Miskolc Mathematical Notes
Vol. 25 (2024), No. 2, pp. 1075–1084
HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4397

NONDECREASING BOUNDED CONTINUOUS SOLUTIONS OF A q-DIFFERENCE EQUATION

HOU YU ZHAO AND SHAN SHAN GUO

Received 03 September, 2022

Abstract. Schauder's fixed point theorem and Banach contraction principle are used to study a q-difference equation. We give sufficient conditions for the existence, uniqueness, and stability of the nondecreasing bounded continuous solutions. We also give the approximate sequences for the corresponding solutions. Finally, some examples are considered for our results.

2010 *Mathematics Subject Classification:* 39B12; 37E05

Keywords: q-difference equation, nondecreasing bounded continuous solutions, fixed point

1. INTRODUCTION

The study about q-difference equations has a long history. For example, the linear ordinary q-difference equations have been investigated in the beginning of the 20 century by Birkhoff [4, 5], Carmichael [6], Jackson [8, 9], Adams [1], Trjitzinsky [16], Mason [11], and other authors [2,3,7,10,12,13,17,18]. However, since the late 1940s, the theory has been relatively little researched. In the last 20 years the field has recovered its original vitality and the theory of q-difference equations or more generally functional equations has witnessed substantial advances. See, for example, [15, 19]. Recently, Si and Zhang [14] studied the existence of analytic solutions of the nonlinear q-difference equation

$$G\Big(f(x), f(qx), \ldots, f(q^n x)\Big) = 0,$$

where f is an unknown function and $G(x)$ is given function.

This work was partially supported by Science and Technology Research Program of Chongqing Municipal Education Commission (Grant No. KJQN 202000536), the Natural Science Foundation of Chongqing(Grant No. cstc2020jcyj-msxmX0606), Research Project of Chongqing Education Commission (Grant No. CXQT21014), Innovation and Entrepreneurship Support Program of Chongqing (Grant No. cx2023094).

In this note, we will be concerned with

$$G\Big(f(x), f(qx), \dots, f(q^n x)\Big) = F(x), \tag{1.1}$$

where f is an unknown function, $F(x)$ and $G(x)$ are given functions. By means of the Schauder's fixed point theorem and Banach contraction principle, we discuss the existence, uniqueness and stability of nondecreasing bounded continuous solutions of equation (1.1). Furthermore, we consider the approximate solutions sequence for the corresponding solutions.

Let $M \geq 1 \geq m \geq 0$ and $C(I)$ consist of all continuous functions on $I = [a,b]$. Define

$$\Phi(I;m,M) = \{f \in C(I) : f(a) = a, f(b) = b, a \leq f(x) \leq b,$$
$$m(x-y) \leq f(x) - f(y) \leq M(x-y),\ \forall x,y \in I, x \geq y\}.$$

Clearly, $C(I)$ is a real Banach space with respect to the uniform norm

$$\|f\| = \max\{|f(x)| : x \in I\} \quad \text{for } f \in C(I).$$

In fact, it is easy to check that $\Phi(I;m,M)$ is a metric space under the uniform norm $\|f\|$. Furthermore, suppose that the sequence $\{f_n\}_{n=1}^{\infty}$ in $\Phi(I;m,M)$ has a limit $\widetilde{f} \in C(I)$. Noting

$$\widetilde{f}(x) - \widetilde{f}(y) = (\widetilde{f}(x) - f_n(x)) + (f_n(x) - f_n(y)) + (f_n(y) - \widetilde{f}(y)), \qquad x \geq y,$$

taking $n \to \infty$, we have

$$m(x-y) \leq \widetilde{f}(x) - \widetilde{f}(y) \leq M(x-y), \qquad \forall x,y \in I,\ x \geq y$$

and

$$\widetilde{f}(a) = \lim_{n\to\infty} f_n(a) = a, \qquad \widetilde{f}(b) = \lim_{n\to\infty} f_n(b) = b.$$

Thus, we have $\widetilde{f} \in \Phi(I;m,M)$ and $\Phi(I;m,M)$ is a a complete metric space. In the class of C^1 functions the conditions in the definition of $\Phi(I;m,M)$ coincide with $m \leq f'(x) \leq M$.

The rest of the paper is organized as follows. In Section 2, we give the existence of nondecreasing bounded continuous solutions of Eq. (1.1) under the monotonicity assumption. Section 3 deals with the uniqueness and stability of those solutions. The final section presents some examples.

2. Nondecreasing bounded continuous solutions

In this section, the existence of a nondecreasing bounded continuous solutions of Eq. (1.1) will be proved. Let us give some lemmas which will be used to prove our theorem.

Lemma 1 ([21, Lemma 1])**.** *$\Phi(I;m,M)$ is a compact convex subset of $C(I)$.*

Lemma 2 ([20, Lemma 2]). *Suppose that $f,g\in\Phi(I;m,M)$, where $M\geq 1\geq m>0$, then the following inequalities hold:*

(i) $\|f^k-g^k\|\leq\sum_{j=0}^{k-1}M^j\|f-g\|$, $k\in\mathbb{Z}^+$.
(ii) $\|f-g\|\leq M\left\|f^{-1}-g^{-1}\right\|$.
(iii) $\left\|f^{-1}-g^{-1}\right\|\leq m^{-1}\|f-g\|$.
(iv) $f^{-1}\in\Phi(I;M^{-1},m^{-1})$.

Now, we shall consider (1.1) on $I=[0,b]$ under the following assumptions.

(H1) $G(x_1,x_2,\ldots,x_{n+1})\in C(I^{n+1},I)$, $G(0,x_2,\ldots,x_{n+1})=0, G(b,x_2,\ldots,x_{n+1})=b$.
(H2) There exist $L_i\geq l_i\geq 0$ such that, for all $x_i,y_i\in I, x_i\geq y_i, i=1,2,\ldots,n+1$,

$$\sum_{i=1}^{n+1}l_i(x_i-y_i)\leq G(x_1,x_2,\ldots,x_{n+1})-G(y_1,y_2,\ldots,y_{n+1})\leq\sum_{i=1}^{n+1}L_i(x_i-y_i).$$

Remark 1. Taking $x_i=b, y_i=0, i=1,2,\ldots,n+1$, we see

$$b\sum_{i=1}^{n+1}l_i\leq G(b,\ldots,b)-G(0,\ldots,0)\leq b\sum_{i=1}^{n+1}L_i,$$

i.e., $\sum_{i=1}^{n}l_i\leq 1\leq\sum_{i=1}^{n}L_i$.

Theorem 1. *Suppose that* (H1) *and* (H2) *hold, $F\in\Phi(I;m_F,M_F)$, $0<m_F\leq 1\leq M_F$. If inequality*

$$0<m\leq\frac{m_F}{L_1+M_0}\leq 1\leq\frac{M_F}{l_1-M_1}\leq M \tag{2.1}$$

holds for constants $M_0=\frac{M}{m}\sum_{i=1}^{n}q^iL_{i+1}, M_1=\frac{m}{M}\sum_{i=1}^{n}q^il_{i+1}$, where $q\in(0,1)$. Then Eq. (1.1) *has a solution $f\in\Phi(I;m,M)$ with $0<m\leq 1\leq M$.*

Proof. Define a mapping $T:\Phi(I;m,M)\to C(I)$ by

$$Tf(x)=G\Big(x,f(qf^{-1}(x)),\ldots,f(q^nf^{-1}(x))\Big),\qquad\forall x\in I.$$

By (H1) and (H2) we know Tf is nondecreasing and $Tf(0)=0, Tf(b)=b$. For any $x,y\in I$ with $x\geq y$, from Lemma 2, we can check that

$$\begin{aligned}
&Tf(x)-Tf(y)\\
&=G\Big(x,f(qf^{-1}(x)),\ldots,f(q^nf^{-1}(x))\Big)-G\Big(y,f(qf^{-1}(y)),\ldots,f(q^nf^{-1}(y))\Big)\\
&\leq L_1(x-y)+L_2\Big(f(qf^{-1}(x))-f(qf^{-1}(y))\Big)+\cdots\\
&\quad+L_{n+1}\Big(f(q^nf^{-1}(x))-f(q^nf^{-1}(y))\Big)\\
&\leq L_1(x-y)+\frac{qM}{m}L_2(x-y)+\cdots+\frac{q^nM}{m}L_{n+1}(x-y)\\
&=\Big(L_1+\frac{M}{m}\sum_{i=1}^{n}q^iL_{i+1}\Big)(x-y)=(L_1+M_0)\,(x-y)
\end{aligned}$$

and

$$\begin{aligned}
&Tf(x)-Tf(y)\\
&\geq l_1(x-y)+l_2\Big(f(qf^{-1}(x))-f(qf^{-1}(y))\Big)+\cdots\\
&\quad+l_{n+1}\Big(f(q^nf^{-1}(x))-f(q^nf^{-1}(y))\Big)\\
&\geq l_1(x-y)-\frac{qm}{M}l_2(x-y)-\cdots-\frac{q^nm}{M}l_{n+1}(x-y)\\
&=\Big(l_1-\frac{m}{M}\sum_{i=1}^{n}q^il_{i+1}\Big)(x-y)=(l_1-M_1)(x-y),
\end{aligned}$$

where $M_0=\frac{M}{m}\sum_{i=1}^{n}q^iL_{i+1}, M_1=\frac{m}{M}\sum_{i=1}^{n}q^il_{i+1}$.

Theeqrefore,

$$0<(l_1-M_1)(x-y)\leq Tf(x)-Tf(y)\leq (L_1+M_0)(x-y), \qquad \forall x\geq y\in I.$$

So Tf is invertible and

$$\frac{1}{L_1+M_0}(x-y)\leq (Tf)^{-1}(x)-(Tf)^{-1}(y)\leq \frac{1}{l_1-M_1}(x-y), \qquad \forall x\geq y\in I, \tag{2.2}$$

$(Tf)^{-1}(0)=0, (Tf)^{-1}(b)=b, 0\leq (Tf)^{-1}(x)\leq b, \forall x\in I$ and $(Tf)^{-1}$ is increasing. Obviously, Tf is a homeomorphism from $I=[0,b]$ to itself.

Define a mapping $\mathcal{T}:\Phi(I;m,M)\to C(I)$ by

$$\mathcal{T}f(x)=(Tf)^{-1}\circ F(x). \tag{2.3}$$

Clearly, $\mathcal{T}f(0)=0, \mathcal{T}f(b)=b$ and $0\le \mathcal{T}f(x)\le b$. From (2.2) and (2.1), for $x\ge y\in I$, we have

$$\mathcal{T}f(x)-\mathcal{T}f(y)\le \frac{M_F}{l_1-M_1}(x-y)\le M(x-y)$$

and

$$\mathcal{T}f(x)-\mathcal{T}f(y)\ge \frac{m_F}{L_1+M_0}(x-y)\ge m(x-y),$$

implying that $\mathcal{T}$ is a self-mapping on $\Phi(I;m,M)$. For $f,g\in\Phi(I;m,M)$, by Lemma 2,

$$\begin{aligned}
\|\mathcal{T}f-\mathcal{T}g\| &\le \left\|(Tf)^{-1}-(Tg)^{-1}\right\| \\
&\le \frac{1}{l_1-M_1}\|Tf-Tg\| \\
&\le \frac{1}{l_1-M_1}\bigg(L_2\Big(f(qf^{-1}(x))-g(qg^{-1}(x))\Big)+\cdots \\
&\qquad\qquad +L_{n+1}\Big(f(q^nf^{-1}(x))-g(q^ng^{-1}(x))\Big)\bigg) \\
&\le \Big(\sum_{i=1}^{n}L_{i+1}\Big)\|f-g\|+M\Big(\sum_{i=1}^{n}q^iL_{i+1}\Big)\|f^{-1}-g^{-1}\| \\
&\le \left(\sum_{i=1}^{n}L_{i+1}+\frac{M}{m}\sum_{i=1}^{n}q^iL_{i+1}\right)\|f-g\| \\
&= (M_0+\sum_{i=1}^{n}L_{i+1})\|f-g\|, \qquad (2.4)
\end{aligned}$$

implying the continuity of $\mathcal{T}$. By Lemma 1 and Schauder's fixed-point theorem, $\mathcal{T}$ has a fixed point $f\in\Phi(I;m,M)$, which gives the desired solution. □

Theorem 2. *In addition to the assumption of Theorem 1, suppose that*

$$M_0+\sum_{i=1}^{n}L_{i+1}<1. \qquad (2.5)$$

Then for any $\varphi_0\in\Phi(I;m,M)$, there exists a sequence $(\varphi_k)_{k=0}^{\infty}\subset\Phi(I;m,M)$ which is defined by $\varphi_k=\mathcal{T}\varphi_{k-1}, k=1,2,\ldots$, convergent to φ^* which is a solution of Eq. (1.1).

Proof. Consider a mapping $\mathcal{T}$ on $\Phi(I;m,M)$ as in (2.3). Furthermore, set

$$\varphi_k=\mathcal{T}\varphi_{k-1}, \qquad \varphi_0\in\Phi(I;m,M)$$

for $k \in \mathbb{N}$. Noting that $\mathcal{T}$ is a self-mapping on $\Phi(I;m,M)$, we have $(\varphi_k)_{k=0}^{\infty}$ is a subset of $\Phi(I;m,M)$ and from (2.4),

$$\sup_{x\in[a,b]} |\mathcal{T}\varphi_{k+1}(x) - \mathcal{T}\varphi_k(x)| \le \left(M_0 + \sum_{i=1}^{n} L_{i+1}\right) \|\varphi_{k+1} - \varphi_k\|,$$

i.e.,

$$\|\mathcal{T}\varphi_{k+1} - \mathcal{T}\varphi_k\| \le \Gamma \|\varphi_{k+1} - \varphi_k\|,$$

where $\Gamma = M_0 + \sum_{i=1}^{n} L_{i+1}$. Theeqrefore,

$$\|\varphi_{k+1} - \varphi_k\| = \|\mathcal{T}\varphi_k - \mathcal{T}\varphi_{k-1}\| \le \Gamma^k \|\varphi_1 - \varphi_0\|.$$

Let

$$\varphi_s(x) = \varphi_0(x) + \sum_{k=0}^{s-1} (\varphi_{k+1}(x) - \varphi_k(x)),$$

we now show that $\sum_{k=0}^{s-1}(\varphi_{k+1}(x) - \varphi_k(x))$ converges on the interval $[0,b]$. This would imply that $\varphi_s(x)$ has a limit on this interval as $s \to \infty$. Clearly, to establish the convergence of $\sum_{k=0}^{\infty}\left(\varphi_{k+1}(x) - \varphi_k(x)\right)$, we note that, in view of (2.5), the series

$$\sum_{k=0}^{\infty} \|\varphi_{k+1} - \varphi_k\| \le \sum_{k=0}^{\infty} \Gamma^k \|\varphi_1 - \varphi_0\| = \frac{1}{1-\Gamma} \|\varphi_1 - \varphi_0\|$$

converges.

This means that $(\varphi_k)_{k=0}^{\infty}$ is a Cauchy sequence under the supreme norm and, theeqrefore, uniformly converges to a continuous function φ^* on $[0,b]$. But we already know that $\Phi(I;m,M)$ is compact, so $(\varphi_k)_{k=0}^{\infty}$ converges to φ^* in $\Phi(I;m,M)$. From ((2.4)) we see that $\mathcal{T}$ is continuous, thus $\varphi^* \leftarrow \varphi_{k+1} = \mathcal{T}\varphi_k \to \mathcal{T}\varphi^*$, we obtain $\mathcal{T}\varphi^* = \varphi^*$. Noting that $\varphi_k \in \Phi(I;m,M)$ for any $\varphi_k = \mathcal{T}\varphi_{k-1}, \quad \varphi_0 \in \Phi(I;m,M), \ k = 1,2,\ldots$. Thus $\|\varphi_k\| = \|\mathcal{T}\varphi_{k-1}\| \le b$ and we see that $\|\varphi^*\| = \|\mathcal{T}\varphi^*\| \le b$. Theeqrefore, the sequence of functions given by $S = (\varphi_0(x), \varphi_1(x), \ldots, \varphi_k(x), \ldots)$ can be regarded as approximate solutions of Eq. (1.1). Theorem (2) is proved. □

3. Uniqueness and stability

In this section, we consider the uniqueness and stability of the nondecreasing bounded continuous solutions of (1.1).

Theorem 3. *In addition to the assumption of Theorem 1, suppose that (2.5) holds. Then Eq. (1.1) has a unique solution in $\Phi(I;m,M)$, and the unique solution depends continuously on the given functions G and F. Furthermore, the unique solution can be obtained by the sequence $(\varphi_k)_{k=0}^{\infty}$, here $\varphi_0 \in \Phi(I;m,M)$, $\varphi_{k+1} = \mathcal{T}\varphi_k, k = 0,1,\ldots$ and $\mathcal{T}$ is defined as in (2.3).*

Proof. From the proof of Theorem (1), the map $\mathcal{T}: \Phi(I;m,M) \to \Phi(I;m,M)$ in (2.3). Moreover, by (2.4), we get

$$\|\mathcal{T}f - \mathcal{T}g\| \le (M_0 + \sum_{i=1}^{n} L_{i+1})\,\|f-g\|\,,$$

where M_0 is defined as in Theorem (1). By (2.5),

$$\Gamma = M_0 + \sum_{i=1}^{n} L_{i+1} < 1.$$

So the fixed point must be unique by the Banach fixed point theorem.

Given G_1, G_2 satisfy (H1)-(H2), $F_1, F_2 \in \Phi(I;m_F,M_F)$, we consider the corresponding operators $\mathcal{T}, \widetilde{\mathcal{T}}$ defined by (2.3). Assuming the corresponding conditions (2.1) and (2.5), there are two unique corresponding functions f_1 and f_2 in $\Phi(I;m,M)$ such that

$$f_1 = \mathcal{T}f_1, \qquad f_2 = \widetilde{\mathcal{T}}f_2.$$

Then we have

$$\begin{aligned}\|f_1 - f_2\| &\le \|\mathcal{T}f_1 - \mathcal{T}f_2\| + \|\mathcal{T}f_2 - \widetilde{\mathcal{T}}f_2\| \\ &\le \Gamma\|f_1 - f_2\| + \|\mathcal{T}f_2 - \widetilde{\mathcal{T}}f_2\|,\end{aligned}$$

which implies

$$\|f_1 - f_2\| \le \frac{1}{1-\Gamma}\|\mathcal{T}f_2 - \widetilde{\mathcal{T}}f_2\|. \tag{3.1}$$

Using (2.2),

$$\begin{aligned}\|\mathcal{T}f_2 - \widetilde{\mathcal{T}}f_2\| &= \|(G_1)^{-1}\circ F_1 - (G_2)^{-1}\circ F_2\| \\ &\le \|(G_1)^{-1}\circ F_1 - (G_1)^{-1}\circ F_2\| + \|(G_1)^{-1}\circ F_2 - (G_2)^{-1}\circ F_2\| \\ &\le \frac{1}{l_1 - M_1}\,\|F_1 - F_2\| + \|G_1^{-1} - G_2^{-1}\|,\end{aligned}$$

by (3.1), we get

$$\|f_1 - f_2\| \le \frac{1}{(1-\Gamma)(l_1 - M_1)}\,\|F_1 - F_2\| + \frac{1}{1-\Gamma}\|G_1^{-1} - G_2^{-1}\|.$$

This proves the continuous dependence of solution f upon G and F, otherwise eqreferred to as stability. From Theorem 2, we can finish the proof. □

4. Examples

In this section, some examples are provided to illustrate that the assumptions of Theorem 1 is not self-contradictory.

Example 1. First, we show that the conditions in Theorem 1 are not self-contradictory. Consider the following equation:

$$f(x)+f(x)\left(\frac{1}{2}-f(x)\right)f\left(\frac{1}{5}x\right)=x^2+\frac{x}{2},\qquad x\in\left[0,\frac{1}{2}\right],\tag{4.1}$$

where $G(x_1,x_2)=x_1+\left(\frac{1}{2}-x_1\right)x_1x_2, G(0,x_2)=0, G\left(\frac{1}{2},x_2\right)=\frac{1}{2}$. We can take $L_1=\frac{5}{4}, L_2=\frac{1}{16}, l_1=\frac{3}{4}, l_2=0$ and $F(x)=x^2+\frac{x}{2}$ and $F\in\Phi\left(\left[0,\frac{1}{2}\right];\frac{1}{2},\frac{3}{2}\right)$. Taking $m=\frac{1}{5}, M=10$, then $M_0=\frac{5}{8}, M_1=0$, and a simple calculation yields

$$0\le m=\frac{1}{5}\le\frac{m_F}{L_1+M_0}=\frac{4}{15}\le 1\le 2=\frac{M_F}{l_1-M_1}\le 10=M,$$

thus (2.1) is satisfied. Theorem 1 gives a nondecreasing bounded continuous solution f of Eq. (4.1) in $\Phi\left(\left[0,\frac{1}{2}\right];\frac{1}{5},10\right)$. Noting $M_0+L_2=\frac{11}{16}<1$, (2.5) is satisfied, hence by Theorem 3, we know that the nondecreasing continuous solution is the unique one in $\Phi\left(\left[0,\frac{1}{2}\right];\frac{1}{5},10\right)$. Furthermore, for any $\varphi_0\in\Phi\left(\left[0,\frac{1}{2}\right];\frac{1}{5},10\right)$, the unique solution of (4.1) in $\Phi\left(\left[0,\frac{1}{2}\right];\frac{1}{5},10\right)$ can be approximated by the sequence $(\varphi_k)_{k=0}^{\infty}$, $\varphi_k=\mathcal{T}\varphi_{k-1}$, $\mathcal{T}$ is defined as in (2.3), $k=1,2,\ldots$.

Example 2. Consider the following equation:

$$f(x)+f(x)\left(\frac{1}{2}-f(x)\right)f(qx)=x^2+\frac{x}{2},\qquad x\in\left[0,\frac{1}{2}\right],\tag{4.2}$$

where $q\in(0,1)$, as in Example 6.1,

$$G(x_1,x_2)=x_1+(\frac{1}{2}-x_1)x_1x_2,\qquad G(0,x_2)=0,\qquad G(\frac{1}{2},x_2)=\frac{1}{2}.$$

We can take

$$L_1=\frac{5}{4},\quad L_2=\frac{1}{16},\quad l_1=\frac{3}{4},\quad l_2=0,\quad F(x)=x^2+\frac{x}{2},\quad F\in\Omega\left(\left[0,\frac{1}{2}\right];\frac{1}{2},\frac{3}{2}\right).$$

We will consider the existence of solution $f\in\Phi\left(\left[0,\frac{1}{2}\right];\delta,M\right)$ for (4.2). Noting that $M_0=\frac{M}{8}qL_2=\frac{qM}{16\delta}, M_1=0$. In order to apply (2.1) in Theorem 1, we need

$$0<\delta\le\frac{m_F}{L_1+M_0}=\frac{8\delta}{20m+qM}\le 1\le 2=\frac{M_F}{l_1-M_1}\le M.$$

then

$$qM\le 8-20\delta\quad\text{and}\quad 2\le M.\tag{4.3}$$

Theeqrefore

$$0<q\le 4-10\delta,$$

then

$$0 < \delta < \frac{2}{5}.$$

Furthermore, in order to apply Theorem 3, we need

$$M_0 + L_2 = \frac{qM}{16\delta} + \frac{1}{16} < 1,$$

from (4.3), we have

$$2 \le M < \frac{1}{q} \min \left\{ 8 - 20\delta, 15\delta \right\}. \tag{4.4}$$

From (4.3) and (4.4), we know that Eq. (4.2) has an unique nondecreasing bounded continuous solution $f \in \Omega\left(\left[0, \frac{1}{2}\right]; \delta, M\right)$ with $0 < q \le 4 - 10\delta$ and $2 \le M < \frac{1}{q} \min \left\{ 8 - 20\delta, 15\delta \right\}$ for $0 \le \delta < \frac{2}{5}$.

By Theorem 3, for any $\varphi_0 \in \Omega\left(\left[0, \frac{1}{2}\right]; \delta, M\right)$, the unique nondecreasing continuous solution of (4.2) can be approximated by the sequence $(\varphi_k)_{k=0}^{\infty}$, $\varphi_k = \mathcal{T}\varphi_{k-1}$, $\mathcal{T}$ as in (2.3), $k = 1, 2, \ldots$.

It is easy to check that $q = \frac{1}{5}, m = \frac{1}{5}, M = 10$ in Example 1 which is a special case for Example 2.

REFERENCES

[1] C. R. Adams, "On the linear ordinary q-difference equation." *Annals of Mathematics*, vol. 30, no. 2, pp. 195–205, 1929, doi: 10.2307/1968274.

[2] Y. Andre, "Series Gevrey de type arithmetique. ii." *Annals of Mathematics*, vol. 151, no. 2, pp. 741–756, 2000, doi: 10.2307/121046.

[3] J.-P. Bezivin and A. Boutabaa, "Sur lesequations fonctionelles p-adiques aux q-differences." *Collectanea Mathematica*, vol. 43, no. 2, pp. 125–140, 1992, doi: 10.1007/BF01835698.

[4] G. Birkhoff, "The generalized Riemann problem for linear differential equations and the allied problem for linear difference and q-difference equations." *Bull. Am. Math. Soc.*, vol. 49, no. 2, pp. 508–509, 1913, doi: 10.1137/050641867.

[5] G. Birkhoff and P. Guenther, "Note on canonical form for the linear q-difference system." *Proc. Natl. Acad Sci. USA*, vol. 27, no. 4, pp. 218–222, 1941, doi: 10.1073/pnas.27.4.218.

[6] R. D. Carmichael, "The general theory of linear q-difference equations." *Am. J. Math.*, vol. 34, no. 2, pp. 147–168, 1912, doi: 10.2307/2369887.

[7] G. M. Golusin, "Solution of the fundamental plance problems of mathematical physics for the Laplace equation and multi-connected domains bounded by circumferences." *Mat. Sb.*, vol. 41, pp. 246–283[in Russian], 1934.

[8] F. H. Jackson, "On q-functions and a certain difference operator." *Trans. R. Soc. Edinb.*, vol. 46, pp. 253–281, 1908, doi: 10.1017/S0080456800002751.

[9] F. H. Jackson, "q-difference equations." *Am. J. Math.*, vol. 32, no. 4, pp. 305–314, 1910, doi: 10.2307/2370183.

[10] D. Levi, J. Negro, and M. A. DelOlmo, "Discrete q-derivatives and symmetries of q-difference equations." *J. Phys. A, Math. Gen.*, vol. 37, no. 10, pp. 3459–3473, 2004, doi: 10.1088/0305-4470/37/10/010.

[11] T. E. Mason, "On properties of the solution of linear q-difference equations with entire fucntion coefficients." *Am. J. Math.*, vol. 37, no. 4, pp. 439–444, 1915, doi: 10.2307/2370216.

[12] V. V. Mityushev, *Some mathematical problems of heat conduction 1.*, Moscow, [in Russian], 1987.

[13] J. Sauloy, "Systemes aux q-differences singuliers reguliers: classification, matrice de connexion et monodromie." *Ann. Inst. Fourier*, vol. 50, no. 4, pp. 1021–101, 2000, doi: 10.5802/aif.1784.

[14] J. G. Si and W. N. Zhang, "Analytic solutions of a q-difference equation and applications to iterative equations." *J. Differ Equ. Appl.*, vol. 10, no. 11, pp. 955–962, 2004, doi: 10.1080/10236190412331272607.

[15] J. G. Si and H. Y. Zhao, "Small divisor problem in dynamical systems and analytic solutions of a q-difference equation with a singularity at the origin." *Results. Math.*, vol. 58, no. 3-4, pp. 337–353, 2010, doi: 10.1007/s00025-010-0060-2.

[16] W. J. Trjitzinsky, "Analytic theory of linea q-difference equations." *Acta Math.*, vol. 61, no. 2, pp. 1–38, 1933, doi: 10.1007/BF02547785.

[17] M. van der Put and M. F. Singer, *Galois theory of difference equations. No. 1666*. Berlin: Springer, 1997. doi: 10.1007/BFb0096118.

[18] L. D. Vizio, "An ultrametric version of the Maillet-Malgrange theorem for nonlinear q-difference equations." *Proc. Am. Math. Soc.*, vol. 136, no. 8, pp. 2803–2814, 2008, doi: 10.1090/S0002-9939-08-09352-0.

[19] B. Xu, , and W. N. Zhang, "Small divisor problem for an analytic q-difference equation." *J. Math. Anal. Appl.*, vol. 342, no. 1, pp. 694–703, 2008, doi: 10.1016/j.jmaa.2007.12.010.

[20] W. N. Zhang and J. A. Baker, "Continuous solutions of a polynomial-like iterative equation with variable coefficients." *Ann. Pol. Math.*, vol. 73, no. 1, pp. 29–36, 2000, doi: 10.4064/ap-73-1-29-36.

[21] W. N. Zhang, K. Nikodem, and B. Xu, "Convex solutionsof polynomial-like interative equations." *J. Math. Anal. Appl.*, vol. 315, pp. 29–40, 2006, doi: 10.1016/j.jmaa.2005.10.006.

Authors' addresses

Hou Yu Zhao
(**Corresponding author**) Chongqing Normal University, School of Mathematics, 401331 Chongqing, P.R.China
E-mail address: `houyu19@gmail.com`

Shan Shan Guo
Chongqing Normal University, School of Mathematics, 401331 Chongqing, P.R.China
E-mail address: `guoshanshan0516@163.com`