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ON ALMOST η-RICCI-BOURGUIGNON SOLITONS

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Abstract. We investigate a Riemannian manifold with almost η-Ricci-Bourguignon soliton structure. We use the Hodge-de Rham decomposition theorem to make a link with the associated vector field of an almost η-Ricci-Bourguignon soliton. Moreover, we show that a nontrivial, compact almost η-Ricci-Bourguignon soliton of constant scalar curvature is isometric to the Euclidean sphere. Using some results obtaining from almost η-Ricci Bourguignon soliton, we give the integral formulas for compact orientable almost η-Ricci-Bourguignon soliton.

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1. Introduction

The notion of Ricci solitons correspond to self-similar of Ricci flow introduced in [15] by R.S. Hamilton. Perelman [19] proved that any compact Ricci solitons is gradient. In the compact case there are nontrivial Ricci solitons [10]. Most of the proofs for compact case are found in [12] or [13]. Moreover, there do not exist gradient Ricci solitons in the noncompact case [3] and [17]. On the other hand, Naber [18] showed that noncompact shrinking solitons are gradient in some special cases.

Recently, Pigola et al. [21] introduced the notion of almost Ricci soliton. By adding the condition on the parameter λ to be a variable function, they modified the definition of Ricci soliton. Likewise, many authors studied the almost η-Ricci solitons, for example, Blaga [1] investigated almost η-Ricci solitons in $(LCS)_n$ manifolds, Siddiqi [22] studied η-Ricci Yamabe solitons on Riemannian submersions from Riemannian manifold. Generalizing for this notion Dwivedi studied in [14] the almost Ricci-Bourguignon solution and Soylu [23] examined Ricci-Bourguignon soliton and almost soliton with concurrent vector field. Blaga and Taştan [7] also studied almost Ricci-Bourguignon solitons with some special potential vector fields and almost η-Ricci-Bourguignon solitons on a doubly warped product. Dwivedi [14] derived integral formulas for compact Ricci-Bourguignon solitons and Ricci-Bourguignon almost solitons. In addition, Aquino et al. [2] and Barros and Riberio

[5, 6] presented integral formula for the compact almost Ricci solitons and generalized m-quasi Einstein metrics.

In the present paper, we give basic background of an almost η-Ricci-Bourguignon solitons and definitions of gradient solitons in section 2. In section 3, we investigated compact almost η-Ricci-Bourguignon solitons using Hodge-de Rham potential decomposition. Moreover, we study gradient η-Ricci-Bourguignon soliton and compact almost η-Ricci-Bourguignon soliton when the potential vector field is conformal. We proved that the potential vector field of a compact almost η-Ricci-Bourguignon soliton is a Killing vector field under some conditions. In section 4, we derived the integral formulas for gradient compact almost η-Ricci-Bourguignon soliton.

2. Preliminaries

In this section, we recall the fundamental definitions and notions for the further study.

On an n-dimensional Riemannian manifold (M^n, g) *Ricci-Bourguignon solitons* are self-similar solutions to *Ricci-Bourguignon flow* [8]

$$\frac{\partial}{\partial t} g(t) = -2(\mathrm{Ric} - \rho R g), \tag{2.1}$$

where Ric is the Ricci tensor of the metric, R is the scalar curvature of the Riemannian metric g and $\rho \in \mathbb{R}$ is a real constant.

A Riemannian manifold (M^n, g) is called *Ricci-Bourguignon soliton* if the metric g satisfies the following equation

$$\mathrm{Ric} + \frac{1}{2}\pounds_{\xi} g = (\lambda + \rho R) g. \tag{2.2}$$

where $\pounds_{\xi} g$ denotes the Lie derivative of the metric g along a vector field ξ, Ric is a Ricci tensor, R is a curvature tensor, ρ and λ are constant. Considering $\eta = df(X)$ is a 1-form, the Riemannian manifold (M^n, g) is called η*-Ricci-Bourguignon soliton* if there exist a vector field ξ, a smooth function f and $\lambda \in \mathbb{R}$ a constant such that

$$\mathrm{Ric} + \frac{1}{2}\pounds_{\xi} g = (\lambda + \rho R) g + \mu df \otimes df, \tag{2.3}$$

It is called expanding, steady or shrinking, respectively, if $\lambda < 0$, $\lambda = 0$, $\lambda > 0$. The manifold is called a gradient η*-Ricci-Bourguignon soliton* when the vector field $\xi = \nabla f$ is a gradient of a differentiable function $f : M^n \to \mathbb{R}$ such that

$$\mathrm{Ric} + \nabla^2 f = (\lambda + \rho R) g + \mu df \otimes df, \tag{2.4}$$

where $\nabla^2 f$ stands for the Hessian of f. The η-Ricci-Bourguignon soliton is called trivial when either the vector field ξ is trivial or the potential f is constant. Hence the vector field is called *Killing vector field*, i.e $\pounds_{\xi} g = 0$. If $n \geq 3$ and ξ is a Killing vector field, η-Ricci-Bourguignon soliton becomes trivial soliton. Then we get an *Einstein manifold* in that case, since λ is constant. If λ is a smooth function in

(2.3), then (M^n, g) is called *almost η-Ricci-Bourguignon soliton* and is denoted by $(M^n, g, \xi, \lambda, \mu)$.

Using the Hodge-de Rham decomposition theorem (see [2]), we shall decompose the vector field ξ over a compact oriented Riemannian manifold as a sum of the gradient of a function h and a free divergence vector field Y, i.e.

$$\xi = \nabla h + Y,$$

where $\operatorname{div} Y = 0$. We may indicate a proof of this decomposition for the understanding of his completeness. In fact, we consider the 1-form $\xi^\flat$. We decompose $\xi^\flat$ with the help of the Hodge-de Rham decomposition theorem as follows

$$\xi^\flat = d\alpha + \delta\beta + \gamma. \tag{2.5}$$

Considering $Y = (\delta\beta + \gamma)^\sharp$ and $(d\alpha)^\sharp = \nabla h$ to arrive at the desired result. For more simplicity let us call h the Hodge-de Rham potential.

3. Main Results

In this section we investigate a compact almost η-Ricci-Bourguignon soliton and we give a characterization for a gradient η-Ricci-Bourguignon soliton.

We remark that the same result obtained in [2] for compact Ricci solitons also works for compact almost η-Ricci-Bourguignon solitons. We give the next theorem for more explicitly.

Theorem 1. *Let $(M^n, g, \xi, \lambda, \mu)$ be a compact almost η-Ricci-Bourguignon soliton. If M^n is also a gradient almost η-Ricci-Bourguignon soliton with potential f, then, up to a constant, it agrees with the Hodge-de Rham potential h.*

Proof. For an almost η-Ricci-Bourguignon soliton $(M^n, g, \xi, \lambda, \mu)$, we have

$$(1 - n\rho)R + \operatorname{div}\xi = n\lambda + \mu|\nabla f|^2. \tag{3.1}$$

The Hodge-de Rham decomposition allows us to write $\operatorname{div}\xi = \Delta h$. Hence we get

$$(1 - n\rho)R + \Delta h = n\lambda + \mu|\nabla f|^2. \tag{3.2}$$

If $(M^n, g, \xi, \lambda, \mu)$ is also a compact gradient almost η-Ricci-Bourguignon soliton, then from equation (2.4) we have

$$(1 - n\rho)R + \Delta f = n\lambda + \mu|\nabla f|^2. \tag{3.3}$$

Substracting equations (3.2) and (3.3), we deduce $\Delta(f - h) = 0$. Using Hopf's theorem we conclude that $f = h + c$, hence the proof is completed. □

On a Riemannian manifold (M, g), consider the function $u = e^{-\mu f}$, then we have $\nabla u = -\mu e^{-\mu f}\nabla f$, which can be found in [11]. Hence we get

$$\nabla^2 f - \mu df \otimes df = -\frac{\nabla^2 u}{\mu u} \tag{3.4}$$

and

$$\frac{\Delta u}{u\mu} = ((1-n\rho)R - n\lambda). \tag{3.5}$$

Then using (2.4), we get

$$\text{Ric} - \frac{\nabla^2 u}{\mu u} = \lambda g + \rho R g. \tag{3.6}$$

Recall that a vector field ∇u on a Riemannian manifold (M,g) is called a *conformal vector field* if there exists a smooth function $\psi : M \to \mathbb{R}$ such that $\frac{1}{2}\pounds_{\nabla u} g = \psi g$. The conformal vector field is nontrivial if $\psi \neq 0$. Suppose ∇u is nontrivial conformal vector field, then we can write $\frac{1}{2}\pounds_{\nabla u} g = \nabla^2 u = \frac{\Delta u}{n} g$. Putting (3.5) in (3.6), we get

$$\text{Ric} = \frac{R}{n} g, \tag{3.7}$$

where R is constant scalar curvature. Therefore, we deduce that M^n is an Einstein manifold if and only if ∇u is a conformal vector field. For more detail, (see [6]).

Suppose ∇u is nontrivial conformal vector field, i.e. $\pounds_{\nabla u} g = 2\psi g$, then (3.6) becomes

$$\text{Ric} = (\lambda + \rho R + \frac{\psi}{\mu u}) g. \tag{3.8}$$

Taking the trace of (3.8) and covariant derivative we get

$$(1-n\rho)\nabla R = n\nabla\left(\lambda + \frac{\psi}{\mu u}\right). \tag{3.9}$$

Now taking the divergence of (3.8) and using $\nabla R = 2\,\text{div}\,\text{Ric}$, we have

$$\left(\frac{1}{2} - \rho\right)\nabla R = \nabla\left(\lambda + \frac{\psi}{\mu u}\right). \tag{3.10}$$

Thus (3.9) and (3.10) imply

$$(1-n\rho)\nabla R = n\left(\frac{1}{2} - \rho\right)\nabla R, \tag{3.11}$$

which implies that R and $\lambda + \frac{\psi}{\mu u}$ are constant.

The next theorem is a characterization for gradient almost η-Ricci-Bourguignon soliton when ξ is a conformal vector field and generalizes Theorem 3 of [2].

Theorem 2. *Let $(M^n, g, \xi, \lambda, \mu)$, $n \geq 3$, be a gradient η-Ricci-Bourguignon soliton and $\xi = \nabla u$ is a conformal vector field. Then the following conditions holds:*

(1) *If M is compact then ∇u is a Killing vector field, so that $(M^n, g, \xi, \lambda, \mu)$ is trivial soliton,*
(2) *If M is noncompact gradient η-Ricci-Bourguignon soliton then $(M^n, g, \xi, \lambda, \mu)$ is isometric to a Euclidean space or ∇u is Killing vector field.*

Proof. If ∇u is a conformal vector field, then there exist a smooth function ψ on M such that

$$\pounds_{\nabla u} g = 2\psi g. \tag{3.12}$$

Therefore, taking the trace of (3.12), we get

$$\operatorname{div} \nabla u = n\psi. \tag{3.13}$$

Since M is compact, integrating (3.13), we obtain

$$0 = \int_M 2 \operatorname{div} \nabla u \, dM = nVol(M)\psi, \tag{3.14}$$

which implies that $\psi = 0$. Then ∇u is Killing vector field. Hence, the first assertion is proved.

For the second, we have $\frac{1}{2}\pounds_{\nabla u} g = \nabla^2 u = \psi g$, since ψ constant from (3.11). Then if $\psi \neq 0$, we may use a result of Tashiro ([24], Theorem 2) to conclude that M^n is isometric to the Euclidean space. If $\psi = 0$, thus ∇u is a Killing vector field and the proof is completed. □

With the help of Theorem 4.2 of [25], we obtain the following theorem for compact almost η-Ricci-Bourguignon soliton.

Theorem 3. *Let $(M^n, g, \xi, \lambda, \mu)$, $n \geq 3$, be a compact almost η-Ricci-Bourguignon soliton with $n \geq 3$. If $\xi = \nabla u$ is a nontrivial conformal vector field, then M^n is isometric to a Euclidean sphere.*

Proof. If $\xi = \nabla u$ is nontrivial conformal vector field, then $\pounds_{\nabla u} g = 2\psi g$, where $\psi \neq 0$. Then from (3.11), R and $\lambda + \frac{\psi}{\mu u}$ are constant. We may use [22, Lemma 2.3, pp. 52] to conclude that $R \neq 0$, otherwise $\psi = 0$. Hence from (3.6), we get

$$\pounds_{\nabla u} \mathrm{Ric} = 2\left(\lambda + \rho R + \frac{\psi}{\mu u}\right)\psi g.$$

Since $\lambda + \rho R + \frac{\psi}{\mu u}$ is constant, we have

$$\pounds_{\nabla u} \mathrm{Ric} = \left(\lambda + \rho R + \frac{\psi}{\mu u}\right)\pounds_{\nabla u} g = 2\left(\lambda + \rho R + \frac{\psi}{\mu u}\right)\psi g.$$

Now we may apply Theorem 4.2 (pp. 54 of [25]) to conclude that M^n is isometric to a Euclidean sphere. Hence the proof is completed. □

Remark 1. M^n is isometric to a Euclidean sphere if it is trivial or ∇u is conformal vector field.

On a Riemannian manifold (M^n, g) the following formulas hold [20]:

$$\operatorname{div}(\pounds_X g)(X) = \frac{1}{2}\Delta|X|^2 - |\nabla X|^2 + \mathrm{Ric}(X,X) + \nabla_X \operatorname{div} X, \tag{3.15}$$

or in $(1,1)$-tensor notation

$$\operatorname{div}\nabla^2 f = \operatorname{Ric}(\nabla f) + \nabla\Delta f \tag{3.16}$$

and

$$\frac{1}{2}\Delta|\nabla f|^2 = |\nabla^2 f|^2 + \operatorname{Ric}\left(\nabla f, \nabla f + g(\nabla\Delta f, \nabla f)\right). \tag{3.17}$$

These previous formulas allows us to obtain the following lemma which is a generalization of an almost η-Ricci-Bourguignon soliton.

Lemma 1. *Let $(M^n, g, \xi, \lambda, \mu)$ be an almost η-Ricci-Bourguignon soliton. Then the following equations hold:*

$$\begin{aligned}\frac{(1-n\rho)}{2}\Delta|\xi|^2 = {} & (1-n\rho)|\nabla\xi|^2 + (n\rho-1)\operatorname{Ric}(\xi,\xi) + n\rho\nabla_\xi \operatorname{div}\xi \\ & + 2\rho(1-n\rho)g(\nabla R,\xi) - (n(2\rho+1)-2)g(\nabla\lambda,\xi) \\ & + 2\mu(1-n\rho)|\xi|^2 \operatorname{div}\xi - \mu n\rho\xi(|\xi|^2)\end{aligned} \tag{3.18}$$

and

$$\begin{aligned}\frac{(1-n\rho)}{2}(\Delta-\nabla_\xi)|\xi|^2 = {} & (1-n\rho)|\nabla\xi|^2 + \lambda(n\rho-1)|\xi|^2 \\ & + \rho(n\rho-1)R|\xi|^2 + \mu(n\rho-1)|\xi|^4 + n\rho\nabla_\xi \operatorname{div}\xi \\ & + 2\rho(1-n\rho)g(\nabla R,\xi) - (n(2\rho+1)-2)g(\nabla\lambda,\xi) \\ & + 2\mu(1-n\rho)|\xi|^2 \operatorname{div}\xi - \mu n\rho\xi(|\xi|^2).\end{aligned} \tag{3.19}$$

Proof. From (2.3), we have

$$2\operatorname{div}\operatorname{Ric} + \operatorname{div}(\pounds_\xi g) = 2\nabla\lambda + 2\rho\nabla R + 2\mu \operatorname{div}(df\otimes df). \tag{3.20}$$

Using $\operatorname{div}(df\otimes df) = \xi\operatorname{div}\xi + \nabla_\xi\xi$ and taking the trace of (2.3), we get $(1-n\rho)R + \operatorname{div}\xi = n\lambda + \mu|\xi|^2$. With the help of covariant derivative operator, we have

$$(1-n\rho)\nabla_\xi R + \nabla_\xi(\operatorname{div}\xi) = n\nabla_\xi\lambda + \mu\nabla_\xi|\xi|^2. \tag{3.21}$$

Using the contracted second Bianchi identity $\nabla R = 2\operatorname{div}\operatorname{Ric}$ and (3.15), (3.20), (3.21), we get

$$\begin{aligned}\nabla_\xi(\operatorname{div}\xi) &= ng(\nabla\lambda,\xi) + (n\rho-1)\nabla_\xi R + \mu\xi(|\xi|^2) \\ &= ng(\nabla\lambda,\xi) + 2(n\rho-1)\operatorname{div}\operatorname{Ric}(\xi) + \mu\xi(|\xi|^2) \\ &= ng(\nabla\lambda,\xi) + \mu\xi(|\xi|^2) - (n\rho-1)\operatorname{div}(\pounds_\xi g)(\xi) + 2(n\rho-1)g(\nabla\lambda,\xi) \\ &\quad + 2\rho(n\rho-1)g(\nabla R,\xi) + 2\mu(n\rho-1)|\xi|^2\operatorname{div}\xi + \mu(n\rho-1)\xi(|\xi|^2) \\ &= (1-n\rho)\left[\frac{1}{2}\Delta|\xi|^2 - |\nabla\xi|^2 + \operatorname{Ric}(\xi,\xi) + \nabla_\xi \operatorname{div}\xi\right] \\ &\quad + 2\rho(n\rho-1)g(\nabla R,\xi) + (n(2\rho+1)-2)g(\nabla\lambda,\xi) + \mu\xi(|\xi|^2)\end{aligned}$$

$$
\begin{aligned}
&+2\mu(n\rho-1)|\xi|^2\operatorname{div}\xi+\mu(n\rho-1)\xi(|\xi|^2)\\
=\ &\frac{(1-n\rho)}{2}\Delta|\xi|^2-(1-n\rho)|\nabla\xi|^2+(1-n\rho)\operatorname{Ric}(\xi,\xi)\\
&+(1-n\rho)\nabla_\xi\operatorname{div}\xi+2\rho(n\rho-1)g(\nabla R,\xi)+(n(2\rho+1)-2)g(\nabla\lambda,\xi)\\
&+2\mu(n\rho-1)|\xi|^2\operatorname{div}\xi+\mu n\rho\xi(|\xi|^2).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
\frac{(1-n\rho)}{2}\Delta|\xi|^2 =\ &(1-n\rho)|\nabla\xi|^2+(n\rho-1)\operatorname{Ric}(\xi,\xi)+n\rho\nabla_\xi\operatorname{div}\xi\\
&+2\rho(1-n\rho)g(\nabla R,\xi)-(n(2\rho+1)-2)g(\nabla\lambda,\xi)\\
&+2\mu(1-n\rho)|\xi|^2\operatorname{div}\xi-\mu n\rho\xi(|\xi|^2).
\end{aligned}\tag{3.22}
$$

Thus (3.18) is proved.

Next using the fundamental equation to write

$$
\operatorname{Ric}(\xi,\xi)=\lambda|\xi|^2+\rho R|\xi|^2+\mu|\xi|^4-\frac{1}{2}\pounds_\xi g(\xi,\xi),
$$

then we get

$$
\begin{aligned}
\frac{(1-n\rho)}{2}\Delta|\xi|^2 =\ &(1-n\rho)|\nabla\xi|^2+(n\rho-1)\left[\lambda|\xi|^2+\rho R|\xi|^2+\mu|\xi|^4\right.\\
&\left.-\frac{1}{2}\pounds_\xi g(\xi,\xi)\right]+n\rho\nabla_\xi\operatorname{div}\xi+2\rho(1-n\rho)g(\nabla R,\xi)\\
&-(n(2\rho+1)-2)g(\nabla\lambda,\xi)+2\mu(1-n\rho)|\xi|^2\operatorname{div}\xi-\mu n\rho\xi(|\xi|^2)\\
=\ &(1-n\rho)|\nabla\xi|^2+\frac{(1-n\rho)}{2}\nabla_\xi|\xi|^2+\lambda(n\rho-1)|\xi|^2+\rho R(n\rho-1)|\xi|^2\\
&+\mu(n\rho-1)|\xi|^4+n\rho\nabla_\xi\operatorname{div}\xi+2\rho(1-n\rho)g(\nabla R,\xi)\\
&-(n(2\rho+1)-2)g(\nabla\lambda,\xi)+2\mu(1-n\rho)|\xi|^2\operatorname{div}\xi-\mu n\rho\xi(|\xi|^2).
\end{aligned}
$$

Hence we obtain

$$
\begin{aligned}
\frac{(1-n\rho)}{2}(\Delta-\nabla_\xi)|\xi|^2 =\ &(1-n\rho)|\nabla\xi|^2+\lambda(n\rho-1)|\xi|^2\\
&+\rho R(n\rho-1)|\xi|^2+\mu(n\rho-1)|\xi|^4+n\rho\nabla_\xi\operatorname{div}\xi\\
&+2\rho(1-n\rho)g(\nabla R,\xi)-(n(2\rho+1)-2)g(\nabla\lambda,\xi)\\
&+2\mu(1-n\rho)|\xi|^2\operatorname{div}\xi-\mu n\rho\xi(|\xi|^2),
\end{aligned}\tag{3.23}
$$

which completes the proof of the lemma. □

Using the diffusion operator $\Delta_\xi=\Delta-\nabla_\xi$ (see [[4], pp. 143]) and taking $\xi=\nabla f$ in the previous lemma, namely, $\Delta f=\Delta-\nabla_{\nabla f}$, we get the next corollary with the help of (3.19).

Corollary 1. *Let $(M^n, g, \nabla f, \lambda, \mu)$ be a gradient almost η-Ricci-Bourguignon soliton. Then we have*

$$\begin{aligned}\frac{(1-n\rho)}{2}\Delta_f|\nabla f|^2 = (1-n\rho)|\nabla^2 f|^2 + \lambda(n\rho-1)|\nabla f|^2 \\ + \rho(n\rho-1)R|\nabla f|^2 + \mu(n\rho-1)|\nabla f|^4 + n\rho\nabla_{\nabla f}(\Delta f) \\ + 2\rho(1-n\rho)g(\nabla R, \nabla f) - (n(2\rho+1)-2)g(\nabla\lambda, \nabla f) \quad (3.24) \\ + 2\mu(1-n\rho)\Delta f|\nabla f|^2 - \mu n\rho\nabla f(|\nabla f|^2).\end{aligned}$$

Remark 2. The similar results of (3.18) and (3.19) for η-Ricci-Bourguignon soliton $(M^n, g, \xi, \lambda, \mu)$ are

$$\begin{aligned}\frac{(1-n\rho)}{2}\Delta|\xi|^2 = (1-n\rho)|\nabla\xi|^2 + (n\rho-1)\,\mathrm{Ric}(\xi,\xi) + n\rho\nabla_\xi \operatorname{div}\xi \quad (3.25) \\ + 2\rho(1-n\rho)g(\nabla R, \xi) + 2\mu(1-n\rho)|\xi|^2 \operatorname{div}\xi - \mu n\rho\xi(|\xi|^2)\end{aligned}$$

and

$$\begin{aligned}\frac{(1-n\rho)}{2}(\Delta-\nabla_\xi)|\xi|^2 = (1-n\rho)|\nabla\xi|^2 + \lambda(n\rho-1)|\xi|^2 + \rho R(n\rho-1)|\xi|^2 \\ + \mu(n\rho-1)|\xi|^4 + n\rho\nabla_\xi \operatorname{div}\xi + 2\rho(1-n\rho)g(\nabla R, \xi) \\ + 2\mu(1-n\rho)|\xi|^2 \operatorname{div}\xi - \mu n\rho\xi(|\xi|^2). \quad (3.26)\end{aligned}$$

Proof. The proofs are the same as Lemma 1 with $\nabla\lambda = 0$. □

In Theorem 3 of [5], the authors proved that for a compact almost Ricci soliton (M^n, g, ξ, λ), $n \geq 3$ satisfying that $\int_M (\mathrm{Ric}(\xi,\xi) + (n-2)g(\nabla\lambda,\xi))\,dM \leq 0$, the potential vector field ξ is Killing and the soliton is trivial. Now, we give a similar result for compact almost η-Ricci-Bourguignon soliton as follows:

Theorem 4. *Let $(M^n, g, \xi, \lambda, \mu)$, $n \geq 3$, be a compact almost η-Ricci-Bourguignon soliton. If $\rho \neq \frac{1}{n}$ and*

$$\begin{aligned}\int_M \Bigg(\mathrm{Ric}(\xi,\xi) + \frac{n\rho}{n\rho-1}\nabla_\xi \operatorname{div}\xi - 2\rho g(\nabla R, \xi) - \frac{(n(2\rho+1)-2)}{n\rho-1}g(\nabla\lambda, \xi) \\ - 2\mu|\xi|^2 \operatorname{div}\xi - \frac{\mu n\rho}{n\rho-1}\xi(|\xi|^2)\Bigg)\,dM \leq 0, \quad (3.27)\end{aligned}$$

then ξ is a Killing vector field and M^n is a trivial soliton.

Proof. It sufficient to integrate (3.18) of Lemma 1 with $\rho \neq \frac{1}{n}$. So we get

$$\begin{aligned}\int_M |\nabla\xi|^2 = \int_M \Bigg(\mathrm{Ric}(\xi,\xi) + \frac{n\rho}{n\rho-1}\nabla_\xi \operatorname{div}\xi - 2\rho g(\nabla R, \xi) \quad (3.28) \\ - \frac{(n(2\rho+1)-2)}{n\rho-1}g(\nabla\lambda, \xi) - 2\mu|\xi|^2 \operatorname{div}\xi - \frac{\mu n\rho}{n\rho-1}\xi(|\xi|^2)\Bigg)\,dM \leq 0.\end{aligned}$$

As we are assuming that the right-hand side of (3.18) is less than or equal to zero, we get $\nabla\xi = 0$, therefore we have $\pounds_\xi g = 0$, which yields that ξ is a Killing vector field. Thus, M^n is trivial, which completes the proof of the theorem. □

As a consequence of this theorem, we give the following corollary when $\nabla\lambda = 0$.

Corollary 2. *Let $(M^n, g, \xi, \lambda, \mu)$, $n \geq 3$, be a compact η-Ricci-Bourguignon soliton. If $\rho \neq \frac{1}{n}$ and*

$$\int_M \left(\mathrm{Ric}(\xi,\xi) + \frac{n\rho}{n\rho - 1}\nabla_\xi \operatorname{div}\xi - 2\rho g(\nabla R, \xi) - 2\mu|\xi|^2 \operatorname{div}\xi - \frac{\mu n\rho}{n\rho - 1}\xi(|\xi|^2) \right) dM \leq 0. \tag{3.29}$$

Then ξ is a Killing vector field and M^n is a trivial.

Proof. It is sufficient to take $\nabla\lambda = 0$. □

Remark 3. Corollary 2 is an analog of Theorem 1.1 in [20] which was for the case of compact Ricci solitons. We obtain Petersen-Wylie's result from our result by taking $\rho = 0$ and ξ is a conformal vector field.

4. Integral formulas for gradient almost η-Ricci-Bourguignon soliton

In this section, we derive some integral formulas for a compact almost η-Ricci-Bourguignon soliton $(M^n, g, \xi, \lambda, \mu)$ which are the generalization formula of a natural extension obtained for an almost Ricci-Bourguignon soliton in [14], as well as a similar one in [21].

Proposition 1. *Let $(M^n, g, \nabla f, \lambda, \mu)$ be a gradient almost η-Ricci-Bourguignon soliton, then the following equations hold:*

$$(1 - n\rho)R + \Delta f = n\lambda + \mu|\nabla f|^2, \tag{4.1}$$

$$\begin{aligned}(1 - 2\rho(n-1))\nabla R = 2(1-\mu)\,\mathrm{Ric}(\nabla f) + 2(n-1)\nabla\lambda \\ + 2\mu(R - (n-1)(\lambda + \rho R))\nabla f,\end{aligned} \tag{4.2}$$

$$\begin{aligned}&(\nabla_Y \mathrm{Ric})(X,Z) - (\nabla_X \mathrm{Ric})(Y,Z) - g(\mathcal{R}(X,Y)Z, \nabla_f) \\ &\quad = Y[\lambda]g(X,Z) - X[\lambda]g(Y,Z) + \rho\left((\nabla_Y R)g(X,Z) - (\nabla_X R)g(Y,Z)\right) \\ &\qquad + \mu\big(\nabla_Y(df \otimes df)(X,Y) - \nabla_X(df \otimes df)(X,Y)\big),\end{aligned} \tag{4.3}$$

for any vector fields X, Y, Z on M and $\mathcal{R}$ is the Riemannian curvature tensor of M.

$$\begin{aligned}&\nabla\big((1 - 2\rho(n-1))R + |\nabla f|^2 - 2(n-1)\lambda\big) \\ &\quad = 2(\lambda + \rho R)\nabla f + 2\mu\big(\nabla_{\nabla f}\nabla f + (|\nabla f|^2 - \Delta f)\nabla f\big).\end{aligned} \tag{4.4}$$

Proof. Taking the trace of equation (2.3), we obtain (4.1). For proving (4.2), taking the divergence of (2.4), we get

$$\operatorname{div}\operatorname{Ric}+\operatorname{div}(\nabla^2 f)=\mu\Delta f\nabla f+\mu\nabla_{\nabla f}\nabla f+\nabla(\lambda+\rho R). \tag{4.5}$$

From equation (4.1), we have $\Delta f = -R+n\lambda+n\rho R+\mu|\nabla f|^2$, remembering that $\nabla|\nabla f|^2=2\nabla_{\nabla f}\nabla f$ and $\nabla_{\nabla f}\nabla f=\lambda\nabla f+\rho R\nabla f+\mu|\nabla f|^2\nabla f-\operatorname{Ric}(\nabla f)$ with the help of equation (3.16), we get

$$\begin{aligned}\frac{1}{2}\nabla R &= -\operatorname{Ric}(\nabla f)-\nabla(-R+n(\lambda+\rho R)+\mu|\nabla f|^2)\\ &\quad +\mu\Delta f\nabla f+\mu\nabla_{\nabla f}\nabla f+\nabla(\lambda+\rho R)\\ &= -\operatorname{Ric}(\nabla f)+\nabla R-\mu\nabla_{\nabla f}\nabla f+\mu\Delta f\nabla f-(n-1)\nabla(\lambda+\rho R).\end{aligned}$$

Hence

$$\frac{1}{2}\nabla R=\operatorname{Ric}(\nabla f)-\mu\Delta f\nabla f+\mu\nabla_{\nabla f}\nabla f+(n-1)\nabla(\lambda+\rho R). \tag{4.6}$$

Using (2.3),

$$\nabla_{\nabla f}\nabla f=(\lambda+\rho R)\nabla f+\mu|\nabla f|^2\nabla f-\operatorname{Ric}(\nabla f). \tag{4.7}$$

Combining (4.6) and (4.7), we obtain

$$\begin{aligned}(1-2\rho(n-1))\nabla R &= (1-\mu)\operatorname{Ric}(\nabla f)+\mu\big((\lambda+\rho R)+\mu|\nabla f|^2)-\Delta f\big)\nabla f+(n-1)\nabla\lambda\\ &= (1-\mu)\operatorname{Ric}(\nabla f)+\mu\big(R-(n-1)(\lambda+\rho R)\nabla f\big)+(n-1)\nabla\lambda.\end{aligned}$$

which gives the second assertion. To get the equation (4.3), using equation (2.4) and covariant derivatives of $\operatorname{Ric}(X,Z)$ and $\operatorname{Ric}(Y,Z)$ where X,Y,Z are any vector fields on M, we get

$$\begin{aligned}(\nabla_Y\operatorname{Ric})(X,Z)-(\nabla_X\operatorname{Ric})(Y,Z) &= \big(\nabla_Y\nabla_X\nabla_Z f-\nabla_X\nabla_Y\nabla_Z f\big)\\ &\quad +(\nabla_Y\lambda)g(X,Z)-(\nabla_X\lambda)g(Y,Z)\\ &\quad +\rho\big((\nabla_Y R)g(X,Z)-(\nabla_X R)g(Y,Z)\big)\\ &\quad +\mu\big(\nabla_Y(df\otimes df)(X,Y)-\nabla_X(df\otimes df)(X,Y)\big)\\ &= g(\mathcal{R}(X,Y)Z,\nabla_f)+Y[\lambda]g(X,Z)-X[\lambda]g(Y,Z)\\ &\quad +\rho\big(Y[R]g(X,Z)-X[R]g(Y,Z)\big)\\ &\quad +\mu\big(\nabla_Y(df\otimes df)(X,Y)-\nabla_X(df\otimes df)(X,Y)\big).\end{aligned}$$

Hence (4.3) is proved. For the last equation, using (4.2), we have

$$\begin{aligned}&\frac{1}{2}(1-2\rho(n-1))\nabla R+\frac{1}{2}\nabla|\nabla f|^2-(n-1)\nabla\lambda\\ &\quad =(1-\mu)\operatorname{Ric}(\nabla f)-\operatorname{Ric}(\nabla f)+\mu(R-(n-1)(\lambda+\rho R))\nabla f\\ &\qquad +\mu|\nabla f|^2\nabla f+(\lambda+\rho R)\nabla f.\end{aligned}$$

Then

$$\begin{aligned}
&\nabla\big(1-2\rho(n-1)R+|\nabla f|^2-2(n-1)\lambda\big)-2(\lambda+\rho R)\nabla f\\
&\quad=2\mu\big((|\nabla f|^2+R-(n-1)(\lambda+\rho R)\nabla f)-\mathrm{Ric}(\nabla f)\big)\\
&\quad=2\mu\big((|\nabla f|^2+R-n(\lambda+\rho R)\nabla f)+(\lambda+\rho R))\nabla f-\mathrm{Ric}(\nabla f)\big)\\
&\quad=2\mu\big(|\nabla f|^2+\mu|\nabla f|^2-\Delta f+(\lambda+\rho R)\nabla f)-\mathrm{Ric}(\nabla f)\big)\\
&\quad=2\mu\big(\nabla_{\nabla f}\nabla f+(|\nabla f|^2-\Delta f)\nabla f\big).
\end{aligned}$$

Which completes the proof of the proposition. □

Corollary 3. *We have the following equations for the gradient* η*-Ricci-Bourguignon solitons* $(M^n,g,\nabla f,\lambda,\mu)$.

$$(1-n\rho)R+\Delta f=n\lambda+\mu|\nabla f|^2, \tag{4.8}$$

$$(1-2\rho(n-1))\,\nabla R=2(1-\mu)\,\mathrm{Ric}(\nabla f)+2\mu(R-(n-1)(\lambda+\rho R))\nabla f, \tag{4.9}$$

$$\begin{aligned}
(\nabla_Y\,\mathrm{Ric})(X,Z)-(\nabla_X\,\mathrm{Ric})(Y,Z)-g(\mathcal{R}(X,Y)Z,\nabla_f)&\\
=\rho\big((\nabla_Y R)g(X,Z)-(\nabla_X R)g(Y,Z)\big)& \qquad (4.10)\\
+\mu\big(\nabla_Y(df\otimes df)(X,Y)-\nabla_X(df\otimes df)(X,Y)\big).& \qquad (4.11)
\end{aligned}$$

$$\nabla\left(1-2\rho(n-1)\,R+|\nabla f|^2-2\lambda f\right)=2\rho R\nabla f+2\mu\left(\nabla_{\nabla f}\nabla f+(|\nabla f|^2-\Delta f)\nabla f\right). \tag{4.12}$$

Proof. The proof is the same as Proposition 1 taking $\nabla\lambda=0$. □

Lemma 2. *Let* $(M^n,g,\nabla f,\lambda,\mu)$ *be a gradient almost* η*-Ricci-Bourguignon soliton. Then we have*

$$\begin{aligned}
\left(\frac{1-2\rho(n-1)}{2}\right)\Delta R=&-\left|\nabla^2 f-\frac{\Delta f}{n}g\right|^2-\left\{\frac{1+n\mu}{n}\right\}(\Delta f)^2-\frac{n}{2}g(\nabla f,\nabla\lambda)\\
&-\frac{n}{2}\rho g(\nabla f,\nabla R)+\left\{\frac{1-2\mu}{2}\right\}g(\nabla f,\nabla\Delta f) \qquad (4.13)\\
&+\mu\,\mathrm{div}(\nabla_{\nabla f}\nabla f)+(n-1)\Delta\lambda+\lambda\Delta f+\rho R\Delta f.
\end{aligned}$$

Proof. First we take the divergence of (4.4) in Proposition 1 to get

$$\begin{aligned}
&\left(\frac{1-2\rho(n-1)}{2}\right)\Delta R+\Delta|\nabla f|^2-(n-1)\Delta\lambda\\
&\qquad=\lambda\Delta f+\rho R\Delta f+\mu\left(g(\nabla(|\nabla f|^2-\Delta f),\nabla f)\right.\\
&\qquad\qquad\left.+(|\nabla f|^2-\Delta f)\Delta f+\mathrm{div}(\nabla_{\nabla f}\nabla f)\right).
\end{aligned}$$

Since $\left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 = |\nabla^2 f|^2 - \frac{1}{n}(\Delta f)^2$ with the help of Bochner's formula, we deduce from the last relation:

$$\begin{aligned}\left(\frac{1-2\rho(n-1)}{2}\right)\Delta R = &-\mathrm{Ric}(\nabla f, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 - \frac{1}{n}(\Delta f)^2 - g(\nabla \Delta f, \nabla f)\\ &+ (n-1)\Delta\lambda + \lambda\Delta f + \rho R \Delta f + 2\mu g(\nabla_{\nabla f}\nabla f, \nabla f)\\ &+ \mu\left((|\nabla f|^2 - \Delta f)\Delta f - g(\nabla \Delta f, \nabla f) + \mathrm{div}(\nabla_{\nabla f}\nabla f)\right).\end{aligned}$$

Thereby, using equation (4.1) to write $g(\nabla \Delta f, \nabla f) = g(\nabla(n(\lambda + \rho R) + \mu|\nabla f|^2 - R), \nabla f)$, then the we have

$$\begin{aligned}\left(\frac{1-2\rho(n-1)}{2}\right)\Delta R &= -\mathrm{Ric}(\nabla f, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 - \frac{1+n\mu}{n}(\Delta f)^2 + (n-1)\Delta\lambda\\ &\quad - g(\nabla(\mu|\nabla f|^2 - R + (\lambda + \rho R)n), \nabla f) + 2\mu g(\nabla_{\nabla f}\nabla f, \nabla f)\\ &\quad + \lambda\Delta f + \rho R \Delta f + \mu\left(|\nabla f|^2 \Delta f - g(\nabla \Delta f, \nabla f)\right.\\ &\quad \left.+ \mathrm{div}(\nabla_{\nabla f}\nabla f)\right)\\ &= -(\mathrm{Ric}(\nabla f, \nabla f) + (n-1)g(\nabla\lambda, \nabla f)) - \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2\\ &\quad - \frac{1+n\mu}{n}(\Delta f)^2 + (n-1)\Delta\lambda + (\lambda + \rho R)\Delta f + g(\nabla R, \nabla f)\\ &\quad - n\rho g(\nabla R, \nabla f) + \rho g(\nabla R, \nabla f) + \mu\left(|\nabla f|^2 \Delta f\right.\\ &\quad \left.- g(\nabla \Delta f, \nabla f) + \mathrm{div}(\nabla_{\nabla f}\nabla)\right).\end{aligned}$$

Hence, using (4.2) and putting into the last equation, we deduce that

$$\begin{aligned}\left(\frac{1-2\rho(n-1)}{2}\right)\Delta R &= \frac{1}{2} g(\nabla R, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 - \frac{1+n\mu}{n}(\Delta f)^2\\ &\quad + (n-1)\Delta\lambda + (\lambda + \rho R)\Delta f + \frac{\mu}{2} g(\nabla|\nabla f|^2, \nabla f)\\ &\quad + \mu\left[-g(\nabla \Delta f, \nabla f) + \mathrm{div}(\nabla_{\nabla f}\nabla f)\right]\\ &= \frac{1}{2} g(\nabla R, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 - \frac{1+n\mu}{n}(\Delta f)^2 + (n-1)\Delta\lambda\\ &\quad + (\lambda + \rho R)\Delta f + \mu\left(g(\nabla_{\nabla f}\nabla f, \nabla f) - g(\nabla \Delta f, \nabla f)\right.\\ &\quad \left.+ \mathrm{div}\,\nabla_{\nabla f}\nabla f\right)\\ &= g(\nabla R, \nabla f) - \left|\nabla^2 f - \frac{\Delta f}{n} g\right|^2 - \frac{1+n\mu}{n}(\Delta f)^2 + (n-1)\Delta\lambda\\ &\quad + (\lambda + \rho R)\Delta f + \frac{1}{2} g(\nabla f, \nabla \Delta f) - \frac{n}{2} g(\nabla\lambda, \nabla f)\end{aligned}$$

$$-\frac{n\rho}{2}g(\nabla R,\nabla f)-\mu g(\nabla\Delta f,\nabla f)+\mu\operatorname{div}(\nabla_{\nabla f}\nabla f).$$

Since $\nabla_{\nabla f}\nabla f=\frac{1}{2}g(\nabla R,\nabla f)+\frac{1}{2}g(\nabla\Delta f,\nabla f)-\frac{n}{2}g(\nabla\lambda,\nabla f)-\frac{n\rho}{2}g(\nabla R,\nabla f)$, then substituting this equation into the above formula, we get the expression in the statement, which completes the proof of the lemma. □

As a consequence of this lemma, we give the following integral formula.

Theorem 5. *Let $(M^n,g,\xi=\nabla f,\lambda,\mu)$ be a compact orientable almost η-Ricci-Bourguignon soliton. Then we have*

(1) *M^n is trivial provided $\int_M\left(\rho g(\nabla R,\nabla f)+g(\nabla f,\nabla\lambda)\right)dM\geq 0$,*
(2) *$\int_M|\nabla^2 f-\frac{\Delta f}{n}g|^2\,d\mu=-\frac{n+2}{2n}\int_M\left(g(\nabla f,\nabla R)+\mu\Delta f|\nabla f|^2\right)dM$,*
(3) *If $\int_M\left(g(\nabla f,\nabla R)+\mu\Delta f|\nabla f|^2\right)dM\geq 0$ then M^n is conformally equivalent to a unit sphere $\mathbb{S}^n$.*

Proof. Since M^n is compact orientable, then using Lemma 2 and Stokes' formula to infer

$$\int_M|\nabla^2 f-\frac{\Delta f}{n}g|^2\,dM=-\left(\frac{1+n\mu}{n}\right)\int_M(\Delta f)^2\,dM-\left(\frac{1-2\mu}{2}\right)\int_M(\Delta f)^2\,dM$$
$$-\frac{n}{2}\int_M(g(\nabla\lambda,\nabla f)+\rho g(\nabla R,\nabla f)\,dM)$$
$$-\rho\int_M g(\nabla f,\nabla R)\,dM-\int_M g(\nabla\lambda,\nabla f)dM.$$

Hence, we get

$$\int_M\left(\left|\nabla^2 f-\frac{\Delta f}{n}g\right|^2+\left(\frac{n+2}{2n}\right)(\Delta f)^2\right)dM \tag{4.14}$$
$$=-\left(\frac{n+2}{2}\right)\int_M(g(\nabla\lambda,\nabla f)+\rho g(\nabla R,\nabla f))\,dM.$$

Then we have

$$\int_M(\rho g(\nabla R,\nabla f)+g(\nabla\lambda,\nabla f))\,dM\geq 0,$$

it implies that if R and λ are constant, we deduce from the first assertion

$$\int_M\left(\left|\nabla^2 f-\frac{\Delta f}{n}g\right|^2+\left(\frac{n+2}{2n}\right)(\Delta f)^2\right)dM=0,$$

which implies that $\nabla^2 f=\frac{\Delta f}{n}g$ and $\Delta f=0$. So, f is constant, then M^n is trivial. Hence the first statement is proved.

For the second assertion, from (4.1), we can write

$$\int_M g(\nabla f,\nabla\lambda)dM=\frac{1}{n}\int_M g(\nabla f,\nabla((1-n\rho)R+\Delta f-\mu|\nabla f|^2))dM.$$

Therefore, using equation (4.14), we infer

$$\begin{aligned}\int_M \left(\left| \nabla^2 f - \frac{\Delta f}{n} g \right|^2 + \left(\frac{n+2}{2n} \right) (\Delta f)^2 \right) dM \\ = -\frac{n+2}{2n} \int_M g(\nabla f, \nabla R) dM + \frac{n+2}{2n} \int_M (\Delta f)^2 dM \\ + \frac{\mu(n+2)}{2} \int_M g\left(\nabla f, \nabla |\nabla f|^2\right) dM.\end{aligned}$$

Therefore, after some calculations and using Stokes' formula, we deduce

$$\int_M |\nabla^2 f - \frac{\Delta f}{n} g|^2 dM = -\frac{n+2}{2n} \int_M \left(g(\nabla f, \nabla R) + \mu \Delta f |\nabla f|^2 \right) dM.$$

Hence the second item is proved.

For the last item, if $\int_M \left(g(\nabla f, \nabla R) + \mu \Delta f |\nabla f|^2 \right) dM \geq 0$, then we have

$$\int_M \left| \nabla^2 f - \frac{\Delta f}{n} g \right|^2 dM = 0.$$

If f is constant, the solution is trivial otherwise $\nabla^2 f = \frac{\Delta f}{n} g$. We may invoke a theorem due to Ishihara and Tashiro [16] to conclude that M^n is conformally equivalent to a sphere $\mathbb{S}^n$, which completes the proof of the theorem. □

For a conformal vector field ξ on a compact orientable Riemannian manifold M^n we have $\int_M \pounds_\xi R dM = \int_M g(\xi, \nabla R) dM = 0$, see [9] and $\int_M |\xi|^2 \operatorname{div} \xi dM = 0$ from Lemma 1 of [6]. From Theorem 5, we give the following corollary.

Corollary 4. *Let $(M^n, g, \xi = \nabla f, \lambda, \mu)$ be a compact orientable almost η-Ricci-Bourguignon soliton. Then we have*

(1) *If $n \geq 0$, $\int_M g(\nabla f, \nabla R) dM = 0$ and $\int_M \Delta f |\nabla f|^2 dM = 0$, then ∇f is a conformal vector field.*
(2) *If $n = 2$ and $\int_M \Delta f |\nabla f|^2 dM = 0$, then f is constant.*

Proof. Using the last item of Theorem 5, we deduce that $\nabla^2 f = \frac{\Delta f}{n} g$, which allows us to say ∇f is conformal. Hence the first statement is proved. Moreover for $n = 2$, and supposing $\int_M \Delta f |\nabla f|^2 dM = 0$,we conclude that f is constant. □

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