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MULTIPLICITY OF POSITIVE SOLUTIONS FOR A THIRD-ORDER BOUNDARY VALUE PROBLEM WITH NONLOCAL CONDITIONS OF INTEGRAL TYPE

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Abstract. We prove the existence of multiple positive solutions for a nonlinear third-order nonlocal boundary value problem by applying Krasnosel'skii's fixed point theorem. To illustrate the applicability of the obtained results, we consider an example.

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1. INTRODUCTION

We study boundary value problem consisting of the nonlinear third-order differential equation

$$x''' + f(t,x) = 0, \quad t \in (0,1), \tag{1.1}$$

and the boundary conditions

$$x(0) = 0, \quad x'(0) = 0, \quad x(1) = \int_0^1 x(\xi)\,d\xi. \tag{1.2}$$

We assume that $f\colon [0,1]\times[0,\infty)\to[0,\infty)$ is continuous.

The purpose of the paper is to give results on the existence of multiple positive solutions to (1.1), (1.2) by applying Krasnosel'skii's fixed point theorem. By a positive solution of (1.1), (1.2) we understand $C^3[0,1]$ function which is positive on $0 < t \le 1$ and satisfies differential equation (1.1) for $0 < t < 1$ and boundary conditions (1.2). However, note that if $f(t,0) = 0$, then boundary value problem (1.1), (1.2) always has the trivial solution. Since $f(t,x)$ is not defined for $x < 0$, every solution of (1.1), (1.2) is nonnegative. In what follows, we will show that every nonnegative nontrivial solution of (1.1), (1.2) is positive.

Actually, our main result states that for each given positive integer n, we can indicate f so that problem (1.1), (1.2) has at least n positive solutions. The approach that we will use to obtain this result is the often used one. First, we rewrite problem (1.1), (1.2) as an equivalent integral equation by constructing the corresponding Green's function. Then, we define an operator in the suitable cone of nonnegative continuous functions, and hence the problem reduces to find fixed points of the operator. Finally, we prove the existence of multiple fixed points in the cone using Krasnosel'skii's cone compression and expansion theorem of norm type [7, 8].

Krasnosel'skii's theorem is a very convenient and effective tool for studying the existence and multiplicity of positive solutions to boundary value problems, and therefore a lot of authors employ this technique in their research. J.R. Graef and B. Yang consider third-order three-point boundary value problem and give sufficient conditions for the existence of multiple positive solutions to this problem in [3]. The same authors, in [4], obtain existence and nonexistence results for positive solutions for a higher-order multi-point boundary value problem, and in [5], obtain sufficient conditions for the existence and nonexistence of positive solutions for a third-order two-point boundary value problem. J.R.L. Webb and G. Infante establish the existence of multiple positive solutions for a second-order nonlocal boundary value problem in [12]. The same authors, in [13], give a method of establishing the existence of multiple positive solutions for a large number of nonlinear differential equations of arbitrary order with any allowed number of nonlocal conditions. The problem we are studying here is a special case of a family of problems studied in [10], but in this paper, we use somewhat different conditions on the nonlinearity in the equation. We will explain this difference in Example 1 (see Section 4). A. Calamai and G. Infante study a parameter-dependent analogue of problem (1.1),(1.2) in the context of delay equations in [1].

Nonlocal boundary conditions make it possible to obtain more accurate models and consequently often appear in physics and various branches of applied mathematics. Much research has been done on nonlocal boundary value problems in the last decades. See, for example, the papers by J.R. Graef and J.R.L. Webb [2], J.R.L. Webb [9–11], J.R. Graef and B. Yang [6] and references therein.

Since our main tool in this paper is Krasnosel'skii's fixed point theorem, let us state this theorem for the reader's convenience.

Theorem 1 (Krasnosel'skii, [8])**.** *Let E be a Banach space and $K \subset E$ be a cone in E. Assume Ω_1 and Ω_2 are open subsets of E with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$, $T\colon K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right) \to K$ is a completely continuous operator such that*

(A) $\|Tx\| \le \|x\|$, $\forall x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \ge \|x\|$, $\forall x \in K \cap \partial\Omega_2$, *or*
(B) $\|Tx\| \ge \|x\|$, $\forall x \in K \cap \partial\Omega_1$ *and* $\|Tx\| \le \|x\|$, $\forall x \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap \left(\overline{\Omega}_2 \setminus \Omega_1\right)$.

The paper contains three sections besides the Introduction. In Section 2, we rewrite the main problem as an equivalent integral equation, by constructing the corresponding Green's function. Also, we give some inequalities for Green's function here. In Section 3, we prove our main theorem on the existence of multiple positive solutions for the problem. In conclusion, we consider an example to illustrate the applicability of our main result in Section 4.

2. Construction and estimation of the Green's function

Our first goal is to rewrite problem (1.1), (1.2) as an equivalent integral equation. So, let us consider the linear equation

$$x''' + h(t) = 0, \quad t \in (0,1), \tag{2.1}$$

together with boundary conditions (1.2).

Proposition 1. *Let $h\colon [0,1] \to \mathbb{R}$ be a continuous function. Then the function defined by*

$$x(t) = \int_0^1 G(t,s)h(s)\,ds$$

is the unique solution of boundary value problem (2.1),(1.2), where

$$G(t,s) = \frac{1}{4}\begin{cases} t^2(1-s)^2(s+2) - 2(t-s)^2, & 0 \le s \le t \le 1, \\ t^2(1-s)^2(s+2), & 0 \le t \le s \le 1. \end{cases} \tag{2.2}$$

Proof. Let $x(t)$ be a solution of problem (2.1),(1.2). Integrating equation (2.1) thrice, we get

$$x(t) = x(0) + t\,x'(0) + \frac{1}{2}t^2x''(0) - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,ds,$$

and, in view of boundary conditions (1.2), we obtain

$$x(t) = \frac{1}{2}t^2x''(0) - \frac{1}{2}\int_0^t (t-s)^2 h(s)\,ds.$$

Since $x(1) = \int_0^1 x(\xi)\,d\xi$, it follows that

$$\frac{1}{2}x''(0) - \frac{1}{2}\int_0^1 (1-s)^2h(s)\,ds = \int_0^1 \left(\frac{1}{2}\xi^2x''(0) - \frac{1}{2}\int_0^\xi (\xi-s)^2h(s)\,ds\right)\,d\xi$$

and hence

$$x''(0) = \frac{1}{2}\int_0^1 (1-s)^2(s+2)h(s)\,ds.$$

Therefore,

$$\begin{aligned}
x(t) &= \frac{1}{4}t^2\int_0^1 (1-s)^2(s+2)h(s)\,ds - \frac{1}{2}\int_0^t (t-s)^2h(s)\,ds \\
&= \frac{1}{4}\int_0^t t^2(1-s)^2(s+2)h(s)\,ds + \frac{1}{4}\int_t^1 t^2(1-s)^2(s+2)h(s)\,ds \\
&\quad - \frac{1}{2}\int_0^t (t-s)^2h(s)\,ds \\
&= \frac{1}{4}\int_0^t \left(t^2(1-s)^2(s+2) - 2(t-s)^2\right)h(s)\,ds + \frac{1}{4}\int_t^1 t^2(1-s)^2(s+2)h(s)\,ds.
\end{aligned}$$

□

Hence boundary value problem (1.1), (1.2) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s)f(s,x(s))\,ds, \quad 0 \le t \le 1, \tag{2.3}$$

in the sense that x is a solution of (1.1), (1.2) iff it is a solution of (2.3). Here $G(t,s)$ denotes the Green's function for the problem $x''' = 0$, (1.2), and is explicitly given by (2.2).

Next, we prove some inequalities for the Green's function $G(t,s)$.

Proposition 2. *For all $(t,s) \in [0,1]\times[0,1]$, we have*

$$G(t,s) \ge 0. \tag{2.4}$$

If $(t,s) \in (0,1)\times(0,1)$, then

$$G(t,s) > 0.$$

Proof. For $0 \le t \le s \le 1$, it is obvious that $t^2(1-s)^2(s+2) \ge 0$.

If $0 \le s \le t \le 1$, we have

$$\begin{aligned}
&t^2(1-s)^2(s+2) - 2(t-s)^2 \ge t^2(1-s)^2(s+2) - (s+2)(t-s)^2 \\
&= (s+2)\left(t^2(1-s)^2 - (t-s)^2\right) = (s+2)\left(s^2(t-1)(t+1) - 2st(t-1)\right) \\
&= s(s+2)(t-1)(t(s-2)+s) = s(s+2)(1-t)(t(2-s)-s) \ge 0.
\end{aligned}$$

□

Proposition 3. *For all* $(t,s) \in [0,1] \times [0,1]$*, we have*

$$G(t,s) \le \frac{s(1-s)^2(s+2)}{2(3-s^2)}. \tag{2.5}$$

Proof. Let us find the maximum of $G(t,s)$ for each s with respect to t.
For $0 \le s \le t \le 1$, the maximum occurs at $t = \dfrac{2}{3-s^2}$ and is equal to $\dfrac{s(1-s)^2(s+2)}{2(3-s^2)}$.
If $0 \le t \le s \le 1$, the maximum occurs at $t = s$ and is equal to $\dfrac{s^2(1-s)^2(s+2)}{4}$.
Since for all $(t,s) \in [0,1] \times [0,1]$,

$$\frac{s(1-s)^2(s+2)}{2(3-s^2)} = \frac{s^2(1-s)^2(s+2)}{2s(3-s^2)} \ge \frac{s^2(1-s)^2(s+2)}{4},$$

we get the proof. □

Proposition 4. *For all* $(t,s) \in [1/2,1] \times [0,1]$*, we have*

$$G(t,s) \ge \frac{1}{4} \cdot \frac{s(1-s)^2(s+2)}{2(3-s^2)}. \tag{2.6}$$

Proof. For $\Lambda_1 = \{(t,s) : 1/2 \le t \le 1,\ 0 \le s \le 1,\ s \le t\}$, we have

$$\inf_{\Lambda_1} \frac{G(t,s)}{\dfrac{s(1-s)^2(s+2)}{2(3-s^2)}} = \inf_{\Lambda_1} \frac{(3-s^2)(4t-2s-t^2(3-s^2))}{2(1-s)^2(s+2)} = \frac{1}{3}.$$

If $\Lambda_2 = \{(t,s) : 1/2 \le t \le 1,\ 0 \le s \le 1,\ t \le s\}$, then

$$\inf_{\Lambda_2} \frac{G(t,s)}{\dfrac{s(1-s)^2(s+2)}{2(3-s^2)}} = \min_{\Lambda_2} \frac{t^2(3-s^2)}{2s} = \frac{1}{4}.$$

Therefore,

$$\frac{G(t,s)}{\dfrac{s(1-s)^2(s+2)}{2(3-s^2)}} \ge \frac{1}{4} \quad \text{for} \quad \frac{1}{2} \le t \le 1,\ 0 \le s \le 1.$$

□

Proposition 5. *Every nonnegative nontrivial solution* $x(t)$ *of* (1.1), (1.2) *is positive.*

Proof. Suppose that there exists $t_0 \in (0,1)$ such that $x(t_0) = 0$. Since boundary value problem (1.1), (1.2) is equivalent to integral equation (2.3) we get

$$x(t_0) = \int_0^1 G(t_0,s) f(s,x(s))\, ds = 0.$$

Since $G(t_0,s)f(s,x(s)) \geq 0$ for all $s \in [0,1]$ then

$$G(t_0,s)f(s,x(s)) = 0 \quad \text{for all } s \in [0,1].$$

Since $G(t_0,s) > 0$ for all $s \in (0,1)$ we get that $x'''(s) = -f(s,x(s)) = 0$ for all $s \in (0,1)$. Therefore $x(s)$ is a polynomial of degree at most two. Since $x(s)$ satisfies boundary conditions (1.2) it follows that $x(s) = 0$ for all $s \in [0,1]$. We get the contradiction. □

3. Existence of multiple positive solutions

In this section, we prove our main result on the existence of multiple positive solutions for boundary value problem (1.1), (1.2) by applying Krasnosel'skii's fixed point theorem.

For our constructions, consider the Banach space $C[0,1]$ with the norm

$$\|x\| = \max_{0 \leq t \leq 1} |x(t)|, \quad x \in C[0,1].$$

Define a cone K in $C[0,1]$ by

$$K = \left\{ x \in C[0,1] \,:\, x(t) \geq 0, \ \min_{\frac{1}{2} \leq t \leq 1} x(t) \geq \frac{1}{4}\|x\| \right\},$$

and an integral operator $T: K \to C[0,1]$ by

$$(Tx)(t) = \int_0^1 G(t,s)f(s,x(s))\,ds, \quad 0 \leq t \leq 1.$$

It is easy to see that boundary value problem (1.1), (1.2) has a solution x if and only if x is a fixed point of the operator T. Also, it is well known that $T: K \to C[0,1]$ is a completely continuous operator.

Proposition 6. $T(K) \subset K$.

Proof. From inequality (2.4), it follows that for $x \in K$, $(Tx)(t) \geq 0$ on $[0,1]$. Also, for $x \in K$, we have from (2.5) that

$$(Tx)(t) = \int_0^1 G(t,s)f(s,x(s))\,ds \leq \int_0^1 \frac{s(1-s)^2(s+2)}{2(3-s^2)} f(s,x(s))\,ds,$$

so that

$$\|Tx\| \leq \int_0^1 \frac{s(1-s)^2(s+2)}{2(3-s^2)} f(s,x(s))\,ds. \tag{3.1}$$

And next, if $x \in K$, we have by (2.6) and (3.1),

$$\min_{\frac{1}{2}\le t\le 1}(Tx)(t) = \min_{\frac{1}{2}\le t\le 1}\int_0^1 G(t,s)f(s,x(s))\,ds$$
$$\ge \frac{1}{4}\int_0^1 \frac{s(1-s)^2(s+2)}{2(3-s^2)} f(s,x(s))\,ds \ge \frac{1}{4}\|Tx\|.$$

□

We shall use the following notations:

$$I_1 = \left(\max_{0\le t\le 1}\int_0^1 G(t,s)\,ds\right)^{-1} < \left(\max_{0\le t\le 1}\int_{1/2}^1 G(t,s)\,ds\right)^{-1} = I_2.$$

The next two propositions will be used in the proof of our main result.

Proposition 7. *Suppose that there exists $r > 0$ such that $f(t,x) \le I_1 r$ for $(t,x) \in [0,1]\times[0,r]$. If $x \in K$ with $\|x\| = r$, then $\|Tx\| \le r$.*

Proof. If $x \in K$ with $\|x\| = r$, then for $t \in [0,1]$ we have

$$(Tx)(t) = \int_0^1 G(t,s)f(s,x(s))\,ds \le I_1 r\int_0^1 G(t,s)\,ds \le I_1 r \max_{0\le t\le 1}\int_0^1 G(t,s)\,ds = r,$$

or $\|Tx\| \le r$.

□

Proposition 8. *Suppose that there exists $r > 0$ such that $f(t,x) \ge I_2 r$ for $(t,x) \in [0,1]\times[r/4,r]$. If $x \in K$ with $\|x\| = r$, then $\|Tx\| \ge r$.*

Proof. If $x \in K$ with $\|x\| = r$, then for every $s \in [1/2,1]$ we have $\min_{\frac{1}{2}\le s\le 1} x(s) \ge \frac{1}{4}\|x\| = \frac{1}{4}r$ and $x(s) \in [r/4,r]$. Therefore,

$$\|Tx\| = \max_{0\le t\le 1}\int_0^1 G(t,s)f(s,x(s))\,ds$$
$$\ge \max_{0\le t\le 1}\int_{1/2}^1 G(t,s)f(s,x(s))\,ds \ge I_2 r \max_{0\le t\le 1}\int_{1/2}^1 G(t,s)\,ds = r.$$

□

Theorem 2. *Suppose that there exist* $2m$ *constants* $0 < r_1 < r_2 < \cdots < r_{2m-1} < r_{2m}$, *and let* $\alpha(n) = 2n - \frac{1-(-1)^n}{2}$, $\beta(n) = 2n - \frac{1+(-1)^n}{2}$, *where* $1 \le n \le m$. *If*

$$f(t,x) \le I_1 r_{\alpha(n)} \quad \text{for } (t,x) \in [0,1]\times[0,r_{\alpha(n)}]$$

and

$$f(t,x) \ge I_2 r_{\beta(n)} \quad \text{for } (t,x) \in [0,1]\times[r_{\beta(n)}/4, r_{\beta(n)}],$$

then boundary value problem (1.1),(1.2) *has at least* m *positive solutions* $x_n(t)$ *such that* $r_{2n-1} \le \|x_n\| \le r_{2n}$.

Proof. If $\Omega_k = \{x \in C[0,1] : \|x\| < r_k\}$, $1 \le k \le 2m$, then, from Proposition 7 and Proposition 8, we have

$$\|Tx\| \le \|x\| \quad \text{for } x \in K \cap \partial\Omega_{\alpha(n)},$$

and

$$\|Tx\| \ge \|x\| \quad \text{for } x \in K \cap \partial\Omega_{\beta(n)}.$$

From Theorem 1, we see that T has fixed point in each of the sets $K \cap \left(\overline{\Omega}_{2n} \setminus \Omega_{2n-1}\right)$. Thus, boundary value problem (1.1), (1.2) has at least m positive solutions. □

4. Example

Example 1. Consider boundary value problem (1.1),(1.2) with

$$f(t,x) = \begin{cases} 28x^2, & 0 \le x \le 1, \\ 622(x-1)^2 + 28, & 1 \le x \le 2, \\ \dfrac{60}{2\sqrt{6}-1}(\sqrt{x-1}-1) + 650, & 2 \le x \le 25, \\ g(x),\ g(25) = 710, & 25 \le x, \end{cases}$$

where $g\colon [25,\infty) \to [0,\infty)$ is a continuous function.

We have $I_1 = \frac{256}{9} \approx 28.44$ and $I_2 = \frac{24}{1057}\left(7537 - 405\sqrt{105}\right) \approx 76.9$.

If we choose $r_1 = 1$, $r_2 = 8$, $r_3 = 8.4$, $r_4 = 25$, we get

$$\begin{aligned} f(t,x) &\le I_1 r_1 && \text{for } (t,x) \in [0,1]\times[0,r_1], \\ f(t,x) &\ge I_2 r_2 && \text{for } (t,x) \in [0,1]\times[r_2/4,r_2], \\ f(t,x) &\ge I_2 r_3 && \text{for } (t,x) \in [0,1]\times[r_3/4,r_3], \\ f(t,x) &\le I_1 r_4 && \text{for } (t,x) \in [0,1]\times[0,r_4]. \end{aligned}$$

Therefore, by Theorem 2, the boundary value problem has at least two positive solutions $x_1(t)$ and $x_2(t)$ such that

$$1 \le \|x_1\| \le 8, \quad 8.4 \le \|x_2\| \le 25.$$

Solutions $x_1(t)$ and $x_2(t)$ are depicted in Figure 1 and Figure 2. These figures were obtained by using the program Wolfram Mathematica 11.1. Each solution has a double zero at $t = 0$ and is equal to its antiderivative at $t = 1$. The initial conditions for these solutions are $x_1(0) = 0$, $x_1'(0) = 0$, $x_1''(0) \approx 6.27314$ and $x_2(0) = 0$, $x_2'(0) = 0$, $x_2''(0) \approx 170.0105$.

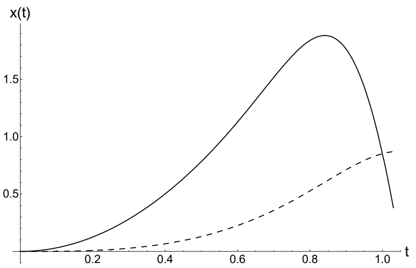

FIGURE 1. Solution x_1 (solid) with its antiderivative (dashed).

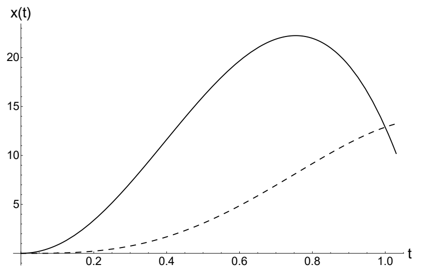

FIGURE 2. Solution x_2 (solid) with its antiderivative (dashed).

Now, let us discuss the difference between conditions on nonlinearity f in Theorem 2 and in [10] (Theorem 2.2). According to [10], the problem has at least two positive solutions if certain conditions are fulfilled, two of which are the following inequalities

$$0 \leq \lim_{x \to 0+} \frac{f(x)}{x} < \mu_1, \quad 0 \leq \lim_{x \to \infty} \frac{f(x)}{x} < \mu_1,$$

where μ_1 is called the principal characteristic value of operator T or the principal eigenvalue of the corresponding boundary value problem. We see, that the first inequality is fulfilled, but the second one is not fulfilled, because $\lim_{x \to \infty} \frac{f(x)}{x}$ can be every nonnegative number in our example. Also, the author would like to mention, that our conditions allow us to get an estimate of the norm for positive solutions to the problem.

REFERENCES

[1] A. Calamai and G. Infante, "An affine Birkhoff-Kellogg-type result in cones with applications to functional differential equations." *Math. Meth. Appl. Sci.*, pp. 1–9, 2022, doi: 10.1002/mma.8665.

[2] J. R. Graef and J. R. L. Webb, "Third order boundary value problems with nonlocal boundary conditions." *Nonlinear Anal.*, vol. 71, pp. 1542–1551, 2009, doi: 10.1016/j.na.2008.12.047.

[3] J. R. Graef and B. Yang, "Multiple positive solutions to a three point third order boundary value problem." *Discrete Contin. Dyn. Syst.*, vol. suppl., pp. 337–344, 2005.

[4] J. R. Graef and B. Yang, "Positive solutions to a multi-point higher order boundary value problem." *J. Math. Anal. Appl.*, vol. 316, pp. 409–421, 2006.

[5] J. R. Graef and B. Yang, "Existence and nonexistence of positive solutions of a nonlinear third order boundary value problem." *Electron. J. Qual. Theory Differ. Equ.*, vol. 9, pp. 1–13, 2008, doi: 10.14232/ejqtde.2008.1.31.

[6] J. R. Graef and B. Yang, "Positive solutions of a third order nonlocal boundary value problem." *Discrete Contin. Dyn. Syst. Ser. S*, vol. 1, pp. 89–97, 2008, doi: 10.3934/dcdss.2008.1.89.

[7] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones.* San Diego: Academic Press, 1988.

[8] M. A. Krasnosel'skii, *Positive Solutions of Operator Equations.* Groningen: Noordhoff, 1964.

[9] J. R. L. Webb, "Multiple positive solutions of some nonlinear heat flow problems." *Discrete Contin. Dyn. Syst. Ser.*, vol. suppl., pp. 895–903, 2005.

[10] J. R. L. Webb, "Higher order non-local $(n-1,1)$ conjugate type boundary value problems." *Mathematical models in engineering, biology and medicine*, vol. 1124, pp. 332–341, 2009.

[11] J. R. L. Webb, "Non-local second-order boundary value problems with derivative-dependent nonlinearity." *Phil. Trans. R. Soc. A*, vol. 379, pp. 1–12, 2021, doi: 10.1098/rsta.2019.0383.

[12] J. R. L. Webb and G. Infante, "Positive solutions of nonlocal boundary value problems involving integral conditions." *NoDEA Nonlinear Differential Equations Appl.*, vol. 15, pp. 45–67, 2008, doi: 10.1007/s00030-007-4067-7.

[13] J. R. L. Webb and G. Infante, "Non-local boundary value problems of arbitrary order." *J. London Math. Soc.*, vol. 79, no. 2, pp. 238–258, 2009, doi: 10.1112/jlms/jdn066.

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