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BIHARMONIC CURVES ALONG RIEMANNIAN SUBMERSIONS

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Abstract. The purpose of this paper is to study biharmonic curves along Riemannian submersions. We first consider a Riemannian submersion from a Riemannian manifold onto Riemannian manifold and investigate under what conditions a biharmonic curve on the total manifold is transformed to a biharmonic curve on the base manifold. We obtain several results with certain restrictions on curvatures. We then consider a Riemannian submersion from a Kaehler manifold onto a Riemannian manifold. Necessary and sufficient conditions were obtained for a curve that is biharmonic in the total manifold of Riemannian submersion to be biharmonic on the base manifold along the Riemannian submersion. In addition, considering the special cases of the curvatures of the curve, the biharmonicity of the curve on the base manifold is discussed.

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1. INTRODUCTION

Riemannian submersions between Riemannian manifolds were studied by O'Neill [14] and Gray [9]. Riemannian submersions have been used as an effective tool to obtain new manifolds with certain curvatures and to compare their geometry when given two manifolds. In addition, Riemannian submersions are very useful tools for the applications of Kaluza-Klein theory [8, 10] and robotic theory [2].

Harmonic maps $F: (M,g) \to (N,g_N)$ between Riemannian manifolds are the critical points of the energy $E(F) = \frac{1}{2}\int_M |dF|^2 \upsilon_g$, and they are therefore the solutions of the corresponding Euler-Lagrange equation. This equation is giving by the vanishing of the tension field $\tau(F) = \text{trace}\nabla dF$. On the other hand, Jiang [11] studied first and second variation formulas of the bienergy functional $E_2(F)$ whose critical points are called as biharmonic maps.

Biharmonic maps have been a very active research area in recent years and many interesting results have been obtained [1, 3–7, 12, 14–16, 18–21], for update results about biharmonic theory, see:[17]. However, as far as we know, there are no studies on whether a given biharmonic curve is biharmonic in the base manifold along a Riemann submersion.

In this paper, we study curves along Riemannian submersions between Riemannian manifolds We first considered the curve as horizontal curve and check the biharmonicity of a curve on the base manifold along a Riemannian submersion. We also study curves along Riemannian submersions from complex space form onto Riemannian manifolds. The paper is organized as follows. In Section 2, we present the basic information needed for this paper. In Section 3, we investigate necessary and sufficient conditions for the curves along Riemannian submersions from Riemannian manifolds to be biharmonic. We show that, if horizontal vector field $\mathcal{A}$ is parallel, then this curve is biharmonic. Then, we investigate necessary and sufficient conditions for the Frenet curves along Riemannian submersions from Riemannian manifolds to be biharmonic. In Section 4, we investigate necessary and sufficient conditions for the curves along Riemannian submersions from complex space forms to be biharmonic. We also investigate necessary and sufficient conditions for the Frenet curves along Riemannian submersions from complex space forms to be biharmonic.

2. PRELIMINARIES

In this section, we recall some basic notions from [3, 8] and [22] which will be needed throughout the paper.

Let F be a Riemannian submersion between Riemannian manifolds (M, g_M) and (N, g_N). The geometry of Riemannian submersion is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields E, F on M by

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \ \mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{H}F \tag{2.1}$$

where ∇ is the Levi-Civita connection of g_M, $\mathcal{H}$ and $\mathcal{V}$ are projections to horizontal and vertical subbundles, respectively. It is easy to see that a Riemannian submersion $F\colon M \to N$ has totally geodesic fibres if and only if $\mathcal{T}$ vanishes identically. For any $E \in \Gamma(TM)$, $\mathcal{T}_E$ and $\mathcal{A}_E$ are skew -symmetric operators on $(\Gamma(TM), g)$ reversing the horizontal and the vertical distributions. $\mathcal{A}_E$ and $\mathcal{T}_E$ are anti-symmetric with respect to g. It is easy to see that $\mathcal{T}$ is vertical, $\mathcal{T}_E = \mathcal{T}_{\mathcal{V}E}$ and $\mathcal{A}$ is horizontal $\mathcal{A}_E = \mathcal{A}_{\mathcal{H}E}$. We note that the tensor field $\mathcal{T}$ is symmetric, $\mathcal{T}_V W = \mathcal{T}_W V$, $\forall V, W \in \Gamma(\ker F_*)$ and $\mathcal{A}$ is anti-symmetric, $A_X Y = -\mathcal{A}_Y X = \frac{1}{2}\mathcal{V}[X,Y]$, $\forall X, Y \in \Gamma((\ker F_*)^\perp)$. On the other hand, from (2.1), we have

$$\begin{aligned} \nabla_X V &= \mathcal{A}_X V + \mathcal{V}\nabla_X V \\ \nabla_X Y &= \mathcal{H}\nabla_X Y + \mathcal{A}_X Y \end{aligned} \tag{2.2}$$

for $X, Y \in \Gamma((\ker F_*)^\perp)$ and $V \in \Gamma(\ker F_*)$. If X is basic, then $\mathcal{H}\nabla_V X = \mathcal{A}_X V$.

Let $\alpha\colon I \to M$ be a curve parametrized by arc length in an n-dimensional Riemannian manifold (M, g). If there exists orthonormal vector fields $E_1 = \alpha' = T, E_2, \ldots, E_r$ along α such that

$$\nabla_T E_1 = \kappa_1 E_2,$$

$$\nabla_T E_2 = -\kappa_1 E_1 + \kappa_2 E_3,$$
$$\dots$$
$$\nabla_T E_r = -\kappa_{r-1} E_{r-1}. \tag{2.3}$$

then α is called a Frenet curve of osculating order r, where $\kappa_1, \dots, \kappa_{r-1}$ are positive functions on I and $1 \leq r \leq n$. A Frenet curve of osculating order 1 is a geodesic. A Frenet curve of osculating order 2 is called a circle if κ_1 is a nonzero positive constant. A Frenet curve of osculating order $r \geq 3$ is called a helix of order r if $\kappa_1, \dots, \kappa_{r-1}$ are nonzero positive constants. A helix of order 3 is shortly called a helix.

Let $F\colon (M,g) \to (N,h)$ be a map between two Riemannian manifolds of dimensions m and n respectively. The second fundamental form of a map is defined by

$$(\nabla F_*)(X,Y) = \overset{N}{\nabla}{}^F{}_X F_* Y - F_*(\overset{M}{\nabla}_X Y) \tag{2.4}$$

for any vector fields X, Y on M, where $\overset{M}{\nabla}$ is the Levi-Civita cennection of M and $\overset{N}{\nabla}{}^F$ is the pull-back of the connection $\overset{N}{\nabla}$ of N to the induced vector bundle $F^{-1}(TN)$. It is well known that ∇F_* is symmetric. It is known that, F is a harmonic map if and only if the tension field $\tau(F) = \text{trace}(\nabla F_*) = 0$, which is called the harmonic equation or the Euler-Lagrange equation.

A map $F\colon (M, g_M) \to (N, g_N)$ between Riemannian manifolds is a biharmonic map if the bitension field of F

$$\tau_2(F) = -\Delta_F \tau(F) + \text{trace} R^N(\tau(F), F_*)F_*$$

vanishes, where R^N denotes the curvature tensor field of N. The operator Δ_F is the rough Laplacian acting on $\Gamma(F^*TM)$ defined by $\Delta_F := -\sum_{i=1}^n (\overset{N}{\nabla}{}^F{}_{e_i} \overset{N}{\nabla}{}^F{}_{e_i} - \overset{N}{\nabla}{}^F{}_{\overset{N}{\nabla}_{e_i} e_i})$, where $\{e_i\}_{i=1}^n$ is a local orthonormal frame field of N.

3. BIHARMONIC CURVES ALONG RIEMANNIAN SUBMERSIONS FROM RIEMANNIAN MANIFOLDS

In this section, we study biharmonic curves along Riemannian submersions from Riemannian manifolds. Then, we will investigate necessary and sufficient conditions for the curves along Riemannian submersions from Riemannian manifolds to be biharmonic. We first recall the biharmonic equation for curves from [3]. Let $\alpha\colon I \to M$ be a curve defined on an open interval I and parametrized by arc-length. Then the bitension field is given by

$$\tau_2(\alpha) = \nabla_T^3 T - R(T, \nabla_T T)T \tag{3.1}$$

where $T = \alpha'$ and $\nabla_T^3 T = \nabla_T \nabla_T \nabla_T T$. Then, using Frenet equations, the bitension field of α becomes

$$\tau_2(\alpha) = -3\kappa_1 \kappa_1' E_1 + (\kappa_1'' - \kappa_1^3 - \kappa_1 \kappa_2^2 + c\kappa_1)E_2 + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2')E_3$$

$$+\kappa_1\kappa_2\kappa_3 E_4.$$

Theorem 1. *Let F be a Riemannian submersion from a space form $(M(c), g_M)$ with constant sectional curvature c to a Riemannian manifold (N, g_N). Let $\alpha\colon I \to (M(c), g_M)$ be a biharmonic horizontal curve. Then $F\circ\alpha\colon \gamma\colon I \to (N, g_N)$ is a biharmonic curve if and only if the following condition is satisfied*

$$-2\kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} = 0, \tag{3.2}$$

where E_{1h} and E_{1v} denote the horizontal part and the vertical part of E_1.

Proof. Let $F\colon (M(c), g_M) \to (N, g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha\colon I \to (M(c), g_M)$ be a biharmonic horizontal curve. Then, we have

$$\alpha' = T = E_{1h}, \qquad \gamma' = F_*T = \tilde{T}, \tag{3.3}$$

where E_{1h} is horizontal part of $T = E_1$. Note that $\gamma' = \tilde{T}$ is the unit tangent vector field along the curve. Since F is Riemannian submersion, then $(\nabla F_*)(X,Y) = 0$, where $X, Y \in \Gamma(\mathcal{H})$, using (2.3) and (3.3) we get

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T} = \kappa_1 F_* E_{2h}.$$

and

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = \kappa_1' F_*E_{2h} + \kappa_1((\nabla F_*)(E_{1h}, E_{2h}) + F_*\overset{M}{\nabla}_{E_{1h}}E_{2h}). \tag{3.4}$$

From (2.2), (2.3) and Frenet formulas, we have

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{2h} = -\kappa_1 E_{1h} + \kappa_2 E_{3h} - \mathcal{A}_{E_{1h}}E_{2v}. \tag{3.5}$$

Using (3.5) in (3.4), we derive

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = \kappa_1' F_*E_{2h} - \kappa_1^2 F_*E_{1h} + \kappa_1\kappa_2 F_*E_{3h} - \kappa_1 F_*\mathcal{A}_{E_{1h}}E_{2v}. \tag{3.6}$$

Taking the covariant dervivative of (3.6) and using the second fundamental form of the Riemannian submersion, we get

$$\begin{aligned}
\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} = {} & \kappa_1'' F_*E_{2h} + \kappa_1' F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{2h} - 2\kappa_1\kappa_1' F_*E_{1h} - \kappa_1^2 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{1h} \\
& + \kappa_1'\kappa_2 F_*E_{3h} + \kappa_1\kappa_2' F_*E_{3h} + \kappa_1\kappa_2 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{3h} \\
& - \kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.
\end{aligned} \tag{3.7}$$

Since

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{3h} = -\kappa_2 E_{2h} + \kappa_3 E_{4h} - \mathcal{A}_{E_{1h}}E_{3v} \tag{3.8}$$

due (3.5), (3.8) and Frenet formulas, using (3.7), we arrive at

$$\begin{aligned}\overset{N}{\nabla}{}^{3}_{\tilde{T}}\tilde{T} &= -3\kappa_1\kappa_1' F_*E_{1h} + (\tilde{\kappa}_1'' - \kappa_1^3 - \kappa_1\kappa_2^2)F_*E_{2h} + (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')F_*E_{3h}\\ &+ \kappa_1\kappa_2\kappa_3 F_*E_{4h} - 2\kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v}\\ &- \kappa_1 F_*\mathcal{H}\overset{1}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}. \end{aligned} \tag{3.9}$$

It is easy to see that

$$R^N(\tilde{T},\overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_*E_{1h},\kappa_1 F_*E_{2h})F_*E_{1h},$$

Taking the vertical and horizontal parts of E_2, we find

$$R^M(T,\overset{M}{\nabla}_T T)T = R^M(E_{1h},\kappa_1 E_{2v})E_{1h} + R^M(E_{1h},\kappa_1 E_{2h})E_{1h}.$$

Hence, we obtain

$$F_*(R^M(T,\overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h},\kappa_1 E_{2v})E_{1h}) + F_*(R^M(E_{1h},\kappa_1 E_{2h})E_{1h}).$$

Since F is Riemannian submersion, we have

$$\begin{aligned}&F_*(R^M(T,\overset{M}{\nabla}_T T)T)\\ &= F_*(R^M(E_{1h},\kappa_1 E_{2v})E_{1h}) + R^N(F_*E_{1h},\kappa_1 F_*E_{2h})F_*E_{1h}.\end{aligned}$$

On the other hand, since M is a space form, we obtain

$$R^N(F_*E_{1h},\kappa_1 F_*E_{2h})F_*E_{1h} = -c\kappa_1 F_*E_{2h}. \tag{3.10}$$

Putting (3.9) and (3.10) in (3.1), we have

$$\begin{aligned}\tau_2(\gamma) &= \overset{N}{\nabla}{}^{3}_{\tilde{T}}\tilde{T} - R^N(\tilde{T},\overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T}\\ &= -3\kappa_1\kappa_1' F_*E_{1h} + (\kappa_1'' - \kappa_1^3 - \kappa_1\kappa_2^2 + c\kappa_1)F_*E_{2h}\\ &+ (2\kappa_1'\kappa_2 + \kappa_1\kappa_2')F_*E_{3h} + \kappa_1\kappa_2\kappa_3 F_*E_{4h} - 2\kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v}\\ &- \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.\end{aligned}$$

Since $\tau_2(\alpha) = 0$, we can write $F_*\tau_2(\alpha) = 0$. Then, using this equation in $\tau_2(\gamma)$, we get

$$\tau_2(\gamma) = -2\kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.$$

Thus $F\circ\alpha$: γ: $I \to (N, g_N)$ is a biharmonic curve if and only if (3.2) is satisfied. □

In particular cases, we have the following results.

Theorem 2. *Let $F\colon (M(c),g_M) \to (N,g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I \to (M(c),g_M)$ be a biharmonic horizontal curve and $\kappa_1 = constant \neq 0$. Then $F \circ \alpha\colon \gamma\colon I \to (N,g_N)$ is a biharmonic curve if and only if*

$$\kappa_2 F_* \mathcal{A}_{E_{1h}} E_{3v} + F_* \mathcal{H} \overset{M}{\nabla}_{E_{1h}} \mathcal{A}_{E_{1h}} E_{2v} = 0.$$

Proof. The assertion follows from Theorem 1. □

Theorem 3. *Let $F\colon (M(c),g_M) \to (N,g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I \to (M(c),g_M)$ be a biharmonic horizontal curve and horizontal vector field $\mathcal{A}$ be a parallel. Then $F \circ \alpha\colon \gamma\colon I \to (N,g_N)$ is a biharmonic curve.*

Proof. Since, horizontal vector field $\mathcal{A}$ is a parallel, we have $\mathcal{A} = 0$. Then the assertion follows from Theorem 1 . □

If the curve is the Frenet curve, the following theorem regarding the biharmonicity of the curve on the base manifold is obtained.

Theorem 4. *Let $F\colon (M(c),g_M) \to (N,g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I \to (M(c),g_M)$ be a horizontal Frenet curve. Then Frenet curve $F \circ \alpha\colon \gamma\colon I \to (N,g_N)$ is a biharmonic curve if and only if*

$$\tilde{\kappa}_1 = \text{constant} \neq 0, \qquad \tilde{\kappa}_2 = \text{constant}, \qquad \tilde{\kappa}_1{}^2 + \tilde{\kappa}_2{}^2 = c, \qquad \tilde{\kappa}_2 \tilde{\kappa}_3 = 0.$$

where $\tilde{\kappa}_1, \ldots, \tilde{\kappa}_{r-1}$ are positive functions of γ on I.

Proof. Let $F\colon (M(c),g_M) \to (N,g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c),g_M)$ to a Riemannian manifold (N,g_N). Since $\alpha\colon I \to (M(c),g_M)$ is a horizontal Frenet curve, we have

$$\alpha' = T = E_{1h}, \qquad \gamma' = F_* T = \tilde{T},$$

where E_{1h} is horizontal part of $T = E_1$. Then for an orthonormal frame $E_1 = \alpha' = T, E_2, \ldots, E_r$ along α, we have Frenet formulas of γ as follows

$$\overset{N}{\nabla}_{\tilde{T}} \tilde{T} = \tilde{\kappa}_1 F_* E_{2h}$$

$$\overset{N}{\nabla}_{\tilde{T}} F_* E_{2h} = -\tilde{\kappa}_1 F_* E_{1h} + \tilde{\kappa}_2 F_* E_{3h}$$

$$\ldots$$

$$\overset{N}{\nabla}_{\tilde{T}} F_* E_{rh} = -\tilde{\kappa}_{r-1} F_* E_{(r-1)h}.$$

Thus, by direct computations, we have

$$\overset{N}{\nabla}_{\tilde{T}} \tilde{T} = \overset{N}{\nabla}_{F_* E_{1h}} F_* E_{1h} = \tilde{\kappa}_1 F_* E_{2h},$$

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = -\tilde{\kappa}_1{}^2 F_* E_{1h} + \tilde{\kappa}_1{}' F_* E_{2h} + \tilde{\kappa}_1\tilde{\kappa}_2 F_* E_{3h},$$

and

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} = & -3\tilde{\kappa}_1\tilde{\kappa}_1{}' F_* E_{1h} + (\tilde{\kappa}_1'' - \tilde{\kappa}_1{}^3 - \tilde{\kappa}_1\tilde{\kappa}_2{}^2) F_* E_{2h} \\ & + (2\tilde{\kappa}_1{}'\tilde{\kappa}_2 + \tilde{\kappa}_1\tilde{\kappa}_2{}') F_* E_{3h} + \tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3 F_* E_{4h}. \end{aligned} \tag{3.11}$$

Then, using the Frenet formulas, we obtain

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_*E_{1h}, \tilde{\kappa}_1 F_* E_{2h})F_*E_{1h} = \tilde{\kappa}_1 R^N(F_*E_{1h}, F_*E_{2h})F_*E_{1h}. \tag{3.12}$$

Now, taking the vertical and horizontal parts of E_2, we find

$$R^M(T, \overset{M}{\nabla}_T T)T = R^M(E_{1h}, \kappa_1 E_2)E_{1h} = R^M(E_{1h}, \kappa_1 E_{2v})E_{1h} + R^M(E_{1h}, \kappa_1 E_{2h})E_{1h}.$$

Thus, we obtain

$$F_*(R^M(T, \overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h}, \kappa_1 E_{2v})E_{1h}) + F_*(R^M(E_{1h}, \kappa_1 E_{2h})E_{1h}).$$

Since F is a Riemannian submersion, we have

$$F_*(R^M(T, \overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h}, \kappa_1 E_{2v})E_{1h}) + R^N(F_*E_{1h}, \kappa_1 F_* E_{2h})F_*E_{1h}.$$

Using, Riemannian curvature tensor of M, we get

$$\begin{aligned} R^N(F_*E_{1h}, F_*E_{2h})F_*E_{1h} &= \frac{1}{\kappa_1} F_*(R^M(T, \overset{M}{\nabla}_T T)T) - \frac{1}{\kappa_1} F_*(R^M(E_{1h}, \kappa_1 E_{2v})E_{1h}). \\ &= -cF_*E_{2h}. \end{aligned} \tag{3.13}$$

Then, using (3.13) into (3.12), we have

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = -c\tilde{\kappa}_1 F_* E_{2h}. \tag{3.14}$$

Thus putting (3.11) and (3.14) in (3.1), we have

$$\begin{aligned}\tau_2(\gamma) = & -3\tilde{\kappa}_1\tilde{\kappa}_1{}' F_*E_{1h} + (\tilde{\kappa}_1'' - \tilde{\kappa}_1{}^3 - \tilde{\kappa}_1\tilde{\kappa}_2{}^2 + c\tilde{\kappa}_1)F_*E_{2h} + (2\tilde{\kappa}_1{}'\tilde{\kappa}_2 + \tilde{\kappa}_1\tilde{\kappa}_2{}')F_*E_{3h} \\ & + \tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3 F_* E_{4h}.\end{aligned}$$

Thus, Frenet curve $F \circ \alpha \colon \gamma \colon I \to (N, g_N)$ is a biharmonic curve if and only if

$$\begin{aligned} & -3\tilde{\kappa}_1\tilde{\kappa}_1{}' F_*E_{1h} + (\tilde{\kappa}_1'' - \tilde{\kappa}_1{}^3 - \tilde{\kappa}_1\tilde{\kappa}_2{}^2 + c\tilde{\kappa}_1)F_*E_{2h} + (2\tilde{\kappa}_1{}'\tilde{\kappa}_2 + \tilde{\kappa}_1\tilde{\kappa}_2{}')F_*E_{3h} \\ & + \tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3 F_* E_{4h} = 0 \end{aligned}$$

which gives the assertion. □

From Theorem 4, we have the following result.

Corollary 1. *Let $F\colon (M(c),g_M)\to(N,g_N)$ be a Riemannian submersion from a Riemannian manifold $(M(c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I\to(M(c),g_M)$ be a horizontal Frenet curve. Then Frenet curve $F\circ\alpha\colon\gamma\colon I\to(N,g_N)$ is a biharmonic curve such that $\tilde{\kappa}_2=0$ if and only if*

- *γ is a circle with $\tilde{\kappa}_1=\sqrt{c}$ such that $n\geq 2$,*
  or
- *γ is a helix with $\tilde{\kappa}_1{}^2+\tilde{\kappa}_2{}^2=c$ such that $n\geq 3$.*

4. BIHARMONIC CURVES ALONG RIEMANNIAN SUBMERSIONS FROM COMPLEX SPACE FORMS

A $2n-$dimensional Riemannian manifold (M,g,J) is called an almost Hermitian manifold if there exist a tensor field J of type $(1,1)$ on M such that $J^2=-I$ and $g(JX,JY)=g(X,Y)$, $\forall X,Y\in\chi(M)$ where I denotes the identity transformation of T_pM. Consider an almost Hermitian manifold (M,g,J) and denote by ∇ the Levi-Civita connection on M with respect to g. Then M is called a Kaehler manifold if J is parallel with respect to ∇, i.e., $(\nabla_X J)Y=0$, for $X,Y\in\chi(M)$. Let $M^m(4c)$ be a complex space form of holomorphic sectional curvature $4c$. Let us denote by J the complex structure and by g the Riemannian metric on $M^m(4c)$. Then its curvature operator is given by

$$\begin{aligned}R^{M^m(4c)}(X,Y)Z=&c\{g(Y,Z)X-g(X,Z)Y+g(JY,Z)JX-g(JX,Z)JY\\&+2g(X,JY)JZ\}\end{aligned}$$

for $X,Y,Z\in\chi(M)$.

In this section, we study biharmonic curves along Riemannian submersions from complex space forms. More precisely, we will investigate necessary and sufficient conditions for the curves along Riemannian submersions from complex space forms to be biharmonic. We first recall the criteria for a curve on a complex manifold to be biharmonic from [3]. Let (M,g) be a complex space form and $\alpha\colon I\to M$ be a curve defined on an open interval I and parametrized by arc-length. Then, using Frenet equations for an orthonormal frame $E_1=\alpha^{'}=T,E_2,\ldots,E_r$ along α, the bitension field of α becomes

$$\begin{aligned}\tau_2(\alpha)=&-3\kappa_1\kappa_1^{'}E_1+(\kappa_1^{''}-\kappa_1^3-\kappa_1\kappa_2^2+c\kappa_1)E_2+(2\kappa_1^{'}\kappa_2+\kappa_1\kappa_2^{'})E_3+\kappa_1\kappa_2\kappa_3E_4\\&-3c\kappa_1\tau_{12}JE_1.\end{aligned}$$

Following S. Maeda and Y. Ohnita [13], we define the complex torsions of the curve α by $\tau_{ij}=g(E_i,JE_j)$, $1\leq i<j\leq r$.

The following theorem gives the appropriate condition for a biharmonic curve given on the total manifold to be biharmonic on the base manifold along a Riemannian submersion from a complex space form to a Riemannian manifold.

Theorem 5. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) and $\alpha\colon I\to(M(4c),g_M)$ a biharmonic horizontal curve. Then $F\circ\alpha\colon \gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$-2\kappa_1' F_*\mathcal{A}_{E_{1h}}E_{2v}-\kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v}-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}-3c\kappa_1\tau_{12mix}F_*JE_{1h}=0,$$

where $\tau_{12mix}=g_M(E_{1h},JE_{2v})$.

Proof. Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I\to(M(4c),g_M)$ be a biharmonic horizontal curve. Then, we have the following equation.

$$\begin{aligned}\tau_2(\alpha)=&-3\kappa_1\kappa_1'E_1+(\kappa_1''-\kappa_1^3-\kappa_1\kappa_2^2+c\kappa_1)E_2+(2\kappa_1'\kappa_2+\kappa_1\kappa_2')E_3+\kappa_1\kappa_2\kappa_3E_4\\&-3c\kappa_1\tau_{12}JE_1.\end{aligned}$$

Since α is horizontal curve, we have

$$\alpha'=T=E_{1h},\qquad \gamma'=F_*T=\tilde{T},$$

where E_{1h} is horizontal part of $T=E_1$. Note that $\gamma'=\tilde{T}$ is the unit tangent vector field along the curve. Using (2.4) we obtain

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T}=\overset{N}{\nabla}_{F_*E_{1h}}F_*E_{1h}=(\nabla F_*)(E_{1h},E_{1h})+F_*\overset{M}{\nabla}_{E_{1h}}E_{1h}.$$

Since F is Riemannian submersion, we get

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T}=\kappa_1F_*E_{2h}. \tag{4.1}$$

From this we derive

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T}=\kappa_1'F_*E_{2h}+\kappa_1((\nabla F_*)(E_{1h},E_{2h})+F_*\overset{M}{\nabla}_{E_{1h}}E_{2h}). \tag{4.2}$$

By using (2.2), (2.3) and Frenet formulas, we have

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{2h}=-\kappa_1E_{1h}+\kappa_2E_{3h}-\mathcal{A}_{E_{1h}}E_{2v}. \tag{4.3}$$

Inserting (4.3) in (4.2), we have

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T}=\kappa_1'F_*E_{2h}-\kappa_1^2F_*E_{1h}+\kappa_1\kappa_2F_*E_{3h}-\kappa_1F_*\mathcal{A}_{E_{1h}}E_{2v}.$$

From (2.4) we obtain,

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T}=&\kappa_1''F_*E_{2h}+\kappa_1'F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{2h}-2\kappa_1\kappa_1'F_*E_{1h}-\kappa_1^2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{1h}\\&+\kappa_1'\kappa_2F_*E_{3h}+\kappa_1\kappa_2'F_*E_{3h}+\kappa_1\kappa_2F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{3h}-\kappa_1'F_*\mathcal{A}_{E_{1h}}E_{2v}\\&-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.\end{aligned} \tag{4.4}$$

Similar to expression (4.3), we have

$$\mathcal{H}\overset{M}{\nabla}_{E_{1h}}E_{3h} = -\kappa_2 E_{2h} + \kappa_3 E_{4h} - \mathcal{A}_{E_{1h}}E_{3v} \tag{4.5}$$

Putting (4.1), (4.3) and (4.5) in (4.4), we have

$$\begin{aligned}\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} = {} & -3\kappa_1\kappa_1^{'}F_*E_{1h} + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2)F_*E_{2h} + (2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})F_*E_{3h} \\ & + \kappa_1\kappa_2\kappa_3 F_*E_{4h} - 2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} \\ & - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}.\end{aligned} \tag{4.6}$$

On the other hand, using (4.1) and (2.4), we obtain

$$R^N(\tilde{T}, \overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_*E_{1h}, \kappa_1 F_*E_{2h})F_*E_{1h}.$$

and

$$R^M(T, \overset{M}{\nabla}_T T)T = R^M(T, \kappa_1 E_2)T, \tag{4.7}$$

respectively. Now, taking the vertical and horizontal parts of E_2 in (4.7), we find

$$R^M(T, \overset{M}{\nabla}_T T)T = R^M(E_{1h}, \kappa_1 E_{2v})E_{1h} + R^M(E_{1h}, \kappa_1 E_{2h})E_{1h}.$$

Since F is a Riemannian submersion, we derive

$$F_*(R^M(T, \overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h}, \kappa_1 E_{2v})E_{1h}) + R^N(F_*E_{1h}, \kappa_1 F_*E_{2h})F_*E_{1h}.$$

Using curvature operator, we get

$$\begin{aligned}R^N(F_*E_{1h}, \kappa_1 F_*E_{2h})F_*E_{1h} = {} & -c\kappa_1 F_*E_{2h} - 3c\kappa_1\tau_{12}F_*JE_{1h} \\ & - 3c\kappa_1 g_M(E_{2v}, JE_{1h})F_*JE_{1h}.\end{aligned} \tag{4.8}$$

Thus putting (4.6) and (4.8) in (2.4), we have

$$\begin{aligned}\tau_2(\gamma) = {} & -3\kappa_1\kappa_1^{'}F_*E_{1h} + (\kappa_1^{''} - \kappa_1^3 - \kappa_1\kappa_2^2 + c\kappa_1)F_*E_{2h} + +(2\kappa_1^{'}\kappa_2 + \kappa_1\kappa_2^{'})F_*E_{3h} \\ & + \kappa_1\kappa_2\kappa_3 F_*E_{4h} - 2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} \\ & + 3c\kappa_1\tau_{12}F_*JE_{1h} + 3c\kappa_1 g_M(E_{2v}, JE_{1h})F_*JE_{1h}.\end{aligned}$$

Since $\tau_2(\alpha) = 0$, we can write $F_*\tau_2(\alpha) = 0$. Then, using this equation in $\tau_2(\gamma)$, we have

$$\tau_2(\gamma) = -2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v} - \kappa_1\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v} - \kappa_1 F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v} - 3c\kappa_1\tau_{12mix}F_*JE_{1h}.$$

Thus, proof is complete. □

In particular, if κ_1 =constant$\neq 0$, then we have the following result.

Theorem 6. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I\to(M(4c),g_M)$ be a biharmonic horizontal curve and κ_1 =constant$\neq 0$. Then $F\circ\alpha\colon \gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$\kappa_2 F_*\mathcal{A}_{E_{1h}}E_{3v}+F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}+3c\tau_{12mix}F_*JE_{1h}=0$$

Proof. Since κ_1 =constant$\neq 0$, we have $\kappa_1^{'}=0$. Then the assertion follows from Theorem 5. □

Theorem 7. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) . Let $\alpha\colon I\to(M(4c),g_M)$ be a biharmonic horizontal curve such that κ_1 =constant$\neq 0$ and horizontal tensor field $\mathcal{A}$ is parallel. Then $F\circ\alpha\colon \gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if either $c=0$ or $\tau_{12mix}=0$, here $\tau_{12mix}=g_M(E_{1h},JE_{2v})$.*

Proof. Since κ_1 =constant$\neq 0$, we have $\kappa_1^{'}=0$. The parallelity of $\mathcal{A}$ implies that $\mathcal{A}=0$. The assertion follows from Theorem 5. □

Theorem 8. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be an invariant Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) . Let $\alpha\colon I\to(M(4c),g_M)$ be a biharmonic horizontal curve. Then $F\circ\alpha\colon \gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$-2\kappa_1^{'}F_*\mathcal{A}_{E_{1h}}E_{2v}-\kappa_1\kappa_2F_*\mathcal{A}_{E_{1h}}E_{3v}-\kappa_1F_*\mathcal{H}\overset{M}{\nabla}_{E_{1h}}\mathcal{A}_{E_{1h}}E_{2v}=0.$$

Proof. Since F is an invariant Riemannian submersion, we have $J(\ker F_*)\subseteq\ker F_*$. The asserion follows from Theorem 5. □

Theorem 9. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be an invariant Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) . Let $\alpha\colon I\to(M(4c),g_M)$ be a biharmonic horizontal curve such that κ_1 =constant$\neq 0$ and horizontal tensor field $\mathcal{A}$ is parallel. Then $F\circ\alpha\colon \gamma\colon I\to(N,g_N)$ is a biharmonic curve.*

Proof. The parallelity of $\mathcal{A}$ implies that $\mathcal{A}=0$. The assertion follows from Theorem 8. □

The following theorem gives the appropriate condition for a horizontal Frenet curve given on the total manifold to be biharmonic on the base manifold along a Riemannian submersion from a complex space form to a Riemannian manifold.

Theorem 10. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N) and $\alpha\colon I\to(M(4c),g_M)$ a horizontal Frenet curve. Then Frenet curve $F\circ\alpha\colon \gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$\tilde{\kappa_1}=\text{constant}\neq 0,$$

$$\tilde{\kappa_1}^2+\tilde{\kappa_2}^2 = c\{1+3(\tau_{12}-\tau_{12mix})g_N(F_*JE_{1h},F_*E_{2h})\},$$
$$\tilde{\kappa_2}' = -c\{3(\tau_{12}-\tau_{12mix})g_N(F_*JE_{1h},F_*E_{3h})\},$$
$$\tilde{\kappa_2}\tilde{\kappa_3} = -c\{3(\tau_{12}-\tau_{12mix})g_N(F_*JE_{1h},F_*E_{4h})\}.$$

where $\tilde{\kappa_1},\ldots,\tilde{\kappa_{r-1}}$ *are positive functions of* γ *on I.*

Proof. Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N). Since $\alpha\colon I\to(M(4c),g_M)$ is a horizontal Frenet curve, we have

$$\alpha' = T = E_{1h}, \qquad \gamma' = F_*T = \tilde{T},$$

where E_{1h} is horizontal part of $T=E_1$. Note that $\gamma'=\tilde{T}$ is the unit tangent vector field along the curve. Then we have Frenet formulas of γ as follows

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T} = \tilde{\kappa_1}F_*E_{2h}$$
$$\overset{N}{\nabla}_{\tilde{T}}F_*E_{2h} = -\tilde{\kappa_1}F_*E_{1h}+\tilde{\kappa_2}F_*E_{3h}$$
$$\cdots$$
$$\overset{N}{\nabla}_{\tilde{T}}F_*E_{rh} = -\tilde{\kappa}_{r-1}F_*E_{(r-1)h}.$$

Thus we have

$$\overset{N}{\nabla}_{\tilde{T}}\tilde{T} = \overset{N}{\nabla}_{F_*E_{1h}}F_*E_{1h} = \tilde{\kappa_1}F_*E_{2h}.$$

Then, using Frenet formulas of γ, we get

$$\overset{N}{\nabla}{}^2_{\tilde{T}}\tilde{T} = -\tilde{\kappa_1}^2F_*E_{1h}+\tilde{\kappa_1}'F_*E_{2h}+\tilde{\kappa_1}\tilde{\kappa_2}F_*E_{3h}.$$

Thus $\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T}$ becomes

$$\overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} = -3\tilde{\kappa_1}\tilde{\kappa_1}'F_*E_{1h}+(\tilde{\kappa_1}''-\tilde{\kappa_1}^3-\tilde{\kappa_1}\tilde{\kappa_2}^2)F_*E_{2h}$$
$$+(2\tilde{\kappa_1}'\tilde{\kappa_2}+\tilde{\kappa_1}\tilde{\kappa_2}')F_*E_{3h}+\tilde{\kappa_1}\tilde{\kappa_2}\tilde{\kappa_3}F_*E_{4h}. \tag{4.9}$$

Then, using the Frenet formulas, we obtain

$$R^N(\tilde{T},\overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} = R^N(F_*E_{1h},\tilde{\kappa_1}F_*E_{2h})F_*E_{1h} = \tilde{\kappa_1}R^N(F_*E_{1h},F_*E_{2h})F_*E_{1h}. \tag{4.10}$$

Now, taking the vertical and horizontal parts of E_2, we find

$$R^M(T,\overset{M}{\nabla}_TT)T = R^M(E_{1h},\kappa_1E_2)E_{1h}$$
$$= R^M(E_{1h},\kappa_1E_{2v})E_{1h}+R^M(E_{1h},\kappa_1E_{2h})E_{1h}. \tag{4.11}$$

From (4.11), we obtain

$$F_*(R^M(T,\overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h},\kappa_1 E_{2v})E_{1h}) + F_*(R^M(E_{1h},\kappa_1 E_{2h})E_{1h}).$$

Since F is a Riemannian submersion, we have

$$F_*(R^M(T,\overset{M}{\nabla}_T T)T) = F_*(R^M(E_{1h},\kappa_1 E_{2v})E_{1h}) + R^N(F_*E_{1h},\kappa_1 F_*E_{2h})F_*E_{1h}.$$

Using curvature operator, we get

$$\begin{aligned} R^N(F_*E_{1h},F_*E_{2h})F_*E_{1h} &= \frac{1}{\kappa_1}F_*(R^M(T,\overset{M}{\nabla}_T T)T) - \frac{1}{\kappa_1}F_*(R^M(E_{1h},\kappa_1 E_{2v})E_{1h}). \\ &= \frac{1}{\kappa_1}F_*(-c\kappa_1 E_2 - 3c\kappa_1\tau_{12}JE_{1h}) \\ &\quad - \frac{1}{\kappa_1}F_*c\{g_M(\kappa_1 E_{2v},E_{1h})E_{1h} - g_M(E_{1h},E_{1h})\kappa_1 E_{2v} \\ &\quad + g_M(JE_{1h},E_{1h})J\kappa_1 E_{2v} - g_M(J\kappa_1 E_{2v},E_{1h})JE_{1h} \\ &\quad + 2g_M(JE_{1h},\kappa_1 E_{2v})JE_{1h}\} \\ &= -cF_*E_{2h} - 3c\tau_{12}F_*JE_{1h} + 3cg_M(JE_{2v},E_{1h})F_*JE_{1h}. \end{aligned} \tag{4.12}$$

Then, using (4.12) into (4.10), we have

$$\begin{aligned} R^N(\tilde{T},\overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} &= \tilde{\kappa}_1(-cF_*E_{2h} - 3c\tau_{12}F_*JE_{1h} + 3cg_M(JE_{2v},E_{1h})F_*JE_{1h}) \\ &= -c\tilde{\kappa}_1 F_*E_{2h} - 3c\tilde{\kappa}_1\tau_{12}F_*JE_{1h} + 3c\tilde{\kappa}_1\tau_{12mix}F_*JE_{1h}. \end{aligned} \tag{4.13}$$

Thus putting (4.9) and (4.13) in (3.1), we have

$$\begin{aligned} \tau_2(\gamma) &= \overset{N}{\nabla}{}^3_{\tilde{T}}\tilde{T} - R^N(\tilde{T},\overset{N}{\nabla}_{\tilde{T}}\tilde{T})\tilde{T} \\ &= -3\tilde{\kappa}_1\tilde{\kappa}_1{}' F_*E_{1h} + (\tilde{\kappa}_1'' - \tilde{\kappa}_1{}^3 - \tilde{\kappa}_1\tilde{\kappa}_2{}^2 + c\tilde{\kappa}_1)F_*E_{2h} + (2\tilde{\kappa}_1{}'\tilde{\kappa}_2 + \tilde{\kappa}_1\tilde{\kappa}_2{}')F_*E_{3h} \\ &\quad + \tilde{\kappa}_1\tilde{\kappa}_2\tilde{\kappa}_3 F_*E_{4h} + 3c\tilde{\kappa}_1\tau_{12}F_*JE_{1h} - 3c\tilde{\kappa}_1\tau_{12mix}F_*JE_{1h}. \end{aligned}$$

Thus proof is complete. □

In particular cases, we have the following results.

Theorem 11. *Let $F\colon (M(4c),g_M)\to(N,g_N)$ be a Riemannian submersion from a complex space form $(M(4c),g_M)$ to a Riemannian manifold (N,g_N). Let $\alpha\colon I\to(M(4c),g_M)$ be a horizontal Frenet curve such that $\tau_{12}=\tau_{12mix}$. Then Frenet curve $F\circ\alpha\colon \gamma\colon I\to(N,g_N)$ is a biharmonic curve if and only if*

$$\tilde{\kappa}_1 = \text{constant} \neq 0, \quad \tilde{\kappa}_2 = \text{constant}, \quad \tilde{\kappa}_1{}^2 + \tilde{\kappa}_2{}^2 = 0, \quad \tilde{\kappa}_2\tilde{\kappa}_3 = 0.$$

Proof. Since $\tau_{12}=\tau_{12mix}$, we have $E_{1h}\perp JE_{2h}$. The assertion follows from Theorem 10. □

Corollary 2. *Let $F\colon (M^m(4c), g_M) \to (N^n, g_N)$ be a Riemannian submersion from a complex space form $(M(4c), g_M)$ to a Riemannian manifold (N, g_N). Let $\alpha\colon I \to (M(4c), g_M)$ be a horizontal Frenet curve such that $\tau_{12} = \tau_{12mix}$. Then Frenet curve $F \circ \alpha\colon \gamma\colon I \to (N, g_N)$ is a biharmonic curve if and only if*

- *γ is a circle with $\tilde{\kappa}_1 = \sqrt{c}$ such that $n \geq 2$ or*
- *γ is a helix with $\tilde{\kappa}_1{}^2 + \tilde{\kappa}_2{}^2 = c$ such that $n \geq 3$.*

REFERENCES

[1] M. A. Akyol and Y. L. Ou, "Biharmonic Riemannian submersions," *Annali di Matematica Pura ed Applicata*, vol. 198, no. 2, pp. 559–570, 2019, doi: 10.1007/s10231-018-0789-x.

[2] C. Altafini, "Redundant robotic chains on Riemannian submersions," *IEEE Transactions on Robotics and Automation*, vol. 20, pp. 335–340, 2004.

[3] A. Balmus, "Biharmonic maps and submanifolds," *University of Cagliari, Geometry Balkan Press.*, 2007.

[4] A. Balmus, S. Montaldo, and C. Oniciuc, "Biharmonic hypersurfaces in 4-dimensional space forms," *Mathematische Nachrichten*, vol. 283, no. 12, pp. 1696–1705, 2010.

[5] A. Bennouar and S. Ouakkas, "Some constructions of biharmonic maps on the warped product manifolds," *Comment. Math. Univ. Carolin.*, vol. 58, no. 4, pp. 481–500, 2017.

[6] N. E. H. Djaa, A. Boulal, and A. Zagane, "Generalized warped product manifolds and biharmonic maps," *Acta Math. Univ. Comenian. (N.S.)*, vol. 81, no. 2, pp. 283–289, 2012, doi: 10.1307/mmj/1028999604.

[7] I. K. Erken and C. Murathan, "Biharmonic pseudo-Riemannian submersions from 3-manifolds," *Filomat*, vol. 32, no. 2, pp. 543–552, 2018, doi: 10.2298/FIL1802543E.

[8] M. Falcitelli, S. Ianus, and A. M. Pastore, *Riemannian submersions and related topics*. World Scientific, 2004. doi: 10.1142/5568.

[9] A. Gray, "Pseudo-Riemannian almost product manifolds and submersions," *Journal of Mathematics and Mechanics*, vol. 16, no. 7, pp. 715–737, 1967.

[10] P. A. Hogan, "Kaluza–Klein theory derived from a Riemannian submersion," *J. Math. Phys.*, vol. 25, pp. 2301–2305, 1984, doi: 10.1063/1.526407.

[11] G. Y. Jiang, "2-harmonic maps and their first and second variational formulas," *Chinese Ann. Math. Ser. A.*, vol. 7, pp. 389–402, 1986, doi: 10.1285/i15900932v28n1supplp209.

[12] K. Madani and S. Ouakkas, "Biharmonic maps on doubly warped product manifolds," *Kyungpook Math. J.*, vol. 60, no. 3, pp. 599–627, 2020, doi: 10.5666/KMJ.2020.60.3.599.

[13] S. Maeda and Y. Ohnita, "Helical geodesic immersions into complex space forms," *Geom. Dedicata*, vol. 30, pp. 93–114, 1983, doi: 10.1007/BF02424315.

[14] B. O'Neill, "The fundamental equations of a submersion," *Michigan Math. J.*, vol. 13, pp. 459–469, 1966, doi: 10.1307/mmj/1028999604.

[15] C. Oniciuc, "Biharmonic maps between Riemannian manifolds," *A1. I. Cuza Iasi. Mat.(N. S.*, vol. 48, no. 2, pp. 237–248, 2002.

[16] C. Oniciuc, "Biharmonic submanifolds in space forms," *Habilitation Thesis, Universitatea Alexandru Ioan Cuza*, p. 149, 2012, doi: 10.13140/2.1.4980.5605.

[17] Y.-L. Ou and B.-Y. Chen, *Biharmonic submanifolds and biharmonic maps in Riemannian geometry*. World Scientific, 2020. doi: 10.1142/11610.

[18] B. Sahin, "Biharmonic Riemannian maps," *Ann. Polon. Math.*, vol. 102, pp. 39–49, 2011, doi: 10.4064/ap102-1-4.

[19] B. Sahin, *Riemannian submersions, Riemannian maps in Hermitian geometry, and their applications*. Cambridge: Elsevier, Academic Press, 2017.

[20] S. M. K. Torbaghan and M. M. Rezaii, “Warped products, biharmonic and semi-conformal maps,” *Math. Rep. (Bucur.)*, vol. 21, no. 4, pp. 441–459, 2019.
[21] H. Urakawa, *Calculus of variation and harmonic maps*. American Mathematical Society, 2013.
[22] K. Yano and M. Kon, *Structures on manifolds*. World Scientific, 1984. doi: 10.1142/0067.

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