X_t$. According to Robichek's [15] trade-off theory, it is known that there is a cash holding target, and the cash holding level will be adjusted when the actual cash holding deviates from the target. [19] pointed out that there is a safe area $[L,H]$ for cash holding. When the cash holding level of enterprisess is not in the safe area, enterprises need to adjust cash assets and risky assets to make it return to the safe area. If there is a parameter μ_t, and $\mu_t(1-\beta_t)X_t$ is the conversion amount. When $\mu_t > 0$, it means that risky assets are converted into cash assets, and when $\mu_t < 0$, the case is just the opposite. There is no need to discuss the case when $\mu_t = 0$.

In addition, it is assumed that enterprises have a consumption function on which wealth depends. Under the condition of income flow, [20] mentioned how to convert the proportional consumption rate into linear consumption, but did not make a detailed discussion. However, if there is no income flow, bankruptcy will never happen under the proportional consumption model. [2] defined bankruptcy under the proportional consumption pattern as the reduction of wealth to a predetermined level (which must be greater than zero). Based on the above references, we assumed that the consumption function is a linear function $c(X_t) = c + \theta X_t$, which not only reflects the basic consumption expenditure of enterprises, but also reflects the general state of enterprises' increasing expenditure with increasing wealth. Where $c > 0$ is the minimum consumption of enterprises to meet daily expenses, and $0 \leq \theta \leq r_0$ is the consumption ratio that increases with wealth (such as dividends paid, rewards given to employees, etc.).

The existence of the safe area requires the cash holding after conversion $\beta_t X_t + \mu_t(1-\beta_t)X_t - (c+\theta X_t)$ must be in the safe area $[L,H]$. As a result, μ_t must be in the interval $\left[\frac{L+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}, \frac{H+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}\right]$. Let

$$D_1 = [a_1, b_1] = \left[\frac{L+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}, \frac{H+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}\right]$$

represent the set of all feasible strategies μ_t.

Let $B(t)$ denote the price of risky-free assets at time t and $S(t)$ denote the price of risky assets at time t, then $B(t)$, $S(t)$ can be characterized by the following stochastic processes:

$$dB(t) = r_0 B(t)dt \tag{2.1}$$

$$dS(t) = S(t)(r_1 dt + \sigma_1 dW(t)) \tag{2.2}$$

Where r_0, r_1 are the expected rate of return of risky-free assets and risky assets respectively, σ_1 is the volatility and $W(t)$ is the one-dimensional Standard Brownian motion.

Then the random variable X_t satisfies the following stochastic equation:

$$dX_t = \frac{\beta_t X_t + \mu_t(1-\beta_t)X_t}{B_t} dB_t + \frac{(1-\mu_t)(1-\beta_t)X_t}{S_t} dS_t - (c+\theta X_t)dt \tag{2.3}$$

By substituting (2.1) and (2.2) into (2.3), the following process can be obtained:

$$\begin{aligned} dX_t = {} & [r_0\beta_t X_t + r_0\mu_t(1-\beta_t)X_t + r_1(1-\mu_t)(1-\beta_t)X_t - (c+\theta X_t)]dt \\ & + (1-\mu_t)(1-\beta_t)X_t\sigma_1 dW_t \end{aligned}$$

Defining the value function $R(x) = \inf_{\mu_t \in D_1} P_{X_t}(\tau < +\infty)$, it's boundary conditions are

$$R(m) = 1, R(+\infty) = 0 \tag{2.4}$$

So far, the problem to be solved in this paper can be expressed by the following model [M-1]:

$$[M-1]: \begin{cases} \inf_{\mu_t \in D_1} P_{X_t}(\tau < +\infty) \\ s.t. \begin{cases} dX_t & = [r_0\beta_t X_t + r_0\mu_t(1-\beta_t)X_t + r_1(1-\mu_t)(1-\beta_t)X_t \\ & \quad -(c+\theta X_t)]dt + (1-\mu_t)(1-\beta_t)X_t\sigma_1 dW_t \\ \mu_t \in D_1 \end{cases} \end{cases}$$

3. Model solving

In order to solve the model, the threshold value of m should be set first. In the previous literature [21], no specific value of m was set. And setting $m = f(L, H, c, \theta, r_1, r_0)$ is more reasonable than setting it as a fixed value. For the convenience of discussion, this paper supposes $m = \frac{(H+c)(r_1-r_0)+c}{r_1-\theta-r_1\theta+r_0\theta}$. $(H+C)(R_1-r_0)$ is the risk premium for maximum cash holding after taking daily minimum consumption. The denominator part can be regarded as the comprehensive interest rate after considering the risky assets interest rate, risky-free asset interest rate and consumption rate, so it is reasonable and practical to regard it as the pricing formula of m.

When the surplus wealth x is less than m, bankruptcy must occur, and $R(m) = 1$. Therefore, we only discuss the solution of the model when $x > m$. Since the value function $R(x) = \inf_{\mu_t \in D_1} P_{X_t}(\tau < \infty)$ is independent on time parameter, our goal is to obtain the minimum ruin probability $R(x)$ and the optimal strategy μ_t^*. In order to solve this problem, the following HJB equation with boundary conditions (2.4) is obtained according to Krylov's [11] theorem 1.4.5:

$$R_t + \inf\{[r_0\beta_t x + r_0\mu_t(1-\beta_t)x + r_1(1-\mu_t)(1-\beta_t)x \\ -(c+\theta x)]R_x + \frac{1}{2}x^2(1-\beta_t)^2(1-\mu_t)^2\sigma_1^2 R_{xx}\} = 0 \tag{3.1}$$

The constraint of the security area requires the cash holding after conversion should be in the safe area. That is, $\mu_t \in D_1$. And then we can get the following verification theorem:

Theorem 1. *If $V(x)$ is a monotonically decreasing second order continuous differentiable convex solution of HJB equation (3.1) on $(0,+\infty)$ and satisfies boundary value condition (2.4), then $V(x)$ and $R(x)$ are consistent. Furthermore, let μ_t^* satisfy the following equation for all $x \in (0,+\infty)$*

$$V_t + [r_0\beta_t x + r_0\mu_t(1-\beta_t)x + r_1(1-\mu_t)(1-\beta_t)x - (c+\theta x)]V_x \\ + \frac{1}{2}x^2(1-\beta_t)^2(1-\mu_t)^2\sigma_1^2 V_{xx} = 0 \tag{3.2}$$

then $\mu_s^* = \mu^*(X_s^{\mu^*})$, where $X_s^{\mu^*}$ is the solution of equation (2.3), $\mu^*(x)$ is called the optimal feedback control function, that is $V(x) = R(x) = R_{\mu_t^*}(x)$.

Proof. The proof of this theorem can be seen in Appendix 1 of [8]. □

For any monotone decreasing second order continuous differentiable convex function V. For convenience, we note that $\mu_{V(x)} = 1 + \frac{r_1 - r_0}{x(1-\beta_t)\sigma_1^2}\frac{V_x}{V_{xx}}$, $l_1 = \frac{(L+c)(r_1-r_0)+2c}{(1-\theta)(r_1-r_0)+2(r_0-\theta)}$ and $l_2 = \frac{(H+c)(r_1-r_0)+2c}{(1-\theta)(r_1-r_0)+2(r_0-\theta)}$. Then we need to discuss the problem in three cases: $m < l_1 < l_2$, $l_1 < m < l_2$ and $l_1 < l_2 < m$.

Case 1. When $m < l_1 < l_2$, we define the set firstly

$$K_1 = \{x : x \in (m,+\infty), \mu_{V(x)} \le a_1\}$$
$$K_2 = \{x : x \in (m,+\infty), a_1 \le \mu_{V(x)} \le b_1\}$$
$$K_3 = \{x : x \in (m,+\infty), \mu_{V(x)} \ge b_1\}$$

Lemma 1. *If $V(x)$ is a monotonically decreasing differentiable convex function on $(m,+\infty)$, and $V(x)$ satisfies the following equation on K_1*

$$\frac{V_{xx}}{V_x} = g_1(x) \triangleq -\frac{2[r_1 x - \theta x - (L+c+\theta x)(r_1-r_0) - c]}{(x-L-c-\theta x)^2\sigma_1^2} \tag{3.3}$$

then V is the solution of HJB equation (3.1) on K_1.

Proof. Because of $x > \frac{(H+c)(r_1-r_0)+c}{r_1-\theta-r_1\theta+r_0\theta} > \frac{(L+c)(r_1-r_0)+c}{r_1-\theta-r_1\theta+r_0\theta}$ and $r_1-\theta-r_1\theta+r_0\theta > r_0-\theta > 0$, we can get $(r_1-\theta-r_1\theta+r_0\theta)x > (L+c)(r_1-r_0)+c$. That is $r_1x-\theta x-(L+c+\theta x)(r_1-r_0)-c>0$. Therefore (3.3) satisfies $\frac{V_{xx}}{V_x}<0$. Taking (3.3) into (3.1),

$$\mu_t^* = \frac{L+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}$$

In this case, μ_t^* takes the boundary value of the security area constraint, so μ_t^* is the optimal solution of (3.1) left-hand optimization problem, and from (3.3) we know that its optimal solution is 0, then V is the solution of HJB equation (3.1) on K_1. □

Lemma 2. *If $V(x)$ is a monotonically decreasing differentiable convex function on $(m,+\infty)$, and $V(x)$ satisfies the following equation on K_2*

$$\frac{V_{xx}}{V_x} = g_2(x) \triangleq \frac{(r_1-r_0)^2}{2\sigma_1^2[(r_0-\theta)x-c]} \tag{3.4}$$

then V is the solution of HJB equation (3.1) *on K_2.*

Proof. Let $x\in K_2$, Because the optimal solution of optimization problem

$$\inf\{[r_0\beta_t x + r_0\mu_t(1-\beta_t)x + r_1(1-\mu_t)(1-\beta_t)x-(c+\theta x)]V_x + \frac{1}{2}x^2(1-\beta_t)^2(1-\mu_t)^2\sigma_1^2V_{xx}\} = 0$$

without any constraints is $\mu_t^* = 1-2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x}$.

So

$$\begin{aligned} K_2 &= \{x : x\in(m,+\infty), a_1 \le \mu_{v(x)} \le b_1\} \\ &= \{x : x\in(m,+\infty), l_1 \le x \le l_2\} \end{aligned}$$

Because $l_2>m$, then we can have $\frac{(H+c)(r_1-r_0)+2c}{(1-\theta)(r_1-r_0)+2(r_0-\theta)} > \frac{(H+c)(r_1-r_0)+c}{r_1-\theta-r_1\theta+r_0\theta}$. By simplifying, we can get $(H+c)(r_1-r_0)(r_0-\theta) < c(r_1-r_0)(1-\theta)$. So there is $Hr_0-H\theta+cr_0-c<0$, therefore $\frac{(H+c)(r_1-r_0)+2c}{(1-\theta)(r_1-r_0)+2(r_0-\theta)} - \frac{c}{r_0-\theta} = \frac{(r_1-r_0)(Hr_0-H\theta+cr_0-c)}{(r_0-\theta)[(1-\theta)(r_1-r_0)+2(r_0-\theta)])} < 0$. That is $\frac{(H+c)(r_1-r_0)+2c}{(1-\theta)(r_1-r_0)+2(r_0-\theta)} < \frac{c}{r_0-\theta}$, then $x<l_2<\frac{c}{r_0-\theta}$. As a result (3.4) is less than 0, satisfying $\frac{V_{xx}}{V_x}<0$.

According to the definition of K_2, when $x\in K_2$, $a_1\le \mu_{V(x)} \le b_1$, so μ_t^* is the optimal solution of the left-hand optimization problem of (3.1), and the optimal solution is 0. According to equation (3.4), V is the solution of HJB equation (3.1) on K_2. □

Lemma 3. *If $V(x)$ is a monotonically decreasing differentiable convex function on $(m,+\infty)$, and $V(x)$ satisfies the following equation on K_3*

$$\frac{V_{xx}}{V_x} = g_3(x) \triangleq -\frac{2[r_1x-\theta x-(H+c+\theta x)(r_1-r_0)-c]}{(x-H-c-\theta x)^2\sigma_1^2} \tag{3.5}$$

then V is the solution of HJB equation (3.1) *on K_3.*

Proof. Because $x > \frac{(H+c)(r_1-r_0)+c}{r_1-\theta-r_1\theta+r_0\theta}$ and $r_1-\theta-r_1\theta+r_0\theta > r_0-\theta > 0$, so $r_1x-\theta x-(H+c+\theta x)(r_1-r_0)-c > 0$. Then (3.5) satisfies $\frac{V_{xx}}{V_x} < 0$. By substituting (3.5) into (3.1), we can get $\mu_t^* = \frac{H+c+\theta X_t-\beta_t X_t}{(1-\beta_t)X_t}$.

In this case, μ_t^* takes the right bound value of the security zone constraint, so μ_t^* is the optimal solution of the left-hand optimization problem of (3.1), and the optimal solution is 0. According to (3.5), V is the solution of HJB equation (3.1) on K_3. □

According to the above three lemmas, we can get the following three sets:

$$K_1 = \{x : x \in (m,+\infty), \mu_{V(x)} \leq a_1\} = \{x : x \in (m,+\infty), x \leq l_1\}$$
$$K_2 = \{x : x \in (m,+\infty), a_1 \leq \mu_{V(x)} \leq b_1\} = \{x : x \in (m,+\infty), l_1 \leq x \leq l_2\}$$
$$K_3 = \{x : x \in (m,+\infty), \mu_{V(x)} \geq b_1\} = \{x : x \in (m,+\infty), x \geq l_2\}$$

Next, we will discuss the solutions of equation (3.1) on these three sets. The key is to deal with the continuity of value function V at $x = l_1$ and $x = l_2$.

(1): When $x \in K_1$, according to Lemma 1, if we want to get the solution of (3.1) on K_1, we need to solve the following equation on K_1

$$\frac{V_1''}{V_1'} = g_1(x), V_1(m) = 1$$

By solving the above equation, we can get

$$V_1(x) = 1 - C_1 \int_m^x exp\left(\int_m^t g_1(s)ds\right) dt, \forall x \in K_1,$$

where C_1 is the undetermined coefficient.

(2): When $x \in K_2$, according to Lemma 2, if we want to get the solution of (3.1) on K_2, we need to solve the following equation on K_2

$$\frac{V_2''}{V_2'} = g_2(x)$$

By solving the above equation, we can get

$$V_2(x) = C_2 + C_3 \int_x^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right) dt, \forall x \in K_2,$$

where C_2, C_3 are the undetermined coefficient.

(3): When $x \in K_3$, according to Lemma 3, if we want to get the solution of (3.1) on K_3, we need to solve the following equation on K_3

$$\frac{V_3''}{V_3'} = g_3(x), V_3(+\infty) = 0$$

By solving the above equation, we can get

$$V_3(x) = C_4 \int_x^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt, \forall x \in K_3,$$

where C_4 is the undetermined coefficient.

Next, we require $V(x)$ to be continuously differentiable at $x = l_1$, i.e

$$1 - C_1 \int_m^{l_1} exp\left(\int_m^t g_1(s)ds\right) dt = C_2 + C_3 \int_{l_1}^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right) dt \tag{3.6}$$

$$C_1 exp\left(\int_m^{l_1} g_1(s)ds\right) = C_3 \tag{3.7}$$

$$C_1 exp\left(\int_m^{l_1} g_1(s)ds\right) g_1(l_1) = C_3 g_2(l_1) \tag{3.8}$$

Meanwhile, we require $V(x)$ to be continuously differentiable at $x = l_2$, i.e

$$C_2 = C_4 \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt \tag{3.9}$$

$$C_3 exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) = C_4 \tag{3.10}$$

$$C_3 exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) g_2(l_2) = C_4 g_3(l_2) \tag{3.11}$$

By solving equations (3.6)-(3.11), we can get

$$\begin{aligned}
C_1 = \Bigg[&\int_m^{l_1} exp\left(\int_m^t g_1(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) \\
&\times \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) \int_{l_1}^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right) dt\Bigg]^{-1} \\
C_2 = \Bigg[&\int_m^{l_1} exp\left(\int_m^t g_1(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) \\
&\times \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) \int_{l_1}^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right) dt\Bigg]^{-1} \\
&\times exp\left(\int_m^{l_1} g_1(s)ds\right) exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt \\
C_3 = \Bigg[&\int_m^{l_1} exp\left(\int_m^t g_1(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) \\
&\times exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt \\
&+ exp\left(\int_m^{l_1} g_1(s)ds\right) \int_{l_1}^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right) dt\Bigg]^{-1} exp\left(\int_m^{l_1} g_1(s)ds\right) \\
C_4 = \Bigg[&\int_m^{l_1} exp\left(\int_m^t g_1(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right) exp\left(\int_{l_1}^{l_2} g_2(s)ds\right) \\
&\times \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt + exp\left(\int_m^{l_1} g_1(s)ds\right)
\end{aligned}$$

$$\times \int_{l_1}^{l_2} exp\left(\int_{l_1}^{t} g_2(s)ds\right)dt\Bigg]^{-1} exp\left(\int_{m}^{l_1} g_1(s)ds\right) exp\left(\int_{l_1}^{l_2} g_2(s)ds\right)$$

Substituting C_1, C_2, C_3, C_4 into (3.8) and (3.11) can verify the second-order continuous differentiability. Next, we will still give the solution of the model [M-1] in the form of theorem when $m < l_1 < l_2$.

Theorem 2. *When $m < l_1 < l_2$, μ_t^* of model [M-1] is as follows:*

$$\mu_t^* = \begin{cases} \frac{L+c+\theta x-\beta_t x}{(1-\beta_t)x} & x \in (m, l_1]; \\ 1 - 2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x} & x \in [l_1, l_2]; \\ \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x} & x \in [l_2, +\infty]. \end{cases}$$

the corresponding value function is as follows:

$$V(x) = \begin{cases} 1 - C_1 \int_m^x exp\left(\int_m^t g_1(s)ds\right)dt & x \in (m, l_1]; \\ C_2 + C_3 \int_x^{l_2} exp\left(\int_{l_1}^t g_2(s)ds\right)dt & x \in [l_1, l_2]; \\ C_4 \int_x^{\infty} exp\left(\int_{l_2}^t g_3(s)ds\right)dt & x \in [l_2, +\infty]. \end{cases}$$

Proof. The proof process of the theorem is the above calculation and derivation process, which is omitted here. □

According to Theorem 2, it can be deduced that when $m < l_1 < l_2$, the optimal cash holding C^* is

$$C^* = \begin{cases} L & x \in (m, l_1]; \\ (1-\theta+2\frac{r_0-\theta}{r_1-r_0})x - (c+\frac{2c}{r_1-r_0}) & x \in [l_1, l_2]; \\ H & x \in [l_2, +\infty]. \end{cases}$$

and it is not difficult to verify that when $x \in [l_1, l_2]$, the value of $(1-\theta+2\frac{r_0-\theta}{r_1-r_0})x - (c+\frac{2c}{r_1-r_0})$ is in $[L, H]$.

Case 2. When $l_1 < m < l_2$, if $l_1 < x < m$, then ”bankruptcy” must occur and the model has no solution. Therefore, we only need to discuss the solution of the model with the value of x in the following two sets.

$$I_1 = \{x : x \in (m, \infty), m \le x \le l_2\} \tag{3.12}$$

$$I_2 = \{x : x \in (m, \infty), x > l_2\} \tag{3.13}$$

According to the proof process of Lemma 2, the solution $V(x)$ of HJB equation (3.1) on I_1 also satisfies the following equation:

$$\frac{V''}{V'} = g_2(x), x \in I_1 \tag{3.14}$$

At the same time, the solution $V(x)$ of HJB equation (3.1) on I_2 also satisfies the following equation:

$$\frac{V''}{V'} = g_3(x), x \in I_2 \tag{3.15}$$

Next, we also need to discuss the solutions of the HJB equation (3.1) on I_1, I_2. At this time, the key problem is to deal with the value function $V(x)$ at $x = l_2$.

From equation (3.14) and $V(m) = 1$, we can get

$$V(x) = 1 - D_1 \int_m^x exp\left(\int_m^t g_2(s)ds\right) dt.$$

By (3.15) and $V(+\infty) = 0$, then $V(x) = D_2 \int_x^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt$.

Because $V(x)$ is continuously differentiable of the second order at $x = l_2$, according to the continuity and the continuous differentiability of the first order, there is

$$1 - D_1 \int_m^{l_2} exp\left(\int_m^t g_2(s)ds\right) dt = D_2 \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt$$

$$D_1 exp\left(\int_m^{l_2} g_2(s)ds\right) = D_2$$

By combining the above two equations, we can get

$$D_1 = \left[\int_m^{l_2} exp\left(\int_m^t g_2(s)ds\right) dt + exp\left(\int_m^{l_2} g_2(s)ds\right) \times \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt\right]^{-1}$$

$$D_2 = \left[\int_m^{l_2} exp\left(\int_m^t g_2(s)ds\right) dt + exp\left(\int_m^{l_2} g_2(s)ds\right) \times \int_{l_2}^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt\right]^{-1} exp\left(\int_m^{l_2} g_2(s)ds\right)$$

and it's not hard to verify that D_1 and D_2 satisfy $D_1 exp\left(\int_m^{l_2} g_2(s)ds\right) g_2(l_2) = D_2 g_3(l_2)$. That is to say, the second order function is also continuously differentiable. According to the above deduction, the following theorem can be obtained:

Theorem 3. *When $l_1 < m < l_2$, μ_t^* of model [M-1] is as follows:*

$$\mu_t^* = \begin{cases} 1 - 2\frac{[c-(r_0-\theta)x]}{(1-\beta_t)(r_1-r_0)x} & x \in (m, l_2] \\ \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x} & x \in [l_2, +\infty) \end{cases}$$

and the corresponding value function is:

$$V(x) = \begin{cases} 1 - D_1 \int_m^x exp\left(\int_m^t g_2(s)ds\right) dt & x \in (m, l_2] \\ D_2 \int_x^{+\infty} exp\left(\int_{l_2}^t g_3(s)ds\right) dt & x \in [l_2, +\infty) \end{cases}$$

Proof. The proof process of the theorem is the above calculation and derivation process, which is omitted here. □

According to Theorem 3, it can also be calculated that when $l_1 < m < l_2$, the optimal cash holding amount of C^* is

$$C^* = \begin{cases} (1-\theta+2\frac{r_0-\theta}{r_1-r_0})x-(c+\frac{2c}{r_1-r_0}) & x\in(m,l_2] \\ H & x\in[l_2,+\infty). \end{cases}$$

Case 3. When $l_1 < l_2 < m$, if $x < m$, then bankruptcy must happen, and the model has no solution. Because $x \geq m$ and $l_1 < l_2 < m$, so $x > l_2$. According to the discussion of the former two cases, the optimal solution of the model [M-1] is $\mu_t^* = \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x}$. In this case, the corresponding value function $V(x)$ satisfies the following equation:

$$\frac{V''(x)}{V'(x)} = g_3(x) = -\frac{2[r_1x-\theta x-(H+c+\theta x)(r_1-r_0)-c]}{(x-H-c-\theta x)^2\sigma_1^2}$$

By solving this equation, we have $V(x) = E_1 + E_2\int_x^{+\infty} exp\left(\int_m^t g_3(s)ds\right)dt$, $\forall x\in(m,+\infty)$, where E_1 and E_2 are the undetermined coefficient. According to $V(m) = 1, V(+\infty) = 0$, we can get $E_1 + E_2\int_m^{+\infty} exp\left(\int_m^t g_3(s)ds\right)dt = 1, E_1 = 0$. By solving this equation, we can get $E_1 = 0, E_2 = \frac{1}{\int_m^{+\infty} exp\left(\int_m^t g_3(s)ds\right)dt}$.

Similarly, according to the above derivation, the following theorem can be obtained:

Theorem 4. *When $l_1 < l_2 < m$, μ_t^* of model [M-1] is:*

$$\mu_t^* = \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x}, \forall x\in(m,+\infty)$$

and the corresponding value function is:

$$V(x) = E_1 + E_2\int_x^{+\infty} exp\left(\int_m^t g_3(s)ds\right)dt, \forall x\in(m,+\infty)$$

4. Numerical examples

4.1. *Examples of model solutions*

In Combination with the actual economic situation, it is assumed that the capital market parameters are $r_0 = 0.002, r_1 = 0.1, \sigma = 0.4$. At the same time we assume that an enterprise has a surplus asset of 1 million yuan at time t and has a consumption function $c(x) = 12000 + 0.01x$. Meanwhile the company can estimate the safety area $[L,H]$ is $[20,30]$ (the unit is ten thousand yuan). Based on the above hypothesis, it can be calculated that $m = 43.05$, $l_1 = 44.82$ and $l_2 = 54.63$, then we can get $m < l_1 < l_2$. According to Theorem 2, we can have the following conclusions.

If $\beta_t = 0.44$, then $C_t = 44 \in (m,l_1]$, $u_t^* = -0.41$, and it is not difficult to verify that the optimal cash holding is L, which is the lower limit of the safe area. If $\beta_t = 0.5$, then $C_t = 50 \in [l_1, l_2]$, $u_t^* = -0.55$, and the optimal cash holding is $21.15 \in [20, 30]$. If $\beta_t = 0.6$, then $C_t = 60 \in [l_2, +\infty)$, and the optimal cash holding is H, which is the upper limit of the safe area. In the above three cases, u_t^* are all negative indicates that some cash holding need to be converted into risky assets, which is consistent with $X_t > H$.

4.2. *The influence of capital market parameters on the optimal cash holding*

In this part, we assume that $X_t = 435000, c = 12000, \theta = 0.001, L = 200000, H = 300000, \beta_t = 0.8, r_1 = 0.1$ at time t.

First, we discuss the influence of r_0 on μ_t^* and the optimal cash holding. It can be

TABLE 1. μ_t^* and the optimal cash holding under different r_0 values

r_0	m	l_1	l_2	μ_t^*	the optimal cash holding
0.002	43.05	44.82	54.63	-1.56	20
0.003	42.73	44.17	53.78	-1.56	20
0.004	42.42	43.52	52.94	-1.56	20
0.005	42.10	42.89	52.13	-1.48	20.66
0.006	41.79	42.28	51.32	-1.40	21.35
0.007	41.47	41.67	50.54	-1.32	22.06
0.008	41.15	41.08	49.76	-1.24	22.79
0.009	40.84	40.49	49.01	-1.15	23.53
0.01	40.52	39.92	48.26	-1.07	24.29
0.02	37.36	34.74	41.52	-0.41	30
0.03	34.21	30.36	35.83	-0.41	30

seen from Table 1 that when $r_0 \in [0.002, 0.004]$, we can get $m < l_1 < l_2$ and $m < x < l_1$. According to Theorem 2, $\mu_t^* = -1.56$. At this time, r_0 has no effect on μ_t^* and the optimal cash holding but has affect on m, l_1, l_2. We can get $m < l_1 < l_2$ and $l_1 < x < l_2$ when $r_0 \in [0.005, 0.007]$. According to Theorem 2, $\mu_t^* = 1 - 2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x}$. When $r_0 \in [0.008, 0.01]$ then we can get $l_1 < m < l_2$ and $m < x < l_2$, μ_t^* can be obtained according to Theorem 3. In both cases, μ_t^* and the optimal cash holding are increasing functions of r_0. Finally, we can get $l_1 < m < l_2$ and $x > l_2$ when $r_0 \in [0.02, 0.03]$. According to Theorem 3, $\mu_t^* = -0.41$. r_0 has no effect on μ_t^* and the optimal cash holding, and the optimal cash holding is H. At the same time, it is assumed that the initial cash holding is $\beta_t x = 34.8 > H$, so we need to convert cash assets to risky assets. This is illustrated by the negative value of μ_t^* in Table 1. Such process can also be shown in Figure 1 and Figure 2. Figure 2 also verifies that under Theorem 2 and Theorem 3, μ_t^* obtained by the model can make the optimal cash holding in the safe area. The discussion of the impact of r_1 on μ_t^* and the optimal cash holding is similar to that of r_0, and will not be repeated here.

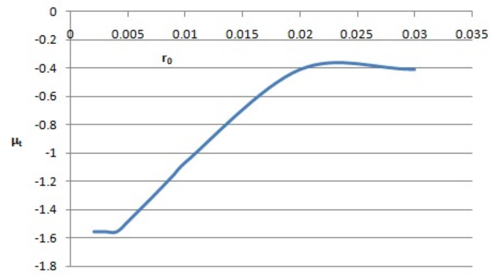

FIGURE 1. The influence of r_0 on μ_t^*

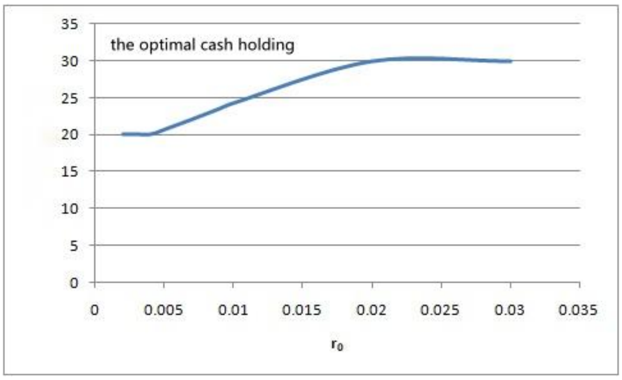

FIGURE 2. The influence of r_0 on the optimal cash holding

4.3. *The influence of c, θ on μ_t^* and the optimal cash holding*

For enterprises, c represents the minimum payment to meet the normal operation needs, and its different values affect m, l_1, l_2 and the size relationship among them. Supposing that $X_t = 50$, $\theta = 0.001$, $L = 200000$, $H = 300000$, $\beta_t = 0.3$, $r_0 = 0.002$, $r_1 = 0.1$. As can be seen from Table 2, when $c \in [0.1, 0.5]$, $l_1 < m < l_2$ and $x > l_2$. According to Theorem 3, $\mu_t^* = \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x}$. c affects μ_t^*, but has no effect on the optimal cash holding. When $c \in [1.1, 1.4]$, we can get $m < l_1 < l_2$ and $l_1 < x < l_2$. According to Theorem 2, $\mu_t^* = 1 - 2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x}$. In this case, μ_t^* and the optimal cash holding decrease with the increase of c. When $c \in [1.5, 1.8]$, $m < l_1 < l_2$ and $m < x < l_1$. According to Theorem 2, then $\mu_t^* = \frac{L+c+\theta x-\beta_t x}{(1-\beta_t)x}$. In this case, μ_t^* increases with the increase of c. However c does not affect the optimal cash holding at this time, and the optimal cash holding is L. When $c = 1.9$, $m < l_1 < l_2$ and $x < m$, bankruptcy has occurred.

TABLE 2. μ_t^* and the optimal cash holding under different c values

$c(\times 10^4)$	m	l_1	l_2	μ_t^*	the optimal cash holding
0.10	30.84	21.72	31.53	-0.99	30
0.50	35.28	30.12	39.93	-0.95	30
1.10	41.94	42.72	52.53	-1.14	27.42
1.20	43.05	44.82	54.63	-1.34	25.28
1.30	44.16	46.92	56.73	-1.55	23.14
1.40	45.27	49.02	58.83	-1.76	20.99
1.50	46.38	51.12	60.93	-1.85	20
1.60	47.49	53.22	63.03	-1.84	20
1.70	48.60	55.32	65.13	-1.83	20
1.80	49.71	57.42	67.23	-1.82	20
1.90	50.82	59.52	69.33	-	-

Next, we will discuss the influence of θ on μ_t^* and the optimal cash holding. At the beginning of this article, we assume that $0 \leq \theta < r_0$. The influence of θ on μ_t^* and optimal cash holding is shown in Table 3.

TABLE 3. μ_t^* and the optimal cash holding under different θ values

θ	m	l_1	l_2	μ_t^*	the optimal cash holding
0.001	30.84	21.72	31.53	-0.99	30
0.0011	35.32	30.18	40.01	-0.94	30
0.0012	42.03	42.90	52.75	-1.16	27.21
0.0013	43.09	45.10	54.98	-1.38	24.96
0.0014	44.36	47.32	57.21	-1.59	22.71
0.0015	45.52	49.54	59.45	-1.81	20.46
0.0016	46.69	51.77	61.71	-1.84	20
0.0017	47.86	54.01	63.97	-1.83	20
0.0018	49.04	56.27	66.24	-1.82	20
0.0019	50.21	58.53	69.53	-	-

When $\theta =\in [0.001, 0.0011]$, we can get $l_1 < m < l_2$ and $x > l_2$. According to Theorem 3, $\mu_t^* = \frac{H+c+\theta x-\beta_t x}{(1-\beta_t)x}$. In this case, μ_t^* is an increasing function of θ, but θ has no effect on the optimal cash holding. The optimal cash holding is H. When $\theta \in [0.0012, 0.0015]$, we can get $m < l_1 < l_2$ and $l_1 < x < l_2$. According to Theorem 2, $\mu_t^* = 1 - 2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x}$. At this time, μ_t^* and the optimal cash holding are decreasing functions of θ, and it is not difficult to verify that the optimal cash holding is in the safe area. When $\theta \in [0.0016, 0.0018]$, we can get $m < l_1 < l_2$ and $m < x < l_1$. According to Theorem 2, $\mu_t^* = \frac{L+c+\theta x-\beta_t x}{(1-\beta_t)x}$. In this case θ has no effect on the optimal cash holding. The optimal cash holding is L. When $\theta = 0.0019$, we can get $m < l_1 < l_2$ and $x < m$, bankruptcy has occurred.

As in Section 4.2, the effects of c and θ can also be shown graphically, but are omitted here due to space constraints.

4.4. *The influence of initial cash holding ratio β_t on the optimal cash holding*

β_t is the proportion of surplus wealth held in the form of cash assets at time t. It is not difficult to find that m, l_1, l_2 and the relationship among them have nothing to do with β_t. Supposing that $m < l_1 < l_2$ and $l_1 < x < l_2$, we can get $\mu_t^* = 1 - 2\frac{[c-(r_0-\theta)x)]}{(1-\beta_t)(r_1-r_0)x}$ according to Theorem 2. As can be seen from Table 4, μ_t^* is a positive value when

TABLE 4. μ_t^* and the optimal cash holding under different β_t values

β_t	m	l_1	l_2	μ_t^*	the optimal cash holding
0.1	43.05	44.82	54.63	0.48	25.28
0.2	43.05	44.82	54.63	0.41	25.28
0.3	43.05	44.82	54.63	0.33	25.28
0.4	43.05	44.82	54.63	0.22	25.28
0.5	43.05	44.82	54.63	0.06	25.28
0.6	43.05	44.82	54.63	-0.17	25.28
0.7	43.05	44.82	54.63	-0.56	25.28
0.8	43.05	44.82	54.63	-1.35	25.28
0.9	43.05	44.82	54.63	-3.69	25.28

$\beta_t \in [0.1, 0.5]$, which indicates that part of the risky assets have been converted into cash assets. When $\beta_t \in [0.6, 0.9]$, μ_t^* is negative, which means that part of the cash assets have been converted into risky assets. μ_t^* is a decreasing function of β_t, which can be seen from the calculation formula, Table 4 or Figure 4.4.

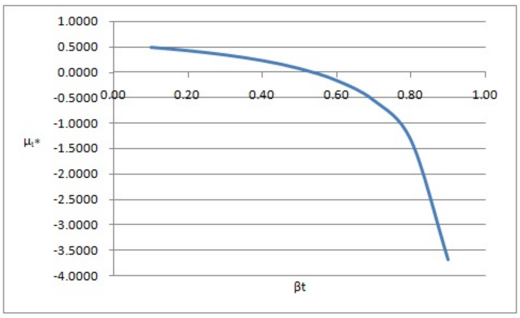

FIGURE 3. The influence of β_t on μ_t^*

5. CONCLUSION

In this paper, aiming at minimizing the ruin probability, we define the ”bankruptcy” time as the time when the residual wealth first reaches the bankruptcy threshold m, and we also study the optimal cash holding decision problem under the security area constraint. The results of this model show that the optimal conversion strategy and the optimal cash holding are related to the consumption function parameters, capital market parameters and the initial cash holding ratio. The model can be regarded as a standard management model. On one hand, it describes the relationship between bankruptcy probability, residual wealth, cash and venture assets of investment objects involved in enterprise management. On the other hand, it can be used to describe the relationship between people, money and things in enterprise management. Although the model is established based on certain assumptions, it does not affect the management significance of the model itself.

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