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LIE TRIPLE DERIVATION AND LIE BI-DERIVATION ON QUATERNION RINGS

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Abstract. In this study, we prove the existence of the central Lie bi-derivation for the ring with identity on the quaternion ring. We also describe the triple Lie derivation using the Jordan derivation on the aforementioned ring. An example is provided to demonstrate that our result is theoretically viable.

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1. INTRODUCTION

Recently, the study of differential operators on quaternion rings has increasingly piqued the interest of researchers. Numerous scholars have investigated the formation of Lie derivations on rings. An essential question arising in this context is; 'On a quaternion ring what are the circumstances that allow the decomposition of a Lie derivation into the sum of a derivation and an additive central map that vanishes on all commutators of rings?' A Lie derivation of certain primitive rings was originally established to be in standard form by Martindale [15]. Miers [16,17] considered similar findings on von Neumann algebras. In [8], Hvala demonstrated that each generalized Lie derivation of a prime ring $\mathcal{W}$ whose characteristic is not 2 is the sum of a generalized derivation and an additive central map that vanishes on all commutators of $\mathcal{W}$. Liao and Liu [12] have extended Hvala's finding to a prime algebra's Lie ideal. In 2020, Ghahramani et al. [6] evaluated Lie derivations on quaternion rings and demonstrated that every Lie derivation is in standard form on acceptable quaternion rings. Again in 2021, Ghahramani et al. [7] demonstrated that every Jordan derivation of $\mathcal{W}$ is a derivation, and that every derivation of $\mathcal{W}$ decomposes into the sum of an inner derivation and a derivation induced by a derivation on $\mathcal{W}$.

On the other hand in 2021, Asif et al. [3] provided an in-depth specification of all Lie triple derivations of $A \otimes F$, where A is a finite dimensional commutative algebra and L is a simple Lie algebra over an algebraically closed field. Inspired by the present ongoing contribution in this direction, it is unsurprising to discuss Lie triple derivation and Lie bi-derivation on quaternion rings and we prove that the existence of the central Lie bi-derivation on the quaternion ring. We also describe the triple Lie derivation using the Jordan derivation on the aforementioned ring. An example is provided to demonstrate that our result is theoretically viable.

Before presenting our results, let's recall some known definitions and related theories established by eminent ring theorist for the sake of completeness. Let $\mathcal{W}$ be a ring and $\mathcal{Z}(\mathcal{W})$ be the center of $\mathcal{W}$. For each $x,y\in\mathcal{W}$, denote the Lie product (or the commutator) of x,y by $[x,y]=xy-yx$, and the Jordan product of x,y by $x\circ y=xy+yx$. An additive map Ω on $\mathcal{W}$ is called as derivation if $\Omega(xy)=\Omega(x)y+x\Omega(y)$ is always true for every $x,y\in\mathcal{W}$. An additive map Ω on $\mathcal{W}$ is called as Lie derivation if $\Omega([x,y])=[\Omega(x),y]+[x,\Omega(y)]$ holds for all $x,y\in\mathcal{W}$. An additive map Ω on $\mathcal{W}$ is called as Lie triple derivation if $\Omega([[x,y],z])=[[\Omega(x),y],z]+[[x,\Omega(y)],z]+[[x,y],\Omega(z)]$ holds for all $x,y,z\in\mathcal{W}$. Furthermore, An additive map Ω on $\mathcal{W}$ is called as Jordan derivation if $\Omega(x\circ y)=\Omega(x)\circ y+x\circ\Omega(y)$ is always true for every $x,y\in\mathcal{W}$. Every derivation is usually both a Lie and Jordan derivation, and also every Lie derivation is a Lie triple derivation. However, the contrary arguments are not true. An essential expression $[[x,y],z]=x\circ(y\circ z)-y\circ(x\circ z)$ for every $x,y,z\in\mathcal{W}$ demonstrates that every Jordan derivation is also a Lie triple derivation. Hence, working on Lie triple derivations allows treating both the major classes, viz., Jordan and Lie derivations at the same time. Previously, numerous authors have contributed significantly to the related concerns (see [1, 2, 4, 5, 9, 10, 13, 14, 17, 18]) for the Lie triple derivation. In fact, usually the researchers obtained that a Lie triple derivation on a unital algebra is proper, meaning it could be represented as the sum of additive derivation and linear functional that vanishes on all Lie triple products.

A bilinear map $\delta:\mathcal{W}\times\mathcal{W}\to\mathcal{W}$ is said to be a bi-derivation (resp. Lie bi-derivation), if it is a derivation (resp. Lie derivation) with respect to both components, that means

$$\delta(xy,z)=\delta(x,z)y+x\delta(y,z)\qquad\text{and}\qquad\delta(x,yz)=\delta(x,y)z+y\delta(x,z)$$

$$(\text{resp. }\delta([x,y],z)=[\delta(x,z),y]+[x,\delta(y,z)]\quad\text{and}\quad\delta(x,[y,z])=[\delta(x,y),z]+[y,\delta(x,z)])$$

for all $x,y,z\in\mathcal{W}$. If $\mathcal{W}$ is a noncommutative ring, then the map $\delta(x,y)=\lambda[x,y]$ for all $x,y\in\mathcal{W}$, where $\lambda\in\mathcal{Z}(\mathcal{W})$, is called an inner bi-derivation. In [7] authors studied the bi-derivation on quaternion rings. In continuation, we study the Lie bi-derivation on Quaternion rings in the present article.

Let $\mathcal{W}$ be a ring with identity. Consider

$$Q=\mathcal{H}(\mathcal{W})=\{s_0+s_1i+s_2j+s_3k\,|\,s_0,s_1,s_2,s_3\in\mathcal{W}\}=\mathcal{W}\oplus\mathcal{W}i\oplus\mathcal{W}j\oplus\mathcal{W}k,$$

where $i^2=j^2=k^2=ijk=-1$, and $ij=-ji$. Upon componentwise addition and multiplication with respect to the given condition, and the conventions that i,j,k commute with $\mathcal{W}$ elementwise, $\mathcal{H}(\mathcal{W})$ is a ring called the quaternion ring over $\mathcal{W}$.

Generally $\mathcal{H}(\mathcal{W})$ is noncommutative except for $\mathcal{W}$ is commutative of $\text{char}(\mathcal{W})=2$ (see [6, Lemma 2.1]). It is significant to mention that the quaternion ring $Q=\mathcal{H}(\mathcal{W})$ turns out to be (isomorphic to) a 2×2 full matrix ring in some specific cases. Assuming that Q has elements with the properties $a^2=f^2=0$ and $af+fa=1$, it follows that Q is isomorphic to a 2×2 complete matrix ring according to [11, Theorem 17.10]. Assume, for instance, that $\mathcal{W}$ is an algebra over the field $\mathbb{Z}_{\mathrm{p}}$, where p is an odd prime integer. According to a familiar result in number theory, there exists $x,y\in\mathbb{Z}_{\mathrm{p}}$ such that $1+x^2+y^2=0$. Assume now that Q has $a=\frac{1}{2}(-yi+xj-kp), f=\frac{1}{2}(-yi+xj+k)$. A simple calculation establishes that $a^2=f^2=0$ and $af+fa=1$. Thus, according to the aforementioned theorem, Q is a 2×2 matrix ring. In these circumstances, any characterization of a map on Q has an analogous specification when such map is examined on the corresponding matrix ring and vice versa.

2. Lie bi-derivation

In this section, we start with some basic facts about Lie bi-derivations as follows.

Lemma 1. *Let $\delta: Q\times Q\to Q$ be a Lie bi-derivation on Q. Then δ satisfies*

$$[\delta(x,y),[u,v]]+[\delta(u,v),[x,y]]=[\delta(x,v),[u,y]]+[\delta(u,y),[x,v]] \textit{ for all } x,y,u,v\in Q.$$

Proof. For any $x,y,u,v\in Q$, we have

$$\begin{aligned}\delta([x,u],[y,v]) &= [\delta(x,[y,v]),u]+[x,\delta(u,[y,v])]\\ &= [[\delta(x,y),v],u]+[[y,\delta(x,v)],u]+[x,[\delta(u,y),v]]+[x,[y,\delta(u,v)]].\end{aligned}$$

On the other way,

$$\begin{aligned}\delta([x,u],[y,v]) &= [\delta([x,u],y),v]+[y,\delta([x,u],v)]\\ &= [[\delta(x,y),u],v]+[[x,\delta(u,y)],v]+[y,[\delta(x,v),u]]+[y,[x,\delta(u,v)]].\end{aligned}$$

On comparing above two expressions, it is easy to obtain that

$$[\delta(x,y),[u,v]]-[[x,y],\delta(u,v)]=[\delta(x,v),[u,y]]-[[x,v],\delta(u,y)].$$

for all $x,y,u,v\in Q$. □

Theorem 1. *Let $\mathcal{W}$ be a ring with identity in which 2 is invertible and let $Q=\mathcal{H}(\mathcal{W})$. Then every Lie bi-derivation of Q is central.*

Proof. Let $X=x_1+y_1i+z_1j+w_1k$ and $Y=x_2+y_2i+z_2j+w_2k$ be arbitrary elements of Q and let $\delta: Q\times Q\to Q$ be a Lie bi-derivation. Since δ is biadditive, we have

$$\begin{aligned}\delta(X,Y) = {} & \delta(x_1,x_2)+\delta(x_1,y_2i)+\delta(x_1,z_2j)+\delta(x_1,w_2k)\\ &+\delta(y_1i,x_2)+\delta(y_1i,y_2i)+\delta(y_1i,z_2j)+\delta(y_1i,w_2k)\\ &+\delta(z_1j,x_2)+\delta(z_1j,y_2i)+\delta(z_1j,z_2j)+\delta(z_1j,w_2k)\end{aligned}$$

$$+\delta(w_1k,x_2)+\delta(w_1k,y_2i)+\delta(w_1k,z_2j)+\delta(w_1k,w_2k).$$

Now we describe all parts of the above expansion. For any $x\in Q$, we have

$$\begin{aligned}\delta(0,x)&=\delta([x,x],x)\\&=[x,\delta(x,x)]+[\delta(x,x),x]=0.\end{aligned}$$

Similarly, we can have

$$\delta(0,i)=\delta(0,j)=\delta(0,k)=0=\delta(i,0)=\delta(j,0)=\delta(k,0)=\delta(x,0). \tag{2.1}$$

In the similar manner,

$$\delta(0,xi)=\delta(0,xj)=\delta(0,xk)=0=\delta(xi,0)=\delta(xj,0)=\delta(xk,0). \tag{2.2}$$

By Lemma 1, we know that

$$[\delta(x,y),[u,v]]+[\delta(u,v),[x,y]]=[\delta(x,v),[u,y]]+[\delta(u,y),[x,v]] \tag{2.3}$$

for all $x,y,u,v\in Q$.
Put $u=i=y$ & $v=j$ and suppose that $\delta(x,i)=a+bi+cj+dk$, we get

$$\begin{aligned}[\delta(x,i),[i,j]]+[\delta(i,j),[x,i]]&=[\delta(x,j),[i,i]]+[\delta(i,i),[x,j]]\\ [\delta(x,i),2k]&=0\\ (a+bi+cj+dk)k-k(a+bi+cj+dk)&=0.\end{aligned}$$

Hence $\delta(x,i)=a+dk$. Again, put $y=i,u=j$ & $v=k$ in (2.3), we get

$$\begin{aligned}[\delta(x,i),[j,k]]+[\delta(j,k),[x,i]]&=0\\ [\delta(x,i),2i]&=0\\ (a+dk)i-i(a+dk)&=0.\end{aligned}$$

This implies that $\delta(x,i)=a$.
In similar manner, $\delta(x,j)=\delta(x,k)=a=\delta(i,x)=\delta(j,x)=\delta(k,x)$. Also, suppose that $\delta(xi,y)=u+vi+wj+zk$, then we get

$$\begin{aligned}\delta(0,x)&=\delta([xi,i],y)\\ 0&=[xi,\delta(i,y)]+[\delta(xi,y),i]\\&=[xi,a]+[u+vi+wj+zk,i]\\&=[x,a]i-2wk+2zj,\end{aligned}$$

which implies to $\delta(xi,y)=u+vi$ and $\delta(i,y)=a\in\mathcal{Z}(\mathcal{W})$. Note that

$$\begin{aligned}\delta(0,j)&=\delta([x,i],j)\\ 0&=[x,\delta(i,j)]+[\delta(x,j),i]\\&=[x,\delta(i,j)]\end{aligned}$$

it follows that $\delta(i,j)\in\mathcal{Z}(\mathcal{W})$. Also with $\delta(x,y)=u'+v'i+w'j+z'k$, we get

$$\delta(0,y)=\delta([i,x],y)$$

$$\begin{aligned}0 &= [i,\delta(x,y)]+[\delta(i,y),x]\\ &= [i,u'+v'i+w'j+z'k]\end{aligned}$$

and hence $\delta(x,y)=u'+v'i$. Further we obtain

$$\begin{aligned}\delta(0,y) &= \delta([j,x],y)\\ 0 &= [j,\delta(x,y)]+[\delta(j,y),x]\\ &= [j,u'+v'i].\end{aligned}$$

This implies $\delta(x,y)=u'$. From (2.3), we arrive at

$$\begin{aligned}[\delta(xi,y),[i,j]]+[\delta(i,j),[xi,y]] &= [\delta(xi,j),[i,y]]+[\delta(i,y),[xi,j]]\\ [\delta(xi,y),2k] &= 0\\ [u+vi,2k] &= 0\end{aligned}$$

and hence $\delta(xi,y)=u$. From (2.1), we have

$$\begin{aligned}\delta(0,y) &= \delta([x,xi],y)\\ 0 &= [x,\delta(xi,y)]+[\delta(x,y),xi]\\ &= [x,u]+[u',x]i\end{aligned}$$

it follows that $\delta(xi,y)=u\in\mathcal{Z}(\mathcal{W})$ and $\delta(x,y)\in\mathcal{Z}(\mathcal{W})$.
In the similar way we can find

$$\delta(x,yi),\delta(xj,y),\delta(x,yj),\delta(xk,y),\delta(x,yk)\in\mathcal{Z}(\mathcal{W}).$$

Also, from (2.2)

$$\begin{aligned}\delta(0,xi) &= \delta([y,i],xi)\\ 0 &= [y,\delta(i,xi)]+[\delta(y,xi),i]\\ &= [y,\delta(i,xi)]\end{aligned}$$

and hence $\delta(i,xi)\in\mathcal{Z}(\mathcal{W})$. Again, we see that

$$\begin{aligned}\delta(0,xi) &= \delta([y,j],xi)\\ 0 &= [y,\delta(j,xi)]+[\delta(y,xi),j]\\ &= [y,\delta(j,xi)]\end{aligned}$$

implies that $\delta(j,xi)\in\mathcal{Z}(\mathcal{W})$.
Assume that $\delta(xi,yi)=u+iv+wj+zk$, then we get

$$\begin{aligned}\delta(0,yi) &= \delta([i,xi],yi)\\ 0 &= [i,\delta(xi,yi)]+[\delta(i,yi),xi]\\ &= [i,\delta(xi,yi)]\end{aligned}$$

and it leads to $\delta(xi, yi) = u + iv$. Further, we have

$$\begin{aligned} \delta(0, yi) &= \delta([j, xi], yi) \\ 0 &= [j, \delta(xi, yi)] + [\delta(j, yi), xi] \\ &= [j, u + vi] \end{aligned}$$

implies that $\delta(xi, yi) = u$. Again, we have

$$\begin{aligned} \delta(0, yi) &= \delta([x, xi], yi) \\ 0 &= [x, \delta(xi, yi)] \\ &= [x, u], \end{aligned}$$

this implies $\delta(xi, yi) \in \mathcal{Z}(\mathcal{W})$.
Similarly, we have $\delta(xj, yj), \delta(xk, yk) \in \mathcal{Z}(\mathcal{W})$. With (2.2), we have

$$\begin{aligned} \delta(0, yj) &= \delta([x, xi], yj) \\ 0 &= [x, \delta(xi, yj)] + [\delta(x, yj), xi] \\ &= [x, u] \end{aligned}$$

and hence $\delta(xi, yj) \in \mathcal{Z}(\mathcal{W})$. Likewise

$$\delta(xi, yk), \delta(xj, yi), \delta(xj, yk), \delta(xk, yi), \delta(xk, yj) \in \mathcal{Z}(\mathcal{W}).$$

Thus, we see that $\delta(X, Y) \in \mathcal{Z}(\mathcal{W})$ and by [7, Lemma 2.4], we have $\delta(X, Y) \in \mathcal{Z}(\mathcal{Q})$. □

3. LIE TRIPLE DERIVATION

In this section, we present one of the main results of this article as follows:

Theorem 2. *Let $f : \mathcal{Q} \to \mathcal{Q}$ be a Lie triple derivation. Then there exists an element A in $\mathcal{Q}$, a Lie triple derivation α on $\mathcal{W}$ and a Jordan derivation β on $\mathcal{W}$ such that $f(t) = \alpha(x) + \beta(y)i + \beta(z)j + \beta(w)k + I_A(t)$ for every element $t = x + yi + zj + wk \in \mathcal{Q}$.*

Proof. Let f be a Lie triple derivation of $\mathcal{Q}$ and assume $f(i) = a + bi + cj + dk$ and $f(j) = x + yi + zj + wk$, we find

$$\begin{aligned} f(j) &= \frac{1}{4} f([[i, j], i]) \\ &= \frac{1}{4}[[f(i), j], i] + \frac{1}{4}[[i, f(j)], i] + \frac{1}{4}[[i, j], f(i)] \\ &= \frac{1}{2}[bk - di, i] + \frac{1}{2}[zk - wj, i] + \frac{1}{2}[k, a + bi + cj + dk] \\ &= bj + zj + wk + bj - ci \\ \Longrightarrow\ & x + yi + zj + wk = -ci + (2b + z)j + wk \end{aligned}$$

and hence $x=0, y=-c, 2b=0$, this leads to $f(i)=a+cj+dk$ and $f(j)=yi+zj+wk$. Again, we see that

$$\begin{aligned}
f(i) &= \frac{1}{4}f([[j,i],j]) \\
&= \frac{1}{4}[[f(j),i],j]+\frac{1}{4}[[j,f(i)],j]+\frac{1}{4}[[j,i],f(j)] \\
&= \frac{1}{4}[[yi+zj+wk,i],j]+\frac{1}{4}[[j,a+cj+dk],j]+\frac{1}{4}[[j,i],yi+zj+wk] \\
&= \frac{1}{2}[-zk+wj,j]+\frac{1}{2}[di,j]+\frac{1}{2}[yi+zj+wk,k] \\
&\implies a+cj+dk = 2zi-yj+dk,
\end{aligned}$$

this implies that $a=0, 2z=0, c=-y$ and hence $f(i)=cj+dk$ and $f(j)=-ci+wk$. In similar manner, we get $f(k)=-di-wj$. Let us suppose $f(1)=a+bi+cj+dk$

$$\begin{aligned}
0 &= f([[j,k],1]) \\
&= [[f(j),k],1]+[[j,f(k)],1]+[[j,k],f(1)] \\
&= [i,f(1)] \\
&= ai+bi^2+cij+dik-ai-bi^2-cji-dki \\
&= 2ck-2dj
\end{aligned}$$

$\implies d=0,\ c=0$ that's why $f(1)=a+bi$. Similarly with $[[i,k],1]=0$, we have $b=0$. Then $f(1)=a$.

Applying f on $[[sj,k],1]=0$, we find that

$$\begin{aligned}
0 &= f([[sj,k],1]) \\
&= [[f(sj),k],1]+[[sj,f(k)],1]+[[sj,k],f(1)] \\
&\implies f(1)\in \mathcal{Z}(\mathcal{W}).
\end{aligned}$$

Now let us take $f(si)=x+yi+zj+wk$ and using $f(i)=cj+dk, f(j)=-ci+bk$ and $f(k)=-di-bj$

$$\begin{aligned}
0 &= f([[si,i],j]) \\
&= [[f(si),i],j]+[[si,f(i)],j]+[[si,i],f(j)] \\
&= [[x+yi+zj+wk,i],j]+[[si,cj+dk],j] \\
&= [2wj-2zk,j]+[(s\circ c)k-(s\circ d)j,j] \\
&= 4zi-2(s\circ c)i.
\end{aligned}$$

Also, we have

$$\begin{aligned}
0 &= f([[si,i],k]) \\
&= [[f(si),i],k]+[[si,f(i)],k]+[[si,i],f(k)]
\end{aligned}$$

$$
\begin{aligned}
&= [[x+yi+zj+wk,i],k]+[[si,cj+dk],k]\\
&= [2wj-2zk,k]+[(s\circ c)k-(s\circ d)j,k]\\
&= 4wi-2(s\circ d)i.
\end{aligned}
$$

Thus, we conclude that $z=\frac{1}{2}(s\circ c)$ and $w=\frac{1}{2}(s\circ d)$ and this leads to

$$f(si)=x+yi+\frac{1}{2}(s\circ c)j+\frac{1}{2}(s\circ d)k.$$

Now

$$
\begin{aligned}
f(si) &= \frac{1}{4}f([[k,si],k])\\
&= \frac{1}{4}[[f(k),si],k]+\frac{1}{4}[[k,f(si)],k]+\frac{1}{4}[[k,si],f(k)]\\
x+yi+\frac{1}{2}(s\circ c)j+\frac{1}{2}(s\circ d)k &= \frac{1}{4}[[-di-bj,si],k]+\frac{1}{4}[[k,x+yi\\
&\quad +\frac{1}{2}(s\circ c)j+\frac{1}{2}(s\circ d)k],k]+\frac{1}{4}[[k,si],-di-bj],
\end{aligned}
$$

this implies $x=\frac{1}{2}I_b(s)$ and by taking $y=\beta(s)$, we obtain

$$f(si)=\frac{1}{2}I_b(s)+\beta(s)i+\frac{1}{2}(s\circ c)j+\frac{1}{2}(s\circ d)k, \tag{3.1}$$

where $\beta:\mathcal{W}\to\mathcal{W}$ is an additive map uniquely determined by f. Similarly, we can find that

$$f(sj)=-\frac{1}{2}I_d(s)-\frac{1}{2}(s\circ c)i+\beta(s)j+\frac{1}{2}(s\circ b)k \tag{3.2}$$

$$f(sk)=\frac{1}{2}I_c(s)-\frac{1}{2}(s\circ d)i-\frac{1}{2}(s\circ b)j+\beta(s)k. \tag{3.3}$$

For $s\in\mathcal{W}$ and let us take $f(s)=x+yi+zj+wk$. Now applying f on $[[i,j],s]=0$, we find that $z=\frac{1}{2}I_d(s)$ and $w=-\frac{1}{2}I_c(s)$. Also with $[[k,i],s]$, we obtain that $y=-\frac{1}{2}I_a(s)$. Therefore,

$$f(s)=\alpha(s)-\frac{1}{2}I_a(s)i+\frac{1}{2}I_d(s)j-\frac{1}{2}I_c(s)k, \tag{3.4}$$

where $\alpha:\mathcal{W}\to\mathcal{W}$ is an additive map uniquely determined by f. Let $s_1,s_2\in\mathcal{W}$. Replacing s with $[[s_1,s_2],s_3]$ in (3.4), we infer that α is a Lie triple derivation of $\mathcal{W}$. Moreover, applying f to the identities

$$
\begin{aligned}
[[s_1j,k],s_2j] &= 2(s_1\circ s_2)k\\
[[s_1j,k],s_2i] &= 2[s_2,s_1]\\
[[s_1j,k],s_2] &= 2[s_1,s_2]i
\end{aligned}
$$

and using the foregoing calculations, we find, respectively, that

$$\begin{aligned}\beta(s_1\circ s_2) &= \beta(s_1)\circ s_2 + s_1\circ\beta(s_2)\\ \alpha([s_1,s_2]) &= [\beta(s_1),s_2]+[s_1,\beta(s_2)]\\ \beta([s_1,s_2]) &= [\alpha(s_1),s_2]+[s_1,\beta(s_2)].\end{aligned}$$

Now let $t = x+yi+zj+wk \in \mathcal{Q}$ be an arbitrary element. Using (3.1), (3.2), (3.3) and (3.4), we find that

$$f(t) = \alpha(x)+\beta(y)i+\beta(z)j+\beta(w)k+h(t),$$

where

$$\begin{aligned}h(t) = &\frac{1}{2}(I_b(y)-I_d(z)+I_c(w))+\frac{i}{2}(-I_b(x)-z\circ c-w\circ d)\\ &+\frac{j}{2}(I_d(x)+y\circ c-w\circ b)+\frac{k}{2}(-I_c(x)+y\circ d+z\circ b).\end{aligned}$$

It is easily seen $h(t) = I_A(t)$, where

$$A = \frac{1}{2}(-bi+dj-ck) = \frac{1}{4}(f(i)i+f(j)j+f(k)k).$$

Consequently,

$$f(t) = \alpha(x)+\beta(y)i+\beta(z)j+\beta(w)k+I_A(t).$$

This proves the required result. □

Here is an example describing how to format some very long formula:

Example 1. Let $\mathcal{W} = \mathbb{Z}_4$ and $\mathcal{Q} = H(\mathbb{Z}_4)$. It is simple to confirm that the mapping $f : \mathcal{Q} \to \mathcal{Q}$ provided by $f(t) = 2t$ is a Lie triple derivation. We assert that f does not have a decomposition as in the theorem. Suppose that $f(t) = \alpha(x)+\beta(y)i+\beta(z)j+\beta(w)k+I_A(t)$ for all $t = x+yi+zj+wk \in \mathcal{Q}$, where β is Jordan derivation of $\mathcal{W}$, α is Lie triple derivation of $\mathcal{W}$ and $A = a+bi+cj+dk$ is an element of $\mathcal{Q}$. Obviously,$\beta = 0$. Furthermore, because α is additive, it is obvious that there exists an element $r \in \mathcal{W}$ provided by $\alpha(x) = rx$ for every $x \in \mathcal{W}$. So as a conclusion, for any $t = x+yi+zj+wk \in \mathcal{Q}$, $2t = f(t) = rx+I_A(t)$. Assuming $t = i$, we get $2i = -2dj+2ck$, a contradiction.

As a direct consequence of the above theorem, we can conclude

Corollary 1. *Suppose that the ring $\mathcal{W}$ is commutative. Then Lie triple derivation f is in standard form.*

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