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ON NON-OSCILLATION FOR TWO DIMENSIONAL SYSTEMS OF NON-LINEAR ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. The paper studies the non-oscillatory properties of two-dimensional systems of non-linear differential equations

$$u' = g(t)|v|^{\frac{1}{\alpha}}\operatorname{sgn} v, \quad v' = -p(t)|u|^{\alpha}\operatorname{sgn} u,$$

where the functions $g\colon [0,+\infty[\to [0,+\infty[$, $p\colon [0,+\infty[\to \mathbb{R}$ are locally integrable and $\alpha > 0$. We are especially interested in the case of $\int^{+\infty} g(s)\,\mathrm{d}s < +\infty$.

In the paper, new non-oscillation criteria are established. Among others, they generalize well-known results for linear systems as well as second order linear and also half-linear differential equations. The criteria presented complement the results of Hartman-Wintner's type for the system in question.

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1. Introduction

On the half-line $\mathbb{R}_+ = [0,+\infty[$, we consider the two-dimensional system of non-linear ordinary differential equations

$$\begin{aligned} u' &= g(t)|v|^{\frac{1}{\alpha}}\operatorname{sgn} v, \\ v' &= -p(t)|u|^{\alpha}\operatorname{sgn} u, \end{aligned} \tag{1.1}$$

where $\alpha > 0$ and $p, g\colon \mathbb{R}_+ \to \mathbb{R}$ are locally Lebesgue integrable functions.

By a solution to system (1.1) on the interval $J \subseteq [0,+\infty[$ we understand a vector function (u,v), where functions $u,v\colon J \to \mathbb{R}$ are absolutely continuous on every compact interval contained in J and satisfy equalities (1.1) almost everywhere in J.

It was proved in [9] that all non-extendable solutions to system (1.1) are defined on the whole interval $[0,+\infty[$. Consequently, speaking about a solution to system (1.1), we assume, without loss of generality, that it is defined on $[0,+\infty[$.

Definition 1. A solution (u,v) of system (1.1) is called *non-trivial*, if

$$|u(t)| + |v(t)| \not\equiv 0 \quad \text{for } t \geq 0.$$

We say that a non-trivial solution (u,v) of system (1.1) is *non-oscillatory* if at least one of its components does not have any sequence of zeroes tending to infinity, and *oscillatory* otherwise.

In [9, Theorem 1.1], it is shown that a certain analogue of Sturm's theorem holds for system (1.1) if the function g is non-negative. Especially if system (1.1) has a non-oscillatory solution, then any other of its non-trivial solutions is also non-oscillatory. Therefore, it is natural to assume

$$g(t) \geq 0 \quad \text{for a. e. } t \geq 0 \tag{1.2}$$

throughout the paper.

On the other hand, if $g(t) \equiv 0$ on some neighborhood of $+\infty$, then all non-trivial solutions to system (1.1) are non-oscillatory. Consequently, we also suppose that the inequality

$$\operatorname{meas}\{\tau \geq t\colon\, g(\tau) > 0\} > 0 \quad \text{for } t \geq 0 \tag{1.3}$$

holds.

Definition 2. We say that system (1.1) is *non-oscillatory* if all its non-trivial solutions are non-oscillatory.

The oscillation and non-oscillation theory for ordinary differential equations is widely studied in the literature. The criteria presented below are close to those established in [1–4, 6–8, 10, 12]. Namely, many of them (see, e.g., the survey given in [1]) are known for the so-called "half-linear" equation

$$\left(r(t)|u'|^{q-1}\operatorname{sgn} u'\right)' + p(t)|u|^{q-1}\operatorname{sgn} u = 0, \tag{1.4}$$

where $q > 1$, $p, r\colon\, [0,+\infty[\to \mathbb{R}$ are continuous and r is positive. We can see that (1.4) is a particular case of system (1.1). Indeed, if the function u, with the properties $u \in C^1$ and $r|u'|^{q-1}\operatorname{sgn} u' \in C^1$, is a solution to equation (1.4), then the vector function $(u, r|u'|^{q-1}\operatorname{sgn} u')$ is a solution to system (1.1) with $g(t) := r^{\frac{1}{1-q}}(t)$ for $t \geq 0$ and $\alpha := q-1$. In the case of $\int_0^{+\infty} g(s)\,\mathrm{d}s = +\infty$, some of the above-mentioned results are generalized in [11].

Throughout the paper, we assume that the function g is integrable on $[0,+\infty[$, i.e.

$$\int_0^{+\infty} g(s)\,\mathrm{d}s < +\infty. \tag{1.5}$$

In this case, the interesting results dealing with the oscillation of the system (1.1) are presented in [2]. Below formulated criteria complement these ones in certain sense.

On the other hand, as far as we know, not many non-oscillation criteria are known under the assumption (1.5). In particular, for the half-linear equation (1.4), one can find some non-oscillation criteria, e.g., in [1,5]. But there are some ”sign” restrictions on the coefficient p.

We introduce the following notations. Let

$$f(t) := \int_t^{+\infty} g(s)\,\mathrm{d}s \quad \text{for } t \geq 0.$$

In view of assumptions (1.2), (1.3) and (1.5), we have

$$\lim_{t\to+\infty} f(t) = 0$$

and

$$f(t) > 0 \quad \text{for } t \geq 0.$$

Further, for any $\lambda > \alpha$, we put

$$c_\alpha(t;\lambda) := (\lambda-\alpha) f^{\lambda-\alpha}(t) \int_0^t \frac{g(s)}{f^{\lambda-\alpha+1}(s)} \left(\int_0^s f^\lambda(\xi) p(\xi)\,\mathrm{d}\xi \right) \mathrm{d}s \quad \text{for } t \geq 0.$$

It is known that some analogy of the Hartmann–Wintner theorem (see [2, Corollary 2.11], where we put $\nu = 1 - \alpha + \lambda$) holds. In particular, if the function $c_\alpha(\cdot;\lambda)$ has no finite limit and $\liminf_{t\to+\infty} c_\alpha(t;\lambda) > -\infty$, then system (1.1) is oscillatory.

In this paper, we provide non-oscillatory criteria for the case where there exists a finite limit of the function $c_\alpha(\cdot;\lambda)$, i.e.,

$$\lim_{t\to+\infty} c_\alpha(t;\lambda) =: c_\alpha^*(\lambda) \in \mathbb{R}.$$

Under this assumption, we put for any $\lambda \in]\alpha,+\infty[$ and $\mu \in [0,\alpha[$

$$Q(t;\alpha,\lambda) := \frac{1}{f^{\lambda-\alpha}(t)} \left(c_\alpha^*(\lambda) - \int_0^t p(s) f^\lambda(s)\,\mathrm{d}s \right) \quad \text{for } t \geq 0,$$

$$H(t;\alpha,\mu) := f^{\alpha-\mu}(t) \int_0^t p(s) f^\mu(s)\,\mathrm{d}s \quad \text{for } t \geq 0.$$

Moreover, let us denote

$$\begin{aligned} Q_*(\alpha,\lambda) &:= \liminf_{t\to+\infty} Q(t;\alpha,\lambda), & H_*(\alpha,\mu) &:= \liminf_{t\to+\infty} H(t;\alpha,\mu), \\ Q^*(\alpha,\lambda) &:= \limsup_{t\to+\infty} Q(t;\alpha,\lambda), & H^*(\alpha,\mu) &:= \limsup_{t\to+\infty} H(t;\alpha,\mu). \end{aligned} \tag{1.6}$$

2. MAIN RESULTS

This section contains formulations of the main results of the paper. Firstly, we formulate the non-oscillation criteria for system (1.1) in terms of the lower and upper limits of the function $Q(\cdot,\alpha,\lambda)$.

For any $\kappa \in \mathbb{R}$, let us denote by $A(\kappa)$ and $B(\kappa)$ the smallest and the greatest roots of the equation

$$\alpha|x|^{\frac{1+\alpha}{\alpha}} + \lambda x + (\lambda-\alpha)\kappa = 0. \tag{2.1}$$

Let us note that, the equation (2.1) has exactly two real roots if $\kappa < \left(\frac{\lambda}{1+\alpha}\right)^{\alpha}\frac{1}{\lambda-\alpha}$. Moreover, $A(\kappa) \in \left]-\infty, \left(\frac{\mu}{1+\alpha}\right)^{\alpha}\right[$, i.e. the smallest one is always negative (see Figure 1(a), where $\alpha = 2$, $\lambda = 3$, $\kappa = \frac{1}{2}$).

Theorem 1. *Let* $\lambda \in]\alpha, +\infty[$,

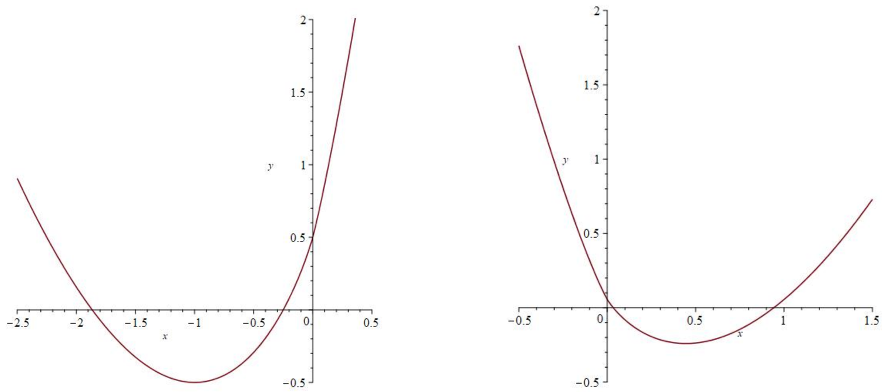
$$A(\kappa) + \kappa < Q_*(\alpha,\lambda) \quad \text{and} \quad Q^*(\alpha,\lambda) < \frac{1}{\lambda-\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} \tag{2.2}$$

be fulfilled, where $\kappa = \frac{\alpha(\lambda-\alpha)+\lambda}{(\lambda-\alpha)(1+\alpha)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$. *Then, system* (1.1) *is non-oscillatory.*

Before we formulate the following statement, we denote by $\widetilde{B}(\eta)$ the greatest root of the equation

$$\alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + \eta = 0. \tag{2.3}$$

Let us note that, the equation (2.3) has exactly two real roots if $\eta < \left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha}$, moreover, $\widetilde{B}(\eta) \in \left]\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}, \infty\right[$, i.e. the greatest one is always positive (see Figure 1(b), where $\alpha = 2$, $\eta = \frac{1}{18}$).



(a) $f(x) = 2|x|^{\frac{3}{2}} + 3x + \frac{1}{2}$ (b) $g(x) = 2|x|^{\frac{3}{2}} - 2x + \frac{1}{18}$

FIGURE 1.

Theorem 2. *Let* $\lambda \in]\alpha, +\infty[$,

$$-\infty < Q_*(\alpha,\lambda) \le A(\kappa) + \kappa \tag{2.4}$$

and

$$Q^*(\alpha,\lambda) < Q_*(\alpha,\lambda) + \widetilde{B}(\eta) + B\left(Q_*(\alpha,\lambda) + \widetilde{B}(\eta)\right) \tag{2.5}$$

be fulfilled, where $\kappa = \frac{\alpha(\lambda-\alpha)+\lambda}{(\lambda-\alpha)(1+\alpha)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$ *and* $\eta = (\lambda-\alpha)Q_*(\alpha,\lambda)$. *Then, system* (1.1) *is non-oscillatory.*

Remark 1. Theorem 2 complements Theorem 1 in certain sense. Indeed, if the first inequality in (2.2) is not satisfied and $Q_*(\alpha,\lambda)$ is finite, then condition (2.4) holds. In such a case, it is sufficient to verify condition (2.5) and the system (1.1) is non-oscillatory according Theorem 2 (see Example 1) .

In the following theorems we established the non-oscillation criteria in terms of the lower and upper limits of the function $H(\cdot,\alpha,\mu)$. Now, we denote by $\bar{A}(\nu)$ and $\bar{B}(\nu)$ the smallest and greatest roots of the equation

$$\alpha|x|^{\frac{1+\alpha}{\alpha}}+\mu x+(\alpha-\mu)\nu=0. \tag{2.6}$$

Let us note that, the equation (2.6) has two real roots if $\nu<\frac{1}{\alpha-\mu}\left(\frac{\mu}{1+\alpha}\right)^{1+\alpha}$, moreover, $\bar{A}(\nu)\in\left]-\infty,\left(\frac{\mu}{1+\alpha}\right)^{\alpha}\right]$, i.e. the smallest one is always negative.

Theorem 3. *Let $\mu\in[0,\alpha[$,*

$$-\frac{\alpha(2\alpha+1)}{(1+\alpha)(\alpha-\mu)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}<H_*(\alpha,\mu)\quad\text{and}\quad H^*(\alpha,\mu)<\nu-\bar{A}(\nu) \tag{2.7}$$

be fulfilled with $\nu=-\frac{\alpha(\alpha+\mu)+\mu}{(\alpha-\mu)(1+\alpha)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$. Then, system (1.1) is non-oscillatory.

Finally, we formulate the statement which completes the previous one in the same sense, as it is mentioned in Remark 1 for Theorem 1 and Theorem 2.

Theorem 4. *Let $\mu\in[0,\alpha[$,*

$$-\infty<H_*(\alpha,\mu)\le-\frac{\alpha(2\alpha+1)}{(1+\alpha)(\alpha-\mu)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} \tag{2.8}$$

and

$$H^*(\alpha,\mu)<\delta-\bar{A}(\delta) \tag{2.9}$$

be fulfilled, where $\delta=\left(\widehat{B}((\alpha-\mu)H_*(\alpha,\mu))\right)^{\frac{\alpha}{1+\alpha}}+H_*(\alpha,\mu)$ and $\widehat{B}(\xi)$ is the greatest root of the equation

$$\alpha|x|^{\frac{\alpha}{1+\alpha}}+\alpha x+\xi=0,\quad\text{for }\ \xi\le 0.$$

Then, system (1.1) is non-oscillatory.

Let us note, that the $\widehat{B}(\xi)\in\left]-\left(\frac{\alpha}{1+\alpha}\right)^{\alpha},+\infty\right[$ and $\widehat{B}(\xi)>0$ for $\xi<0$.

Remark 2. In [1, Section 3.1], there are functions Q, H and also non-oscillatory criteria defined for the equation (1.4) with the particular parameters $\lambda=\alpha+1$ and $\mu=0$. However, in this paper, they are formulated in a more general way, where $\lambda\in]\alpha,+\infty[$ and $\mu\in[0,\lambda[$. One can see (e.g. Example 2) that it is meaningful, since we can decide on non-oscillation in more cases of the system in question (1.1).

Example 1. Let $\alpha = 2$, $\lambda = 3$,

$$g(t) = \frac{1}{(1+t)^2},$$

and

$$p(t) = \left(\frac{89}{432} + \frac{5\sqrt{105}}{144}\right)\left(\sin\left(\ln(1+t)\right) + \cos\left(\ln(1+t)\right) - \frac{15\sqrt{105}-19}{89+15\sqrt{105}}\right) \cdot (1+t)$$

for $t \geq 0$.

One can verify that

$$f(t) = \int_t^{+\infty} g(s)\,\mathrm{d}s = \frac{1}{1+t} \quad \text{for } t \geq 0$$

and

$$\begin{aligned} c_2(t,3) &= \frac{1}{1+t}\int_0^t \left(\int_0^s p(\xi)\,\mathrm{d}\xi\right) ds \\ &= \frac{(-15\sqrt{105}-69)\sin(\ln(1+t)) + (-19+15\sqrt{105})\ln(t+1) + 108t}{432(t+1)} \end{aligned}$$

for $t > 0$. Hence, we get

$$c_2^*(3) = \lim_{t\to+\infty} c_2(t,3) = \frac{1}{4}.$$

Moreover,

$$\begin{aligned} Q(t;2,3) &= (t+1)\left(\frac{1}{4} - \int_0^t \frac{p(s)}{(1+s)^3} ds\right) \\ &= \frac{(89+15\sqrt{105})\left(\cos\left(\frac{\ln(1+t)}{2}\right)\right)^2 - 15\sqrt{105} - 35}{216} \qquad \text{for } t \geq 0, \end{aligned}$$

therefore,

$$Q_*(2,3) = \liminf_{t\to+\infty} Q(t;2,3) = -\frac{15\sqrt{105}+35}{216}$$

and

$$Q^*(2,3) = \limsup_{t\to+\infty} Q(t;2,3) = \frac{1}{4}.$$

On the other hand, for $\alpha = 2$, $\lambda = 3$, we have $\kappa = \frac{20}{27}$ and the equation (2.1) is of the form

$$2|x|^{\frac{3}{2}} + 3x + \frac{20}{27} = 0.$$

It is not difficult to verify that

$$A\left(\frac{20}{27}\right)+\frac{20}{27}=-\frac{(5+\sqrt{105})^2}{144}+\frac{20}{27}=-\frac{15\sqrt{105}+35}{216}=Q_*(2,3).$$

Consequently, we cannot apply Theorem 1, since the first inequality in (2.2) is not satisfied. However, one can show that Theorem 2 guarantees non-oscillation of the system in question. Indeed, for $\lambda=3$ and $\alpha=2$, we have $\eta=Q_*(2,3)$ and equation (2.3) is of the form

$$2|x|^{\frac{3}{2}}-2x-\frac{15\sqrt{105}+35}{216}=0.$$

One can verify that

$$\widetilde{B}\left(-\frac{15\sqrt{105}+35}{216}\right)=\frac{65+5\sqrt{105}}{72}$$

and

$$B\left(Q_*(\alpha,\lambda)+\widetilde{B}(\eta)\right)=-\frac{4}{9}.$$

Hence,

$$Q^*(2,3)=\frac{1}{4}<\frac{8}{27}=Q_*(2,3)+\widetilde{B}\left(-\frac{15\sqrt{105}+35}{216}\right)+B\left(Q_*(\alpha,\lambda)+\widetilde{B}(\eta)\right).$$

Consequently, according to Theorem 2, system (1.1) is non-oscillatory.

Example 2. Let $\alpha=2$,

$$g(t)=\frac{1}{(1+t)^2},\quad\text{and}\quad p(t)=\left(\frac{2\cos t}{3}-\frac{7}{9(1+t)}\right)(1+t)^2\quad\text{for } t\ge 0.$$

If we put $\mu=1$, then one can calculate that

$$f(t)=\int_t^{+\infty}g(s)\,\mathrm{d}s=\frac{1}{1+t}\quad\text{for } t\ge 0$$

and

$$\begin{aligned}H(t;2,1)=f(t)\int_0^t p(s)f(s)\,\mathrm{d}s&=\frac{1}{1+t}\int_0^t\left(\frac{2(1+s)\cos s}{3}-\frac{7}{9}\right)\mathrm{d}s\\&=\frac{6(1+t)\sin t+6\cos t-7t-6}{9(1+t)}\quad\text{for } t\ge 0.\end{aligned}$$

Hence,

$$H_*(2,1)=\liminf_{t\to+\infty}H(t;2,1)=-\frac{13}{9},$$

and

$$H^*(2,1)=\limsup_{t\to+\infty}H(t;2,1)=-\frac{1}{9}.$$

For $\alpha = 2$, $\mu = 1$, we have $\nu = -\frac{8}{27}$ and the equation (2.6) is of the form

$$2|x|^{\frac{3}{2}} + x - \frac{8}{27} = 0.$$

One can verify that

$$\bar{A}\left(-\frac{8}{27}\right) = -\left(\frac{(57+4\sqrt{203})^{\frac{1}{3}}+1}{6} + \frac{1}{6(57+4\sqrt{203})^{\frac{1}{3}}}\right)^2,$$

thus,

$$\nu - \bar{A}(\nu) = -\frac{8}{27} + \left(\frac{(57+4\sqrt{203})^{\frac{1}{3}}+1}{6} + \frac{1}{6(57+4\sqrt{203})^{\frac{1}{3}}}\right)^2$$
$$\approx -0.019 > -\frac{1}{9} = H^*(2,1).$$

Clearly,

$$-\frac{\alpha(2\alpha+1)}{(1+\alpha)(\alpha-\mu)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} = -\frac{40}{27} < -\frac{13}{9} = H_*(2,1).$$

We see that both conditions in (2.7) are satisfied and, therefore, according to Theorem 3, system (1.1) is non-oscillatory.

On the other hand, if we put $\mu = 0$, then

$$-\frac{2\alpha+1}{1+\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} = -\frac{20}{27} > -\frac{19}{18} = H_*(2,0),$$

and now we cannot apply Theorem 3. Consequently, it is meaningful to consider our criteria with the "weight" f^{μ}.

3. Proofs of the main results

Firstly, we present an auxiliary lemma, which we use to prove the main theorems.

Lemma 1 ([11, Lemma 3.1])**.** *Let there exist a locally absolutely continuous function* $\sigma\colon [a,+\infty[\ \to \mathbb{R}$ *satisfying the inequality*

$$\sigma'(t) \le -p(t) - \alpha g(t)|\sigma(t)|^{\frac{1+\alpha}{\alpha}} \quad \textit{for a. e. } t \ge a, \tag{3.1}$$

where $a \ge 0$*. Then, system* (1.1) *is non-oscillatory.*

It is not difficult to verify the next lemma by a direct calculation.

Lemma 2. *Let*

$$y(x) := \alpha|x|^{\frac{1+\alpha}{\alpha}} + \beta x + \gamma,$$

where α, $\beta > 0$ *and* $\gamma \in \mathbb{R}$*. Then,*

$$y'(x) < 0 \ \text{ for }]-\infty, x_1[\,, \quad y'(x) > 0 \ \text{ for }]x_1, \infty[\,, \tag{3.2}$$

where $x_1 = -\left(\frac{\beta}{1+\alpha}\right)^{\alpha}$, *and*

$$\lim_{x\to-\infty} y(x) = +\infty, \qquad \lim_{x\to+\infty} y(x) = +\infty.$$

Proof of Theorem 1. In view of (1.6) and (2.2), there exists $t_0 > 0$ such that

$$A(\kappa)+\kappa < Q(t;\alpha,\lambda) < \frac{1}{\lambda-\alpha}\left(\frac{\alpha}{1+\alpha}\right)^{1+\alpha} \quad \text{for } t \geq t_0.$$

Hence,

$$A(\kappa) < Q(t;\alpha,\lambda)-\kappa < -\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} \quad \text{for } t \geq t_0. \tag{3.3}$$

One can show that $x_2 = -\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$ is the root of the equation (2.1). Moreover, by virtue of the hypothesis $\lambda > \alpha$ and Lemma 2 (with $\beta = \lambda$ and $\gamma = \kappa(\lambda-\alpha)$), we get

$$A(\kappa) < x_1 < x_2, \quad \text{and} \quad \alpha|x|^{\frac{1+\alpha}{\alpha}} + \lambda x + (\lambda-\alpha)\kappa < 0 \quad \text{for } x \in \,]A(\kappa), x_2[, \tag{3.4}$$

where $x_1 = -\left(\frac{\lambda}{1+\alpha}\right)^{\alpha}$. The latter inequalities, together with (3.3), yield

$$\alpha|Q(t;\alpha,\lambda)-\kappa|^{\frac{1+\alpha}{\alpha}} + \lambda(Q(t;\alpha,\lambda)-\kappa) + (\lambda-\alpha)\kappa \leq 0 \quad \text{for } t \geq t_0. \tag{3.5}$$

Let us introduce the function σ as follows

$$\sigma(t) := \frac{1}{f^{\alpha}(t)}\left(Q(t;\alpha,\lambda)-\kappa\right) \quad \text{for a. e. } t \geq t_0. \tag{3.6}$$

It is clear that

$$\sigma'(t) = \frac{g(t)}{f^{1+\alpha}(t)}\left(\lambda(Q(t;\alpha,\lambda)-\kappa)+(\lambda-\alpha)\kappa\right) - p(t) \quad \text{for } t \geq t_0.$$

The latter equality, together with (3.5), implies

$$\sigma'(t) \leq \frac{g(t)}{f^{1+\alpha}(t)}\left(-\alpha|Q(t;\alpha,\lambda)-\kappa|^{\frac{1+\alpha}{\alpha}}\right) - p(t) \quad \text{for a. e. } t \geq 0.$$

Hence, in view of (3.6), we get that inequality (3.1) is satisfied with $a = t_0$. Consequently, according to Lemma 1, system (1.1) is non-oscillatory. $\square$

Proof of Theorem 2. By virtue of (1.6), (2.4) and (2.5), there exist $\varepsilon > 0$ and $t_\varepsilon > 0$ such that

$$Q_*(\alpha,\lambda)-\varepsilon < Q(t;\alpha,\lambda) < Q^*(\alpha,\lambda)+\varepsilon \tag{3.7}$$

and

$$Q^*(\alpha,\lambda)+\varepsilon < Q_*(\alpha,\lambda)-\varepsilon+\widetilde{B}(\eta_\varepsilon)+B\left(Q_*(\alpha,\lambda)-\varepsilon+\widetilde{B}(\eta_\varepsilon)\right) \quad \text{for } t \geq t_\varepsilon \tag{3.8}$$

hold with $\eta_\varepsilon = (\lambda-\alpha)(Q_*(\alpha,\lambda)-\varepsilon)$.

An analysis similar to that in the proof of Theorem 1 shows that (3.4) holds, where $x_1 = -\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$ and $x_2 = -\left(\frac{\lambda}{1+\alpha}\right)^{\alpha}$. Therefore,

$$A(\kappa) < -\left(\frac{\lambda}{1+\alpha}\right)^{\alpha} \quad \text{and} \quad \alpha|x_1|^{\frac{1+\alpha}{\alpha}} + \lambda x_1 + (\lambda-\alpha)\kappa < 0.$$

The latter inequality guarantees that

$$\kappa < \frac{1}{\lambda-\alpha}\left(\frac{\lambda}{1+\alpha}\right)^{1+\alpha}.$$

Hence, in view of (2.4), we obtain

$$Q_*(\alpha,\lambda) \le A(\kappa)+\kappa < \left(\frac{\lambda}{1+\alpha}\right)^{\alpha} \frac{\alpha-\alpha\lambda+\alpha^2}{(1+\alpha)(\lambda-\alpha)}$$

and, consequently,

$$Q_*(\alpha,\lambda)(\lambda-\alpha) < \left(\frac{\lambda}{1+\alpha}\right)^{\alpha} \frac{\alpha-\alpha\lambda+\alpha^2}{1+\alpha}. \tag{3.9}$$

On the other hand, the function $z\colon x \mapsto \alpha|x|^{\frac{1+\alpha}{\alpha}} - \alpha x + \eta_\varepsilon$ is decreasing on $\left]-\infty, \left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\right[$, and increasing on $\left]\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}, \infty\right[$. Moreover, by virtue of (3.9), we get

$$z\left(\left(\frac{\lambda}{1+\alpha}\right)^{\alpha}\right) < 0.$$

Hence, $\widetilde{B}(\eta_\varepsilon) > \left(\frac{\lambda}{1+\alpha}\right)^{\alpha}$ and, consequently,

$$-\widetilde{B}(\eta_\varepsilon) < -\left(\frac{\lambda}{1+\alpha}\right)^{\alpha}. \tag{3.10}$$

If we put

$$\kappa_\varepsilon = Q_*(\alpha,\lambda) - \varepsilon + \widetilde{B}(\eta_\varepsilon), \tag{3.11}$$

then it is not difficult to verify that $-\widetilde{B}(\eta_\varepsilon)$ is the root of equation (2.1) with $\kappa = \kappa_\varepsilon$. Moreover, (3.10) and Lemma 2 (with $\beta = \lambda$ and $\gamma = (\lambda-\alpha)\kappa_\varepsilon$) imply, that $-B(\eta_\varepsilon) = A(\kappa_\varepsilon)$ and

$$\alpha|x|^{\frac{1+\alpha}{\alpha}} + \lambda x + (\lambda-\alpha)\kappa_\varepsilon < 0 \quad \text{for } x \in \left]A(\kappa_\varepsilon), B(\kappa_\varepsilon)\right[. \tag{3.12}$$

In view of (3.7), (3.8) and (3.11), we get

$$A(\kappa_\varepsilon) = -\widetilde{B}(\eta_\varepsilon) \le Q(t;\alpha,\lambda) - \kappa_\varepsilon \le B(\kappa_\varepsilon) \quad \text{for } t \ge t_\varepsilon.$$

The latter inequalities and (3.12) yield (3.5) with $\kappa = \kappa_\varepsilon$ and $t_0 = t_\varepsilon$.

Now, let the function σ be defined by formula (3.6) with $\kappa = \kappa_\varepsilon$ and $t_0 = t_\varepsilon$. Analogously, as in the proof of Theorem 1, one can verify that inequality (3.1) with $a = t_\varepsilon$ holds and, consequently, according to Lemma 1, system (1.1) is non-oscillatory.

□

Proof of Theorem 3. In view of (1.6) and (2.7), there exist $t_0 > 0$ such that

$$-\frac{\alpha(2\alpha+1)}{(1+\alpha)(\alpha-\mu)}\left(\frac{\alpha}{1+\alpha}\right)^{\alpha} < H(t;\alpha,\mu) < \nu - \bar{A}(\nu) \quad \text{for } t \geq t_0. \tag{3.13}$$

According to Lemma 2 (with $\beta = \mu$ and $\gamma = (\alpha-\mu)\nu$), one can see that function

$$y(x) := \alpha|x|^{\frac{1+\alpha}{\alpha}} + \mu x + (\alpha-\mu)\nu \quad \text{for } x \in \mathbb{R} \tag{3.14}$$

satisfies relations (3.2) with $x_1 = -\left(\frac{\mu}{\alpha+1}\right)^{\alpha}$. Moreover, it is not difficult to verify, that $\left(\frac{\alpha}{1+\alpha}\right)^{\alpha}$ is the greatest root of the equation (2.6). Hence, by virtue of (3.2), we have

$$y(x) < 0 \quad \text{for } x \in \left]\bar{A}(\nu), \left(\frac{\alpha}{1+\alpha}\right)^{\alpha}\right[. \tag{3.15}$$

On the other hand, from (3.13), we obtain

$$\bar{A}(\nu) < \nu - H(t;\alpha,\mu) < \left(\frac{\alpha}{1+\alpha}\right)^{\alpha} = \bar{B}(\nu) \quad \text{for } t \geq t_0.$$

The latter inequalities, together with (3.14) and (3.15), yield

$$\alpha|\nu - H(t;\alpha,\mu)|^{\frac{1+\alpha}{\alpha}} + \mu(\nu - H(t;\alpha,\mu)) + (\alpha-\mu)\nu \leq 0 \quad \text{for } t \geq t_0. \tag{3.16}$$

Now, we put

$$\sigma(t) := \frac{1}{f^{\alpha}(t)}(\nu - H(t;\alpha,\mu)) \quad \text{for } t \geq t_0. \tag{3.17}$$

One can show that

$$\sigma'(t) = \frac{g(t)}{f^{1+\alpha}(t)}(\mu(\nu - H(t;\alpha,\mu)) + (\alpha-\mu)\nu) - p(t) \quad \text{for a. e. } t \geq t_0.$$

Hence, in view of (3.16), we obtain

$$\sigma'(t) \leq \frac{g(t)}{f^{1+\alpha}(t)}\left(-\alpha|\nu - H(t;\alpha,\mu)|^{\frac{1+\alpha}{\alpha}}\right) - p(t) \quad \text{for a. e. } t \geq t_0.$$

Consequently, by virtue of (3.17), we get that (3.1) holds with $a = t_0$ and, according to Lemma 1, system (1.1) is non-oscillatory. □

Proof of Theorem 4. In view of (1.6) and (2.9), there exist $\varepsilon > 0$ and $t_\varepsilon > 0$ such that

$$H_*(\alpha,\mu) - \varepsilon < H(t;\alpha,\mu) < H^*(\alpha,\mu) + \varepsilon \quad \text{for } t \geq t_\varepsilon \tag{3.18}$$

and

$$H^*(\alpha,\mu) + \varepsilon < \delta_\varepsilon - \bar{A}(\delta_\varepsilon) \quad \text{for } t \geq t_\varepsilon \tag{3.19}$$

hold, where

$$\delta_\varepsilon = \left(\widehat{B}((\alpha-\mu)(H_*(\alpha,\mu)-\varepsilon))\right)^{\frac{\alpha}{1+\alpha}} + H_*(\alpha,\mu) - \varepsilon. \tag{3.20}$$

From (2.8), we get

$$\widehat{B}((\alpha-\mu)(H_*(\alpha,\mu)-\varepsilon)) > 0.$$

Moreover, in view of the latter inequality, one can show that

$$\left(\widehat{B}((\alpha-\mu)(H_*(\alpha,\mu)-\varepsilon))\right)^{\frac{\alpha}{1+\alpha}}$$

is the greatest root of the equation (2.6) with $\nu = \delta_\varepsilon$, i.e.,

$$\bar{B}(\delta_\varepsilon) = \left(\widehat{B}((\alpha-\mu)(H_*(\alpha,\mu)-\varepsilon))\right)^{\frac{\alpha}{1+\alpha}}.$$

Consequently, from (3.18), (3.19) and (3.20), we get

$$\bar{A}(\delta_\varepsilon) < \delta_\varepsilon - H(t;\alpha,\mu) < \bar{B}(\delta_\varepsilon). \tag{3.21}$$

On the other hand, an analysis similar to that in the proof of Theorem 3 shows that function

$$y_\varepsilon(x) := \alpha|x|^{\frac{1+\alpha}{\alpha}} + \mu x + (\alpha-\mu)\delta_\varepsilon$$

satisfies relations

$$y(x) < 0 \quad \text{for } x \in \left]\bar{A}(\delta_\varepsilon), \bar{B}(\delta_\varepsilon)\right[.$$

The latter inequality together with (3.21) yield (3.16) with $\nu = \delta_\varepsilon$ and $t_0 = t_\varepsilon$. Analogously, as in the proof of Theorem 3, one can verify that function

$$\sigma(t) := \frac{1}{f^\alpha(t)}\left(\delta_\varepsilon - H(t;\alpha,\mu)\right) \quad \text{for } t \geq t_\varepsilon$$

satisfies inequality (3.1) with $a = t_\varepsilon$ and, consequently, according to Lemma 1, system (1.1) is non-oscillatory. □

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