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ON THE NUMBERS OF WEAK CONGRUENCES OF SOME FINITE LATTICES

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This paper is dedicated to Gábor Czédli on his 70th birthday.

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Abstract. We determine the two greatest numbers of weak congruences of lattices. The number of weak congruences of some special lattices are deduced via combinatorial considerations.

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1. INTRODUCTION, PRELIMINARIES

A weak congruence on the algebra A is a compatible weak equivalence on A, i.e., symmetric and transitive relation on A being a subuniverse of A^2. The collection $\mathrm{Cw}A$ of weak congruences on an algebra A is an algebraic lattice under inclusion. The congruence lattice $\mathrm{Con}A$, the subuniverse lattice $\mathrm{Sub}A$ are sublattices of $\mathrm{Cw}A$ as well as $\mathrm{Con}B$, for every subalgebra B of A. See [7] and the book of B. Šešelja and A. Tepavčević [8] for more details. The lattice of weak congruences of a lattice was studied in [9]. Weak congruence lattices of groups have been studied recently with the aim of characterizing various types of groups by weak congruence lattices [4, 5].

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The greatest numbers of subuniverses of lattices were determined in [3] and [1], the greatest numbers of congruences of lattices were determined in [2] and [6] .

Let P and Q be posets with disjoint underlying sets. Then the *ordinal sum* $P +_{ord} Q$ is the poset on $P \cup Q$ with $s \leq t$ if, either $s,t \in P$ and $s \leq t$; or $s,t \in Q$ and $s \leq t$; or $s \in P$ and $t \in Q$. In other words, every element of P is less than every element of Q, and the relations in P and Q stay the same; to draw the Hasse diagram of $P +_{ord} Q$, we place the Hasse diagram of Q above that of P and then connect any minimal element of Q with any maximal element of P. If K with 1 and L with 0 are finite posets, then their *glued sum* $K +_{\mathrm{glu}} L$ is the ordinal sum of the posets $K \setminus \{1_K\}$, the singleton poset, and $L \setminus \{0_L\}$, in this order. Note that $+_{\mathrm{glu}}$ and $+_{ord}$ are associative but not commutative operations.

2. The greatest number of weak congruences of finite lattices

Theorem 1. *If L is a finite lattice of size $n = |L|$, then L has at most $\frac{3^n+1}{2}$ weak congruences. Furthermore, $|\operatorname{Cw} L| = \frac{3^n+1}{2}$ if and only if L is a chain.*

Proof. First, we prove that if L is a chain, then $|\operatorname{Cw} L| = \frac{3^n+1}{2}$. By [3], an n-element lattice can have at most 2^n subuniverses. Furthermore, $|\operatorname{Sub} L| = 2^n$ if and only if it is a chain. By [2], an n-element lattice can have at most 2^{n-1} congruences; furthermore, $|\operatorname{Con} L| = 2^{n-1}$ if and only if it is a chain. Now

$$\begin{aligned} |\operatorname{Cw} L| &= 1 + \sum_{\substack{L^* \in \operatorname{Sub} L \\ L^* \neq \varnothing}} |\operatorname{Con} L^*| = 1 + \sum_{i=1}^{n} \binom{n}{i} 2^{i-1} \\ &= 1 + \frac{\sum_{i=1}^{n} \binom{n}{i} 2^i}{2} = 1 + \frac{-1 + \sum_{i=0}^{n} \binom{n}{i} 2^i}{2} \\ &= 1 + \frac{-1 + (1+2)^n}{2} = \frac{3^n + 1}{2}. \end{aligned}$$

We have to show that all the n-element lattices have fewer weak congruences than $\frac{3^n+1}{2}$. We denote the elements of L by $a_1 \prec \cdots \prec a_n$. If L' is not a chain, then it has at least two incomparable elements, say $p||q$. Of course $p \vee q \in L'$ and $p \wedge q \in L'$. We denote the remaining elements of L' by $b_1, \ldots, b_{n-4}$ arbitrarily. Now

$$\left|\operatorname{Cw} L'\right| = 1 + \sum_{\substack{L^* \in \operatorname{Sub} L' \\ L^* \neq \varnothing}} |\operatorname{Con} L^*|$$

By [3], the sum $|\operatorname{Cw} L'|$ has less summands than the sum $|\operatorname{Cw} L|$. We make an injection from the summands of $|\operatorname{Cw} L'|$ to the summands of $|\operatorname{Cw} L|$ in such a way that the image of each summand is not greater than the summand itself. For this, we define a bijective map $\varphi \colon L' \to L$, $(p \wedge q) \mapsto a_1$, $p \mapsto a_2$, $q \mapsto a_3$, $(p \vee q) \mapsto a_4$, and if $x \notin \{p, q, p \wedge q, p \vee q\}$, then $\varphi(x) \in L \setminus \{a_1, a_2, a_3, a_4\}$ arbitrarily. This φ induces an injective map from the set of sublattices of L' to the set of sublattices of L in a natural way (notice that all the subsets of L are subuniverses), and also an injective map from the summands of $|\mathrm{Cw}L'|$ to the summands of $|\mathrm{Cw}L|$ in such a way that the image of each summand of $|\mathrm{Cw}L'|$ is not greater than the summand itself.

□

3. Lantern: the n-element lattice M_{n-2}

We use the notation M_1 for the 3-element chain and M_2 for the 4-element Boolean lattice. For $n \geq 3$, M_{n-2} consists of $n-2$ atoms, which are also coatoms, and of 0 and 1. So, the lattice M_{n-2} has $n-2$ atoms and n elements. We call the lattice M_{n-2} a *lantern.*

Theorem 2. *For $n \geq 3$, the lantern M_{n-2} has $2^{n-1}+n^2+2n-5$ weak congruences.*

Proof. In the considered lantern M_{n-2} there are n congruences of n one-element sublattices, they provide n weak congruences.

Moreover, there are $2n-3$ two-element sublattices, and each has two congruences, so here we have $4n-6$ weak congruences.

Further, there are $n-2$ three element sublattices (3-element chains) and each has 4 congruences, providing $4n-8$ weak congruences.

Each sublattice with 4 elements contains 1 and 0 and two more middle elements. There are $\frac{(n-2)(n-3)}{2}$ sublattices of 4 elements and each has 4 congruences, so there are $2(n-2)(n-3)$ weak congruences here.

Each sublattice of 5 and more elements contains 1 and 0 and three or more middle elements. Those sublattices have only 2 trivial congruences. There are

$$\sum_{k=3}^{n-2} \binom{n-2}{k}$$

such sublattices. Since

$$\sum_{k=0}^{n-2} \binom{n-2}{k} = 2^{n-2},$$

the number of congruences on sublattices with five and more elements is: $2 \cdot (2^{n-2} - 1 - (n-2) - \binom{n-2}{2})$ which is equal to $2^{n-1} - n^2 + 3n - 4$.

Summing all together, there are $1+n+4n-6+4n-8+2n^2-10n+12+2^{n-1} - n^2+3n-4 = 2^{n-1}+n^2+2n-5$ weak congruences of the lantern M_{n-2}. □

4. The number of weak congruences of ordinal sum of lattices

Lemma 1. *Given finite lattices L_1 and L_2, let $L = L_1 +_{ord} L_2$. Then*

$$|\operatorname{Sub} L| = |\operatorname{Sub} L_1||\operatorname{Sub} L_2|.$$

Proof. It is easy to see that any subuniverse of L can be obtained by the union of a subuniverse chosen from L_1 with a subuniverse chosen from L_2. □

Lemma 2. *Given finite lattices L_1 and L_2, let $L = L_1 +_{glu} L_2$. Then*

$$|\operatorname{Con} L| = |\operatorname{Con} L_1||\operatorname{Con} L_2|.$$

Proof. It is well-known that the congruence classes of lattices are convex. It is easy to see that any congruence of L can be obtained by chosing a congruence of L_1 and choosing a congruence of L_2. □

Lemma 3. *Given finite lattices L_1 and L_2, Let $L = L_1 +_{ord} L_2$. Then*

$$|\operatorname{Cw} L| = 2 \cdot (|\operatorname{Cw} L_1| - 1)(|\operatorname{Cw} L_2| - 1) + |\operatorname{Cw} L_1| + |\operatorname{Cw} L_2| - 1.$$

Proof. Any weak congruence of L_i is a weak congruence of L. Now $|\operatorname{Cw} L_1| + |\operatorname{Cw} L_2|$ count the empty set twice, so we have to subtract 1. All the remaining subuniverses are ordinal sums of nonempty subuniverses of form $B_1 \subseteq L_1$ and $B_2 \subseteq L_2$. Ordinal sum is nothing else than glued sum of them with a 2-element chain in the middle:

$$L_1 +_{ord} L_2 = L_1 +_{glu} C_2 +_{glu} L_2.$$

Now we can note that any weak congruence on L which have some elements from both L_1 and L_2 we can get in two ways. First, as the union of a nonempty weak congruence on L_1 and a nonempty weak congruence on L_2. Second, the transitive closure of the relation which is the union of the full relation on the two element chain with the union of a nonempty weak congruence on L_1 and a nonempty weak congruence on L_2. Therefore, we get $2 \cdot (|\operatorname{Cw} L_1| - 1)(|\operatorname{Cw} L_2| - 1)$ such weak congruences. □

5. Lantern on chain

Theorem 3. *If $L \simeq C_1 +_{ord} M_{k-2} +_{ord} C_2$, where C_1 and C_2 are chains or the empty set, and $|C_1| + |C_2| = l$, then*

$$|\operatorname{Cw} L| = \frac{(2^k + 2k^2 + 4k - 11) \cdot 3^l + 1}{2}$$

Proof. We use Theorem 1, Theorem 2 and Lemma 3. Let $|C_1| = l_1$ and $|C_1| = l_2$ and $L^* \simeq C_1 +_{ord} M_{k-2}$. First

$$\begin{aligned} |\operatorname{Cw} L^*| &= 2 \cdot \left(\frac{3^{l_1}+1}{2} - 1\right)(2^{k-1} + k^2 + 2k - 5 - 1) + \frac{3^{l_1}+1}{2} \\ &\quad + 2^{k-1} + k^2 + 2k - 5 - 1 \\ &= \frac{(2^k + 2k^2 + 4k - 11) \cdot 3^{l_1} + 1}{2}. \end{aligned}$$

Second

$$
\begin{aligned}
|\operatorname{Cw} L| = 2\cdot\left(\frac{3^{l_2}+1}{2}-1\right)\left(\frac{(2^k+2k^2+4k-11)\cdot 3^{l_1}+1}{2}-1\right)\\
+\frac{3^{l_2}+1}{2}+\frac{(2^k+2k^2+4k-11)\cdot 3^{l_1}+1}{2}-1=\\
=\frac{(2^k+2k^2+4k-11)\cdot 3^{l}+1}{2}.
\end{aligned}
$$

□

6. THE SECOND GREATEST NUMBER OF WEAK CONGRUENCES OF FINITE LATTICES

Theorem 4. *If $|L| = n \geq 4$ and L has less than $\frac{3^n+1}{2}$ weak congruences, then the second greatest value in weak congruences is $\frac{53\cdot 3^{n-4}+1}{2}$. Furthermore, L has $\frac{53\cdot 3^{n-4}+1}{2}$ weak congruences if and only if $L \simeq C_1 +_{glu} B_4 +_{glu} C_2$, where C_1 and C_2 are chains or the empty set and B_4 is the four element Boolean lattice.*

Proof. By Theorem 3, we obtain the result $\frac{53\cdot 3^{n-4}+1}{2}$.

We prove that all the other n-element lattices have less weak congruences. To show this, first we calculate the above number in a different way. By [3], L has $13\cdot n^{n-4}$ sublattices. By [2], this form of an n-element lattice L has 2^{n-2} congruences. We denote the non-comparable elements of B_4 by a and b. Now

$$
\begin{aligned}
|\operatorname{Cw} L| &= 1+\sum_{\substack{L^*\in \operatorname{Sub} L\\ L^*\neq\varnothing}} |\operatorname{Con} L^*|\\
&= 1+\sum_{\substack{L^*\in \operatorname{Sub} L\\ B_4\subseteq L^*}} |\operatorname{Con} L^*| + \sum_{\substack{L^*\in \operatorname{Sub} L\\ b\notin L^*\\ a\in L^*}} |\operatorname{Con} L^*| + \sum_{\substack{L^*\in \operatorname{Sub} L\\ \{a\}\cap L^*=\varnothing}} |\operatorname{Con} L^*| = (**)
\end{aligned}
$$

Now

$$
1+\sum_{\substack{L^*\in \operatorname{Sub} L\\ \{a\}\cap L^*=\varnothing}} |\operatorname{Con} L^*| = |\operatorname{Cw} C_{n-1}|
$$

so

$$
\begin{aligned}
(**) &= \sum_{i=0}^{n-4}\binom{n-4}{i}2^{i+4-2}+\sum_{i=0}^{n-2}\binom{n-2}{i}2^{i+1-1}+\frac{3^{n-1}+1}{2}\\
&= 4(1+2)^{n-4}+(1+2)^{n-2}+\frac{3^{n-1}+1}{2}=\frac{53\cdot 3^{n-4}+1}{2}.
\end{aligned}
$$

Consider an arbitrary n-element lattice L' that is neither a chain, nor of form $C_1 +_{glu} B_4 +_{glu} C_2$,. Clearly

$$\left|\mathrm{Cw}\, L'\right| = 1 + \sum_{\substack{L^* \in \mathrm{Sub}\, L' \\ L^* \neq \varnothing}} |\mathrm{Con}\, L^*|\,.$$

This sum contains not more summands than that of L by [3].

We show that $|\mathrm{Cw}\, L'| \leq |\mathrm{Cw}\, L|$. If L' is neither a chain, nor of the form $C_1 +_{glu} B_4 +_{glu} C_2$, then it has antichains, let $a||b$ one of them. We make an injection from the summands of $|\mathrm{Cw}\, L'|$ to the summands of $|\mathrm{Cw}\, L|$ in such a way that the image of each summand is not greater than the summand itself. For this, we define a bijective map $\varphi\colon L' \to L$. Denote the elements of B_4 in L by $\{p, q, p \wedge q, p \vee q\}$. Let $a\varphi = p$, $b\varphi = q$, $(a \wedge b)\varphi = p \wedge q$, $(a \vee b)\varphi = p \vee q$; otherwise we define φ arbitrarily but bijectively. The image of any sublattice of L' is a sublattice of L because if the considered sublattice contains both a and b, then the image of it is a sublattice of form $C_1 +_{glu} B_4 +_{glu} C_2$. If the considered sublattice contains at most one of a and b, then its image is a chain. Now clearly by [2], the image of each summand is not greater than the summand itself because the image of a sublattice is a chain or of form $C_1 +_{glu} B_4 +_{glu} C_2$, but the latter case happens only when the sublattice is not a chain. □

7. Chandelier

Let $N_{m_1, m_2, \ldots, m_n}$ be a lattice of width n, containing n chains with m_1, m_2,...,m_n elements. They have intersection $\{0, 1\}$, any other element of it belongs exactly to one chain. The index i in m_i denote the i-th chain. We call the lattice $N_{m_1, m_2, \ldots, m_n}$ a *chandelier*.

Theorem 5. *The chandelier $N_{m,k}$ has*

$$\frac{3^m - 1}{2} \frac{3^k - 1}{2} + 3 \cdot (2^m - 1)(2^k - 1) + \frac{3^{m+2} + 3^{k+2}}{2} - 4$$

weak congruences.

Proof. This lattice has $m + k + 2$ elements. By Theorem 1 the number of weak congruences on the chain with $m + 2$ elements is $\frac{3^{m+2}+1}{2}$, and the number of weak congruences on the chain with $k + 2$ elements is $\frac{3^{k+2}+1}{2}$. Here, we counted twice the weak congruences on the sublattice $\{0, 1\}$, and there are 5 of them.

Further, we calculate the sublattices having non-comparable elements and congruences on them.

We note that 0 and 1 must belong to any of such sublattices L_1 and at least one element different from 0 and 1 must belong to L_1 from each of the two chains. When we fix such a sublattice with non-comparable elements, we can note that there are two types of congruences on such a sublattice, depending on whether 0 and 1 are one-element congruence classes or not.

First, suppose that in a fixed sublattice L_1, $\{0\}$ and $\{1\}$ are one-element classes in a congruence and we calculate the number of all such congruences. We can chose independently the congruence classes on each of the chains and if this chain has e.g. m_1 and k_1 elements in a sublattice L_1, then there are $2^{m_1-1}\cdot 2^{k_1-1}$ different congruences. Moreover, on each sublattice L_1 we have the congruence L_1^2. When we calculate all congruences on all combinations of such sublattices, there are

$$\sum_{l=1}^{m}\sum_{s=1}^{k}\binom{m}{l}\binom{k}{s}(2^{l-1}2^{s-1}+1)$$

weak congruences of this type.

Second, if $\{0\}$ or $\{1\}$ are not one-element classes, then due to the properties of congruences, if 0 is in a congruence relation with one of the elements of one of the chains, then the whole chain is a congruence block, and 1 is in the relation with all elements of the other chain. Hence, for each of the sublattices there are two possibilities (depending on the chain which is in the congruence relation with 0). Hence, there are

$$2\cdot\sum_{l=1}^{m}\sum_{s=1}^{k}\binom{m}{l}\binom{k}{s}$$

such weak congruences.

Altogether there are

$$\sum_{l=1}^{m}\sum_{s=1}^{k}\binom{m}{l}\binom{k}{s}2^{l-1}2^{s-1}+3\cdot\sum_{l=1}^{m}\sum_{s=1}^{k}\binom{m}{l}\binom{k}{s}+\frac{3^{m+2}+3^{k+2}}{2}-4$$

weak congruences. This is equal to

$$\frac{3^m-1}{2}\frac{3^k-1}{2}+3\cdot(2^m-1)(2^k-1)+\frac{3^{m+2}+3^{k+2}}{2}-4.$$

□

Let $N_{m_1,m_2,\dots,m_n}$ be a chandelier of width n, containing n chains with m_1, m_2,...,m_n elements. Let $w^{(k)}(m_{l_1},\dots,m_{l_k})$ be the number of special weak congruences on $N_{m_1,m_2,\dots,m_n}$, which are congruences of sublattices of $N_{m_1,m_2,\dots,m_n}$ of width k where $\{m_{l_1},\dots m_{l_k}\}$ is a fixed subset of the set $\{m_1,m_2,\dots,m_n\}$ containing k different elements. If $n=1$, then $w^{(n)}(m)$ is the number of weak congruences on the chain with n-elements (not counting the empty set).

It is easy to see that

$$|\,\mathrm{Cw}N_{m,k}|=w^{(1)}(m)+w^{(1)}(k)+w^{(2)}(m,k)-3.$$

Further, $|\,\mathrm{Cw}N_{m,k,l}|=w^{(1)}(m)+w^{(1)}(k)+w^{(1)}(l)+w^{(2)}(m,k)+w^{(2)}(m,l)+w^{(2)}(k,l)+w^{(3)}(m,k,l)-7.$

Lemma 4. *Let $k \geq 3$. Then,*

$$w^{(k)}(m_1,\dots m_n) = \prod_{i=1}^{k} \frac{3^{m_i}-1}{2} + (2^{m_1}-1)\cdot(2^{m_2}-1)\cdot\ldots\cdot(2^{m_k}-1)$$

Proof. $w^{(k)}(m_1,\dots m_k)$ is the set of congruences on sublattices of $N_{m_1,m_2,\dots,m_n}$ of width k. This means that 0 and 1 belong to each of these sublattices as well as at least one element from each of k chains. We obtain different weak congruences as union of weak congruences on chains. Besides the squares of all subalgebras are also weak congruences. 0 and 1 are one-element congruence classes unless the weak congruence is the square of the sublattice.

The number of sublattices of width k is $(2^{m_1}-1)\cdot(2^{m_2}-1)\cdot\ldots\cdot(2^{m_k}-1)$ and the number of weak congruences (which are not square) is $\prod_{i=1}^{k}\frac{3^{m_i}-1}{2}$ □

We use the notation $w^{(i)}A$ for $w^{(i)}(m_1,\dots m_i)$, if $A=\{m_1,\dots m_i\}$.

The proof of the following theorem is straightforward.

Theorem 6. *The number of weak congruences of a chandelier of width n is*

$$|\operatorname{Cw} N_{m_1,m_2,\dots,m_n}| = \sum_{i=1}^{n} \sum_{A\in\mathcal{P}^i(\{m_1,m_2,\dots,m_n\})} w^{(i)}A - 4n + 5,$$

where $\mathcal{P}^i(\{m_1,m_2,\dots,m_n\})$ is the set of all subsets of $\{m_1,m_2,\dots,m_n\}$ with i elements.

8. Appendix: N_5, M_3 and M_4 on chains

In this chapter, we determine the number of weak congruences of some special lattices, which play significant role in lattice theory. These results can be obtained by using Lemma 3, but we prove them without it.

Theorem 7. *If $|L| \geq 5$ and $L \simeq C_1 +_{glu} N_5 +_{glu} C_2$, where C_1 and C_2 are chains or the empty set, then L has $\frac{125\cdot 3^{n-5}+1}{2}$ weak congruences.*

Proof. By [6], the n-element lattice of form $C_1 +_{glu} N_5 +_{glu} C_2$ has $5\cdot 2^{n-5}$ congruences. We denote by a and b the atoms of N_5 and by c and b the coatoms of N_5; of course $a \prec c$.

$$\begin{aligned}|\operatorname{Cw} L| &= 1 + \sum_{\substack{L^*\in \operatorname{Sub} L\\ L^*\neq\varnothing}} |\operatorname{Con} L^*| = \\ &= \sum_{\substack{L^*\in\operatorname{Sub}L\\ N_5\subseteq L^*}} |\operatorname{Con} L^*| + \sum_{\substack{L^*\in\operatorname{Sub}L\\ a,b\in L^*\\ c\notin L^*}} |\operatorname{Con} L^*| + \sum_{\substack{L^*\in\operatorname{Sub}L\\ c,b\in L^*\\ a\notin L^*}} |\operatorname{Con} L^*| + \sum_{\substack{L^*\in\operatorname{Sub}L\\ b\in L^*\\ a,c\notin L^*}} |\operatorname{Con} L^*| \\ &\quad + |\operatorname{Cw} C_{n-1}| = \\ &= \sum_{i=0}^{n-5}\binom{n-5}{i} 5\cdot 2^{i+5-5} + 2\sum_{i=0}^{n-5}\binom{n-5}{i} 2^{i+4-2} + \sum_{i=0}^{n-3}\binom{n-3}{i} 2^{i+1-1}\end{aligned}$$

$$
\begin{aligned}
&+\frac{3^{n-1}+1}{2}= \\
&=5\cdot(1+2)^{n-5}+8\cdot(1+2)^{n-5}+9\cdot(1+2)^{n-5}+\frac{3^{n-1}+1}{2} \\
&=\frac{125\cdot 3^{n-5}+1}{2}.
\end{aligned}
$$

□

Remark 1. We conjecture that $\frac{125\cdot 3^{n-5}+1}{2}$ is the third greatest number of weak congruences of finite lattices, and the corresponding lattice is $L\simeq C_1+_{glu}N_5+_{glu}C_2$, where C_1 and C_2 are chains or the empty set.

Theorem 8. *If $|L|\geq 5$ and $L\simeq C_1+_{glu}M_3+_{glu}C_2$, where C_1 and C_2 are chains or the empty set, then L has $\frac{91\cdot 3^{n-5}+1}{2}$ weak congruences.*

Proof. It is easy to see that the n-element lattice of form $C_1+_{glu}M_3+_{glu}C_2$ has 2^{n-4} congruences. We denote by a, b and c the atoms of M_3, which are also coatoms of M_3. We denote the sublattice of this lattice by deleting one coatom (denoted by a) of M_3 by $\hat{L}$. Now

$$
\begin{aligned}
|\mathrm{Cw}L| &= 1+\sum_{\substack{L^*\in \mathrm{Sub}L\\ L^*\neq\varnothing}}|\mathrm{Con}L^*| \\
&=\sum_{\substack{L^*\in \mathrm{Sub}L\\ M_3\subseteq L^*}}|\mathrm{Con}L^*|+\sum_{\substack{L^*\in \mathrm{Sub}L\\ a\in L^*\\ b,c\notin L^*}}|\mathrm{Con}L^*|+\sum_{\substack{L^*\in \mathrm{Sub}L\\ a,b\in L^*\\ c\notin L^*}}|\mathrm{Con}L^*|+\sum_{\substack{L^*\in \mathrm{Sub}L\\ a,c\in L^*\\ b\notin L^*}}|\mathrm{Con}L^*|+ \\
&\quad+|\mathrm{Cw}\hat{L}| \\
&=\sum_{i=0}^{n-5}\binom{n-5}{i}2^{i+5-4}+\sum_{i=0}^{n-3}\binom{n-3}{i}2^{i+1-1}+2\sum_{i=0}^{n-5}\binom{n-5}{i}2^{i+4-2} \\
&\quad+\frac{53\cdot 3^{n-5}+1}{2} \\
&=2\cdot(1+2)^{n-5}+(1+2)^{n-3}+8\cdot(1+2)^{n-5}+\frac{53\cdot 3^{n-5}+1}{2} \\
&=\frac{91\cdot 3^{n-5}+1}{2}
\end{aligned}
$$

□

Theorem 9. *If $|L|\geq 5$ and $L\simeq C_1+_{glu}M_4+_{glu}C_2$, where C_1 and C_2 are chains or the empty set, then L has $\frac{149\cdot 3^{n-6}+1}{2}$ weak congruences.*

Proof. It is easy to see that the n-element lattice of form $C_1+_{glu}M_4+_{glu}C_2$ has 2^{n-5} congruences. We denote by a, b, c and d the atoms of M_4, which are also coatoms of M_4. We denote the sublattice of this lattice by deleting one coatom (denoted by a) of M_4 by $\hat{L}$. Now

$$
\begin{aligned}
|\operatorname{Cw} L| &= 1 + \sum_{\substack{L^*\in \operatorname{Sub} L\\ L^*\neq\varnothing}} |\operatorname{Con} L^*| = \sum_{\substack{L^*\in \operatorname{Sub} L\\ M_4\subseteq L^*}} |\operatorname{Con} L^*| + \sum_{\substack{L^*\in \operatorname{Sub} L\\ a\in L^*\\ b,c,d\notin L^*}} |\operatorname{Con} L^*| \\
&\quad + \sum_{\substack{L^*\in \operatorname{Sub} L\\ a,b\in L^*\\ c,d\notin L^*}} |\operatorname{Con} L^*| + \sum_{\substack{L^*\in \operatorname{Sub} L\\ a,c\in L^*\\ b,d\notin L^*}} |\operatorname{Con} L^*| + \sum_{\substack{L^*\in \operatorname{Sub} L\\ a,d\in L^*\\ b,c\notin L^*}} |\operatorname{Con} L^*| \\
&\quad + \sum_{\substack{L^*\in \operatorname{Sub} L\\ a,b,c\in L^*\\ d\notin L^*}} |\operatorname{Con} L^*| + \sum_{\substack{L^*\in \operatorname{Sub} L\\ a,b,d\in L^*\\ c\notin L^*}} |\operatorname{Con} L^*| + \sum_{\substack{L^*\in \operatorname{Sub} L\\ a,c,d\in L^*\\ b\notin L^*}} |\operatorname{Con} L^*| + |\operatorname{Cw} \hat{L}| \\
&= \sum_{i=0}^{n-6} \binom{n-6}{i} 2^{i+6-5} + \sum_{i=0}^{n-4} \binom{n-4}{i} 2^{i+1-1} + 3\sum_{i=0}^{n-6} \binom{n-6}{i} 2^{i+4-2} \\
&\quad + 3\sum_{i=0}^{n-6} \binom{n-6}{i} 2^{i+5-4} + \frac{91\cdot 3^{n-6}+1}{2} \\
&= 2\cdot(1+2)^{n-6} + (1+2)^{n-4} + 12\cdot(1+2)^{n-6} + 6\cdot(1+2)^{n-6} \\
&\quad + \frac{91\cdot 3^{n-6}+1}{2} \\
&= \frac{149\cdot 3^{n-6}+1}{2.}
\end{aligned}
$$

□

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