



THE SOLUTIONS TO SOME DUAL MATRIX EQUATIONS

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Abstract. The solvability conditions for the dual matrix equation $AXB = D$, a pair of dual matrix equations $AX = B, XC = D$ and the dual matrix equation $AX + YB = D$ are established by using the generalized inverses and the singular value decompositions of some real matrices, and the expressions of the general solutions to these dual matrix equations are presented. Finally, a numerical experiment is given to validate the accuracy of our result.

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1. INTRODUCTION

Throughout this paper, $\mathbb{R}^{m \times n}$ denotes the set of all $m \times n$ real matrices, and the symbols A^T, A^-, A^\dagger and $\|A\|$ denote the transpose, the g-inverse, the Moore-Penrose inverse and the Frobenius norm of a real matrix A , respectively. I_n represents the identity matrix of order n . For a matrix $A \in \mathbb{R}^{m \times n}$, E_A and F_A stand for the orthogonal projectors: $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$.

In real and complex matrix spaces, many scholars have considered the following linear matrix equations:

$$AXB = D, \tag{1.1}$$

$$AX = B, \quad XC = D, \tag{1.2}$$

and

$$AX + YB = D. \tag{1.3}$$

For Eq. (1.1), Penrose [15] obtained the solvability condition and the general solution of Eq. (1.1) by using the generalized inverse. Tian and Wang [18] analyzed the relations between least-squares and least-rank solutions of Eq. (1.1). Yuan and Dai [22] considered the reflexive solutions of Eq. (1.1) and the associated optimal approximation problem by applying the generalized singular value decomposition. For Eq. (1.2), Mitra [11] provided the solvability conditions and the general solution of

Eq. (1.2) by using the g -inverses of matrices. Dajić and Koliha [7] considered the common Hermitian and positive solutions to Eq. (1.2) for Hilbert space operators. For Eq. (1.3), Baksalary and Kala [1] and Chu [2] have been derived its general solution by using the g -inverse and the singular value decomposition (SVD), respectively. For inconsistent equations, l_p and Chebyshev solutions of Eq. (1.3) were considered in [24] and [25].

The dual number originally was introduced by Clifford [3], and then was further developed to represent the dual angle in spatial geometry [17]. For any dual number a can be written as $a = a_1 + \varepsilon a_2$, where $a_1, a_2 \in \mathbb{R}$ are the real part and the dual part of the dual number a , respectively, and ε denotes the dual unit with the operational rules: $\varepsilon \neq 0$, $\varepsilon 0 = 0\varepsilon = 0$, $\varepsilon 1 = 1\varepsilon = \varepsilon$, $\varepsilon^2 = 0$. A matrix whose elements are dual numbers is called a dual matrix, that is, for an $m \times n$ dual matrix A can be written as $A = A_1 + \varepsilon A_2$, where the matrices A_1 and $A_2 \in \mathbb{R}^{m \times n}$. Today, dual matrices are used in a various areas of engineering like the kinematic analysis and synthesis of spatial mechanisms [4, 16] and the robotics [8]. The growing applicability of dual matrices in science and engineering has sparked a renewal of interest in the linear algebra and numerical algorithms [12, 20].

The solution to some linear dual equations is also a task often required in the kinematic analysis of many spatial mechanisms [13, 14]. For example, Li et al.[9] used dual-quaternions to solve homogeneous transformation equation $AX = ZB$ for the robot-to-world and hand-eye calibration problem. Wang et al.[21] converted the multi-coordinate calibration problem of a dual-robot system into solving the $AXB = YCZ$ problem. Condurache and Burlacu [5] provided a new approach for solving $AX = XB$ sensor calibration problem by applying orthogonal dual tensor methods. Udwardia [19] explored the solution of the linear dual equation $Ax = \beta$ which is commonly encountered in areas of kinematics and robotics, where A is a p -by- q dual matrix and β is a p -by-1 dual vector.

We observe that the dual matrix equations of (1.1), (1.2) and (1.3) are seldom considered in the literature. In this paper, we will discuss the solvability conditions and general solutions of these dual matrix equations, which can be stated as the following problems:

Problem 1. Given dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m \times n}, B_i \in \mathbb{R}^{p \times q}$ and $D_i \in \mathbb{R}^{m \times q}$. Find a dual matrix $X = X_1 + \varepsilon X_2$, where $X_i \in \mathbb{R}^{n \times p}$ ($i = 1, 2$) such that Eq. (1.1) is satisfied.

Problem 2. Given dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2, C = C_1 + \varepsilon C_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m \times n}, B_i \in \mathbb{R}^{m \times p}, C_i \in \mathbb{R}^{p \times q}$ and $D_i \in \mathbb{R}^{n \times q}$. Find a dual matrix $X = X_1 + \varepsilon X_2$, where $X_i \in \mathbb{R}^{n \times p}$ ($i = 1, 2$) such that Eq. (1.2) is satisfied.

Problem 3. Given dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m \times n}, B_i \in \mathbb{R}^{p \times q}$ and $D_i \in \mathbb{R}^{m \times q}$. Find dual matrices $X = X_1 + \varepsilon X_2$ and $Y = Y_1 + \varepsilon Y_2$, where $X_i \in \mathbb{R}^{n \times q}$ and $Y_i \in \mathbb{R}^{m \times p}$ ($i = 1, 2$) such that Eq. (1.3) is satisfied.

In the following sections, each dual matrix equation will firstly be decomposed into two real matrix equations. Then, by making use of the generalized inverses and the SVDs of matrices, the solvability conditions and explicit solutions of Problems 1, 2 and 3 are derived. A numerical example is given to verify its correctness.

2. THE SOLUTION OF PROBLEM 1

To begin with, we introduce some lemmas.

Lemma 1 ([23, page 907]). *Suppose that $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times q}$ and $D \in \mathbb{R}^{m \times q}$. Then Eq. (1.1) has a solution $X \in \mathbb{R}^{n \times p}$ if and only if $AA^\dagger DB^\dagger B = D$. In this case, the general solution is $X = A^\dagger DB^\dagger + F_A V_1 + V_2 E_B$, where V_1 and V_2 are arbitrary matrices.*

Lemma 2 ([6, page 30]). *Let $A \in \mathbb{R}^{m \times p}, B \in \mathbb{R}^{m \times q}$ and $C = (I_m - AA^-)B$. Then the g-inverse of the matrix $[A, B]$ is*

$$[A, B]^- = \begin{bmatrix} A^- - A^- B C^- (I_m - AA^-) \\ C^- (I_m - AA^-) \end{bmatrix}.$$

Lemma 3 ([1, page 145]). *Given $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{p \times q}, C \in \mathbb{R}^{m \times r}, D \in \mathbb{R}^{s \times q}$ and $L \in \mathbb{R}^{m \times q}$. Then the matrix equation $AXB + CYD = L$ is consistent if and only if*

$$E_G E_A L = 0, \quad E_A L F_D = 0, \quad E_C L F_B = 0, \quad L F_B F_H = 0,$$

where $G = E_A C$ and $H = D F_B$. In this case, the general solution is

$$\begin{aligned} X &= A^- (L - CYD) B^- + Z - A^- A Z B B^-, \\ Y &= G^- E_A L D^- + (F_G C^- + F_C G^- E_A) L F_B H^- + W - C^- C F_G W H H^- - G^- G W D D^-, \end{aligned}$$

where W and Z are arbitrary matrices.

Theorem 1. *Let dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m \times n}, B_i \in \mathbb{R}^{p \times q}$ and $D_i \in \mathbb{R}^{m \times q}$ ($i = 1, 2$), and let $H = E_{A_1} A_2 F_{A_1}, K = E_{B_1} B_2 F_{B_1}$. Then Eq. (1.1) is solvable if and only if*

$$A_1 A_1^\dagger D_1 B_1^\dagger B_1 = D_1, \quad (D_2 - A_1 A_1^\dagger D_1 B_1^\dagger B_2) F_{B_1} F_K = 0, \quad (2.1)$$

$$E_{A_1} D_2 F_{B_1} = 0, \quad E_H E_{A_1} (D_2 - A_2 A_1^\dagger D_1 B_1^\dagger B_1) = 0. \quad (2.2)$$

In this case, the solution set S_1 of Problem 1 can be expressed as

$$S_1 = \{ X = X_1 + \varepsilon X_2 \mid X_1, X_2 \in \mathbb{R}^{n \times p} \},$$

where

$$\begin{aligned} X_1 &= A_1^\dagger D_1 B_1^\dagger + H^\dagger (D_2 - A_2 A_1^\dagger D_1) B_1^\dagger + A_1^\dagger (D_2 - D_1 B_1^\dagger B_2) K^\dagger \\ &\quad + F_{A_1} W_{22} - H^\dagger H W_{22} B_1 B_1^\dagger + W_1 E_{B_1} - A_1^\dagger A_1 W_1 K K^\dagger, \\ X_2 &= A_1^\dagger (I_m - A_2 H^\dagger) (D_2 - A_2 A_1^\dagger D_1) B_1^\dagger - A_1^\dagger D_1 B_1^\dagger B_2 B_1^\dagger \end{aligned} \quad (2.3)$$

$$\begin{aligned} & -A_1^\dagger(D_2 - D_1B_1^\dagger B_2)K^\dagger B_2B_1^\dagger - A_1^\dagger A_1 W_{21} B_1 B_1^\dagger + W_{21} \\ & - A_1^\dagger A_1 W_1 E_K E_{B_1} B_2 B_1^\dagger - A_1^\dagger A_2 F_{A_1} F_H W_{22} B_1 B_1^\dagger, \end{aligned} \quad (2.4)$$

and W_1, W_{21}, W_{22} are arbitrary matrices.

Proof. Clearly, Eq. (1.1) can be equivalently written as

$$A_1 X_1 B_1 = D_1, \quad (2.5)$$

$$A_2 X_1 B_1 + A_1 X_1 B_2 + A_1 X_2 B_1 = D_2. \quad (2.6)$$

By Lemma 1, Eq. (2.5) is solvable if and only if the first condition of (2.1) is satisfied, and the general solution is

$$X_1 = A_1^\dagger D_1 B_1^\dagger + F_{A_1} V_1 + V_2 E_{B_1}, \quad (2.7)$$

where V_1 and $V_2 \in \mathbb{R}^{n \times p}$ are arbitrary matrices. Inserting (2.7) into Eq. (2.6), we have

$$[A_1, A_2 F_{A_1}] \begin{bmatrix} X_2 \\ V_1 \end{bmatrix} B_1 + A_1 V_2 E_{B_1} B_2 = D_2 - A_2 A_1^\dagger D_1 B_1^\dagger B_1 - A_1 A_1^\dagger D_1 B_1^\dagger B_2. \quad (2.8)$$

By Lemma 2, we know that $\begin{bmatrix} A_1^\dagger - A_1^\dagger A_2 F_{A_1} H^\dagger E_{A_1} \\ H^\dagger E_{A_1} \end{bmatrix} = \begin{bmatrix} A_1^\dagger - A_1^\dagger A_2 H^\dagger \\ H^\dagger \end{bmatrix}$ is one of the g-inverse of $[A_1, A_2 F_{A_1}]$. Thus

$$E_{[A_1, A_2 F_{A_1}]} = I_m - [A_1, A_2 F_{A_1}][A_1, A_2 F_{A_1}]^- = E_H E_{A_1}.$$

By Lemma 3, Eq. (2.8) is solvable if and only if the second condition of (2.1) and conditions (2.2) are satisfied. In which case, the general solution is

$$V_2 = A_1^\dagger(D_2 - D_1B_1^\dagger B_2)K^\dagger + W_1 - A_1^\dagger A_1 W_1 K K^\dagger, \quad (2.9)$$

$$\begin{aligned} \begin{bmatrix} X_2 \\ V_1 \end{bmatrix} &= [A_1, A_2 F_{A_1}]^-(D_2 - A_2 A_1^\dagger D_1 - A_1^\dagger A_1 D_1 B_1^\dagger B_2 - A_1 V_2 E_{B_1} B_2) B_1^\dagger \\ &+ \begin{bmatrix} W_{21} \\ W_{22} \end{bmatrix} - [A_1, A_2 F_{A_1}]^- [A_1, A_2 F_{A_1}] \begin{bmatrix} W_{21} \\ W_{22} \end{bmatrix} B_1 B_1^\dagger, \end{aligned} \quad (2.10)$$

where W_1, W_{21} and $W_{22} \in \mathbb{R}^{n \times p}$ are arbitrary matrices. From (2.9) and (2.10), we can get the relation of (2.4) and

$$V_1 = H^\dagger(D_2 - A_2 A_1^\dagger D_1) B_1^\dagger + W_{22} - H^\dagger H W_{22} B_1 B_1^\dagger. \quad (2.11)$$

Substituting (2.9) and (2.11) into (2.7), we can get (2.3). \square

Clearly, the dual matrix equation $AXA = A$ is a special case of the dual matrix equation (1.1). According to Theorem 1, we can get the following result about $\{1\}$ -dual generalized inverse of the dual matrix A , which was discussed by Udvardia in [19].

Corollary 1. *Given the dual matrix $A = A_1 + \varepsilon A_2$, where $A_1, A_2 \in \mathbb{R}^{m \times n}$. Then the dual matrix equation $AXA = A$ is solvable if and only if $H = E_{A_1} A_2 F_{A_1} = 0$. In this case, the general solution is $X = X_1 + \varepsilon X_2$, where*

$$\begin{aligned} X_1 &= A_1^\dagger + W_1 E_{A_1} + F_{A_1} W_{22}, \\ X_2 &= -A_1^\dagger A_2 A_1^\dagger - A_1^\dagger A_1 W_1 E_{A_1} A_2 A_1^\dagger + W_{21} - A_1^\dagger A_1 W_{21} A_1 A_1^\dagger - A_1^\dagger A_2 F_{A_1} W_{22} A_1 A_1^\dagger, \end{aligned}$$

and W_1, W_{21}, W_{22} are arbitrary matrices.

3. THE SOLUTION OF PROBLEM 2

Before solving Problem 2, we need the following Lemma.

Lemma 4 ([10, page 215]). *Suppose that $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{m \times p}, C \in \mathbb{R}^{p \times q}$ and $D \in \mathbb{R}^{n \times q}$. Then Eq. (1.2) has a solution $X \in \mathbb{R}^{n \times p}$ if and only if $E_A B = 0, D F_C = 0$ and $AD = BC$. In which case, the general solution is $X = A^\dagger B + F_A D C^\dagger + F_A W E_C$, where $W \in \mathbb{R}^{n \times p}$ is an arbitrary matrix.*

Theorem 2. *Assume that dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2, C = C_1 + \varepsilon C_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m \times n}, B_i \in \mathbb{R}^{m \times p}, C_i \in \mathbb{R}^{p \times q}$ and $D_i \in \mathbb{R}^{n \times q}$ ($i = 1, 2$). Let the SVDs of the matrices A_1 and C_1 be*

$$A_1 = P \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} Q^\top, \quad C_1 = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top, \quad (3.1)$$

where $\Omega = \text{diag}(\omega_1, \dots, \omega_r), \Sigma = \text{diag}(\sigma_1, \dots, \sigma_s), r = \text{rank}(A_1), s = \text{rank}(C_1)$, and $P = [P_1, P_2] \in \mathbb{R}^{m \times m}, Q = [Q_1, Q_2] \in \mathbb{R}^{n \times n}, U = [U_1, U_2] \in \mathbb{R}^{p \times p}, V = [V_1, V_2] \in \mathbb{R}^{q \times q}$ are orthogonal matrices with $P_2 \in \mathbb{R}^{m \times (m-r)}, Q_2 \in \mathbb{R}^{n \times (n-r)}, U_2 \in \mathbb{R}^{p \times (p-s)}$ and $V_2 \in \mathbb{R}^{q \times (q-s)}$. If write

$$\begin{aligned} G_1 &= B_2 - A_2 A_1^\dagger B_1 - A_2 F_{A_1} D_1 C_1^\dagger, & G_2 &= D_2 - A_1^\dagger B_1 C_2 - F_{A_1} D_1 C_1^\dagger C_2, \\ M_1 &= U_2^\top C_2 V_2, & N_1 &= P_2^\top A_2 Q_2, & H_1 &= Q_2 F_{N_1}, & H_2 &= E_{M_1} U_2^\top, \\ S &= Q_2 N_1^\dagger P_2^\top (B_2 - A_2 A_1^\dagger B_1) E_{C_1} + Q_2 F_{N_1} Q_2^\top G_2 V_2 M_1^\dagger U_2^\top. \end{aligned}$$

Then Eq. (1.2) is solvable if and only if

$$E_{A_1} B_1 = 0, \quad D_1 F_{C_1} = 0, \quad A_1 D_1 = B_1 C_1, \quad (3.2)$$

$$A_1 G_2 = G_1 C_1, \quad A_1^\dagger A_1 G_2 F_{C_1} = 0, \quad E_{A_1} G_1 C_1 C_1^\dagger = 0, \quad (3.3)$$

$$E_{N_1} P_2^\top G_1 U_2 = 0, \quad Q_2^\top G_2 V_2 F_{M_1} = 0, \quad N_1 Q_2^\top G_2 V_2 = P_2^\top G_1 U_2 M_1. \quad (3.4)$$

The solution set S_2 of Problem 2 can be expressed as

$$S_2 = \{ X = X_1 + \varepsilon X_2 \mid X_1, X_2 \in \mathbb{R}^{n \times p} \},$$

where

$$X_1 = A_1^\dagger B_1 + F_{A_1} D_1 C_1^\dagger + S + H_1 J_1 H_2, \quad (3.5)$$

$$\begin{aligned} X_2 = & A_1^\dagger (G_1 - A_2 S) + F_{A_1} G_2 C_1^\dagger - S C_2 C_1^\dagger - A_1^\dagger A_2 H_1 J_1 H_2 \\ & - H_1 J_1 H_2 C_2 C_1^\dagger + F_{A_1} R_1 E_{C_1}, \end{aligned} \quad (3.6)$$

and J_1, R_1 are arbitrary matrices.

Proof. Eq. (1.2) can be equivalently written as

$$A_1 X_1 = B_1, \quad X_1 C_1 = D_1, \quad (3.7)$$

$$A_2 X_1 + A_1 X_2 = B_2, \quad X_1 C_2 + X_2 C_1 = D_2. \quad (3.8)$$

By using Lemma 4, we know Eq. (3.7) is solvable if and only if the conditions (3.2) are satisfied, and the general solution is

$$X_1 = A_1^\dagger B_1 + F_{A_1} D_1 C_1^\dagger + F_{A_1} W_1 E_{C_1}, \quad (3.9)$$

where $W_1 \in \mathbb{R}^{n \times p}$ is an arbitrary matrix. Substituting (3.9) into Eq. (3.8), we can get

$$A_1 X_2 = G_1 - A_2 F_{A_1} W_1 E_{C_1}, \quad X_2 C_1 = G_2 - F_{A_1} W_1 E_{C_1} C_2. \quad (3.10)$$

Using Lemma 4 again, Eq. (3.10) with respect to X_2 is solvable if and only if the first condition of (3.3) is satisfied and

$$F_{A_1} W_1 E_{C_1} C_2 F_{C_1} = G_2 F_{C_1}, \quad (3.11)$$

$$E_{A_1} A_2 F_{A_1} W_1 E_{C_1} = E_{A_1} G_1. \quad (3.12)$$

In this case, the general solution is

$$X_2 = A_1^\dagger (G_1 - A_2 F_{A_1} W_1 E_{C_1}) + F_{A_1} (G_2 - F_{A_1} W_1 E_{C_1} C_2) C_1^\dagger + F_{A_1} R_1 E_{C_1}, \quad (3.13)$$

where $R_1 \in \mathbb{R}^{n \times p}$ is an arbitrary matrix. Inserting (3.1) to Eq. (3.11) and Eq. (3.12), we can obtain

$$\begin{aligned} & Q \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^\top W_1 U \begin{bmatrix} 0 & 0 \\ 0 & I_{p-s} \end{bmatrix} U^\top C_2 V \begin{bmatrix} 0 & 0 \\ 0 & I_{q-s} \end{bmatrix} V^\top \\ & = G_2 V \begin{bmatrix} 0 & 0 \\ 0 & I_{q-s} \end{bmatrix} V^\top, \end{aligned} \quad (3.14)$$

$$\begin{aligned} & P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top A_2 Q \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^\top W_1 U \begin{bmatrix} 0 & 0 \\ 0 & I_{p-s} \end{bmatrix} U^\top \\ & = P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top G_1. \end{aligned} \quad (3.15)$$

If let

$$\begin{aligned} Q^\top W_1 U &= \begin{bmatrix} W_{11} & W_{12} \\ W_{13} & W_{14} \end{bmatrix}, \quad U^\top C_2 V = \begin{bmatrix} C_{21} & C_{22} \\ C_{23} & C_{24} \end{bmatrix}, \quad Q^\top G_2 V = \begin{bmatrix} G_{21} & G_{22} \\ G_{23} & G_{24} \end{bmatrix}, \\ P^\top A_2 Q &= \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix}, \quad P^\top G_1 U = \begin{bmatrix} G_{11} & G_{12} \\ G_{13} & G_{14} \end{bmatrix}. \end{aligned}$$

Then Eq. (3.14) and Eq. (3.15) can be equivalently written as

$$G_{22} = 0, \quad G_{13} = 0, \quad (3.16)$$

$$W_{14}M_1 = Q^\top G_2V_2, \quad N_1W_{14} = P_2^\top G_1U_2. \quad (3.17)$$

We note that

$$G_{22} = 0 \Leftrightarrow Q_1^\top G_2V_2 = 0 \Leftrightarrow Q_1Q_1^\top G_2V_2V_2^\top = 0 \Leftrightarrow A_1^\dagger A_1 G_2 F_{C_1} = 0,$$

which is the second condition of (3.3). Similarly, the condition $G_{13} = 0$ is equivalent to the third condition of (3.3). Using Lemma 4, Eq. (3.17) with respect to W_{14} is solvable if and only if conditions (3.4) hold, and the general solution is

$$W_{14} = N_1^\dagger P_2^\top G_1U_2 + F_{N_1} Q_2^\top G_2V_2M_1^\dagger + F_{N_1} J_1 E_{M_1},$$

where $J_1 \in \mathbb{R}^{(n-r) \times (p-s)}$ is an arbitrary matrix. Thus, we have

$$F_{A_1} W_1 E_{C_1} = Q_2 W_{14} U_2^\top = S + H_1 J_1 H_2, \quad (3.18)$$

Substituting (3.18) into (3.9) and (3.13), we can obtain (3.5) and (3.6). \square

4. THE SOLUTION OF PROBLEM 3

Theorem 3. Assume that dual matrices $A = A_1 + \varepsilon A_2, B = B_1 + \varepsilon B_2$ and $D = D_1 + \varepsilon D_2$, where $A_i \in \mathbb{R}^{m \times n}, B_i \in \mathbb{R}^{p \times q}$ and $D_i \in \mathbb{R}^{m \times q} (i = 1, 2)$. Let the SVDs of the matrices A_1 and B_1 be

$$A_1 = P \begin{bmatrix} \Omega & 0 \\ 0 & 0 \end{bmatrix} Q^\top, \quad B_1 = M \begin{bmatrix} \Lambda & 0 \\ 0 & 0 \end{bmatrix} N^\top, \quad (4.1)$$

where $\Omega = \text{diag}(\omega_1, \dots, \omega_r), \Lambda = \text{diag}(\lambda_1, \dots, \lambda_t), r = \text{rank}(A_1), t = \text{rank}(B_1)$, and $P = [P_1, P_2] \in \mathbb{R}^{m \times m}, Q = [Q_1, Q_2] \in \mathbb{R}^{n \times n}, M = [M_1, M_2] \in \mathbb{R}^{p \times p}, N = [N_1, N_2] \in \mathbb{R}^{q \times q}$ are orthogonal matrices with $P_2 \in \mathbb{R}^{m \times (m-r)}, Q_2 \in \mathbb{R}^{n \times (n-r)}, M_2 \in \mathbb{R}^{p \times (p-t)}$ and $N_2 \in \mathbb{R}^{q \times (q-t)}$. If write

$$J = D_2 - A_2 A_1^\dagger D_1 - E_{A_1} D_1 B_1^\dagger B_2, \quad K_1 = P_2^\top A_2 Q_2, \quad K_2 = M_2^\top B_2 N_2. \quad (4.2)$$

Then Eq. (1.3) is solvable if and only if

$$E_{A_1} D_1 F_{B_1} = 0, \quad E_{K_1} P_2^\top J N_2 F_{K_2} = 0. \quad (4.3)$$

The solution set S_3 of Problem 3 can be expressed as

$$S_3 = \{X = X_1 + \varepsilon X_2, Y = Y_1 + \varepsilon Y_2 | X_1, X_2 \in \mathbb{R}^{n \times q}, Y_1, Y_2 \in \mathbb{R}^{m \times p}\},$$

where

$$X_1 = A_1^\dagger D_1 + Q_2 K_1^\dagger P_2^\top J F_{B_1} - A_1^\dagger P_1 V_{11} M_1^\top B_1 + Q_2 V_{23} N_1^\top + Q_2 (F_{K_1} W_4 - K_1^\dagger W_3 K_2) N_2^\top, \quad (4.4)$$

$$Y_1 = E_{A_1} D_1 B_1^\dagger + P_2 E_{K_1} P_2^\top J N_2 K_2^\dagger M_2^\top + P_1 (V_{11} M_1^\top + V_{12} M_2^\top) + P_2 (W_3 - E_{K_1} W_3 K_2 K_2^\dagger) M_2^\top, \quad (4.5)$$

$$\begin{aligned}
X_2 = & A_1^\dagger J - A_1^\dagger A_2 Q_2 (K_1^\dagger P_2^\top J N_2 - K_1^\dagger W_3 K_2 + F_{K_1} W_4) N_2^\top \\
& - A_1^\dagger P_1 (V_{11} M_1^\top + V_{12} M_2^\top) B_2 - A_1^\dagger R_1 B_1 + F_{A_1} R_2 \\
& + A_1^\dagger A_2 (A_1^\dagger P_1 V_{11} M_1^\top B_1 - Q_2 V_{23} N_1^\top), \tag{4.6}
\end{aligned}$$

$$\begin{aligned}
Y_2 = & E_{A_1} J B_1^\dagger + E_{A_1} A_2 (A_1^\dagger P_1 V_{11} M_1^\top B_1 - Q_2 V_{23} N_1^\top) B_1^\dagger \\
& - P_2 E_{K_1} P_2^\top J N_2 K_2^\dagger M_2^\top B_2 B_1^\dagger - P_2 (W_3 - E_{K_1} W_3 K_2 K_2^\dagger) M_2^\top B_2 B_1^\dagger \\
& + R_1 - E_{A_1} R_1 B_1 B_1^\dagger, \tag{4.7}
\end{aligned}$$

and $V_{11}, V_{12}, V_{23}, W_3, W_4, R_1, R_2$ are arbitrary matrices.

Proof. Obviously, Eq. (1.3) can be equivalently written as

$$A_1 X_1 + Y_1 B_1 = D_1, \tag{4.8}$$

$$A_2 X_1 + A_1 X_2 + Y_2 B_1 + Y_1 B_2 = D_2. \tag{4.9}$$

From Lemma 3, Eq. (4.8) is solvable if and only if the first condition of (4.3) is satisfied, and the general solution is

$$Y_1 = E_{A_1} D_1 B_1^\dagger + V_1 - E_{A_1} V_1 B_1 B_1^\dagger, \tag{4.10}$$

$$X_1 = A_1^\dagger D_1 - A_1^\dagger V_1 B_1 + F_{A_1} V_2, \tag{4.11}$$

where V_1 and V_2 are arbitrary matrices. Plugging (4.10) and (4.11) into Eq. (4.9), we have

$$A_1 X_2 + Y_2 B_1 = J - A_2 F_{A_1} V_2 + A_2 A_1^\dagger V_1 B_1 - V_1 B_2 + E_{A_1} V_1 B_1 B_1^\dagger B_2. \tag{4.12}$$

Using Lemma 3 again, Eq. (4.12) with respects to X_2 and Y_2 is solvable if and only if

$$E_{A_1} A_2 F_{A_1} V_2 F_{B_1} + E_{A_1} V_1 E_{B_1} B_2 F_{B_1} = E_{A_1} J F_{B_1}. \tag{4.13}$$

In this case, the general solution is

$$\begin{aligned}
Y_2 = & E_{A_1} J B_1^\dagger + E_{A_1} A_2 A_1^\dagger V_1 B_1 B_1^\dagger - E_{A_1} V_1 E_{B_1} B_2 B_1^\dagger \\
& - E_{A_1} A_2 F_{A_1} V_2 B_1^\dagger + R_1 - E_{A_1} R_1 B_1 B_1^\dagger, \tag{4.14}
\end{aligned}$$

$$X_2 = A_1^\dagger (J - A_2 F_{A_1} V_2 + A_2 A_1^\dagger V_1 B_1 - V_1 B_2 - Y_2 B_1) + F_{A_1} R_2, \tag{4.15}$$

where R_1 and R_2 are arbitrary matrices. Inserting (4.1) to Eq. (4.13), we can get

$$\begin{aligned}
& P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top A_2 Q \begin{bmatrix} 0 & 0 \\ 0 & I_{n-r} \end{bmatrix} Q^\top V_2 N \begin{bmatrix} 0 & 0 \\ 0 & I_{q-t} \end{bmatrix} N^\top \\
& + P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top V_1 M \begin{bmatrix} 0 & 0 \\ 0 & I_{p-t} \end{bmatrix} M^\top B_2 N \begin{bmatrix} 0 & 0 \\ 0 & I_{q-t} \end{bmatrix} N^\top \\
& = P \begin{bmatrix} 0 & 0 \\ 0 & I_{m-r} \end{bmatrix} P^\top J N \begin{bmatrix} 0 & 0 \\ 0 & I_{q-t} \end{bmatrix} N^\top. \tag{4.16}
\end{aligned}$$

If let

$$P^\top A_2 Q = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix}, \quad Q^\top V_2 N = \begin{bmatrix} V_{21} & V_{22} \\ V_{23} & V_{24} \end{bmatrix}, \quad P^\top V_1 M = \begin{bmatrix} V_{11} & V_{12} \\ V_{13} & V_{14} \end{bmatrix},$$

$$M^\top B_2 N = \begin{bmatrix} B_{21} & B_{22} \\ B_{23} & B_{24} \end{bmatrix}, \quad P^\top J N = \begin{bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{bmatrix}.$$

Then Eq. (4.16) can be equivalently written as

$$K_1 V_{24} + V_{14} K_2 = P_2^\top J N_2. \tag{4.17}$$

Using Lemma 3, Eq. (4.17) is solvable if and only if the second condition of (4.3) holds, and the general solution is

$$V_{14} = E_{K_1} P_2^\top J N_2 K_2^\dagger + W_3 - E_{K_1} W_3 K_2 K_2^\dagger, \tag{4.18}$$

$$V_{24} = K_1^\dagger P_2^\top J N_2 - K_1^\dagger W_3 K_2 + F_{K_1} W_4, \tag{4.19}$$

where W_3 and W_4 are arbitrary matrices. Thus,

$$V_1 = P_1 V_{11} M_1^\top + P_2 V_{13} M_1^\top + P_1 V_{12} M_2^\top + P_2 V_{14} M_2^\top, \tag{4.20}$$

$$V_2 = Q_1 V_{21} N_1^\top + Q_2 V_{23} N_1^\top + Q_1 V_{22} N_2^\top + Q_2 V_{24} N_2^\top, \tag{4.21}$$

where $V_{11}, V_{12}, V_{13}, V_{21}, V_{22}$ and V_{23} are arbitrary matrices. Substituting (4.18)–(4.21) into (4.10), (4.11), (4.14), (4.15), and noticing

$$E_{A_1} P_1 V_{11} M_1^\top = 0, \quad P_1 V_{11} M_1^\top E_{B_1} = 0, \quad A_1^\dagger P_2 V_{13} M_1^\top = 0, \quad P_2 V_{13} M_1^\top E_{B_1} = 0,$$

$$E_{A_1} P_1 V_{12} M_2^\top = 0, \quad P_1 V_{12} M_2^\top B_1 = 0, \quad A_1^\dagger P_2 V_{14} M_2^\top = 0, \quad P_2 V_{14} M_2^\top B_1 = 0,$$

$$F_{A_1} Q_1 V_{21} N_1^\top = 0, \quad Q_1 V_{21} N_1^\top F_{B_1} = 0, \quad A_1 Q_2 V_{23} N_1^\top = 0, \quad Q_2 V_{23} N_1^\top F_{B_1} = 0,$$

$$F_{A_1} Q_1 V_{22} N_2^\top = 0, \quad Q_1 V_{22} N_2^\top B_1^\dagger = 0, \quad A_1 Q_2 V_{24} N_2^\top = 0, \quad Q_2 V_{24} N_2^\top B_1^\dagger = 0,$$

we can obtain (4.4)–(4.7). □

Based on Theorem 3, we can formulate the following algorithm to solve Problem 3.

Algorithm 1.

- (1) Input matrices A_i, B_i and D_i ($i = 1, 2$).
- (2) Compute the SVDs of the matrices A_1 and B_1 by (4.1).
- (3) Calculate J, K_1 and K_2 by (4.2).
- (4) If the conditions (4.3) are satisfied, go to (5); otherwise, the Eq. (1.3) has no solution, and stop.
- (5) Randomly choose the matrices $V_{11}, V_{12}, V_{23}, W_3, W_4, R_1$ and R_2 .
- (6) Compute dual matrix $X = X_1 + \epsilon X_2$ by (4.4)–(4.7).

Example 1. Let $m = 7$, $n = 6$, $p = 5$, $q = 8$ and the matrices A_i , B_i and D_i ($i = 1, 2$) be given by

$$A_1 = \begin{bmatrix} 0.9295 & 0.8406 & 0.3948 & 0.8918 & 1.1059 & 0.6344 \\ 1.6982 & 1.7141 & 0.7129 & 1.7326 & 1.6930 & 1.3104 \\ 1.0853 & 1.1207 & 0.5291 & 0.9446 & 0.9248 & 0.7579 \\ 1.3420 & 1.2970 & 0.7564 & 0.8606 & 1.1432 & 0.8302 \\ 1.1715 & 1.2305 & 0.4552 & 1.3524 & 1.1679 & 0.7921 \\ 0.7213 & 0.8033 & 0.3125 & 0.7792 & 0.5855 & 0.4736 \\ 1.0188 & 1.1001 & 0.3871 & 1.2182 & 0.9771 & 0.6932 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0.4243 & 0.7691 & 0.9493 & 0.8620 & 0.4070 & 0.1117 \\ 0.2703 & 0.3968 & 0.3276 & 0.9899 & 0.7487 & 0.1363 \\ 0.1971 & 0.8085 & 0.6713 & 0.5144 & 0.8256 & 0.6787 \\ 0.8217 & 0.7551 & 0.4386 & 0.8843 & 0.7900 & 0.4952 \\ 0.4299 & 0.3774 & 0.8335 & 0.5880 & 0.3185 & 0.1897 \\ 0.8878 & 0.2160 & 0.7689 & 0.1548 & 0.5341 & 0.4950 \\ 0.3912 & 0.7904 & 0.1673 & 0.1999 & 0.0900 & 0.1476 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0.6079 & 1.1159 & 0.5956 & 0.6270 & 1.2325 & 1.0054 & 0.4038 & 0.9286 \\ 1.1726 & 1.7527 & 1.2573 & 1.0184 & 1.7030 & 1.8189 & 1.2731 & 1.1029 \\ 1.1426 & 1.8468 & 1.1734 & 1.0666 & 1.8938 & 1.8022 & 1.0471 & 1.3055 \\ 1.2331 & 1.8517 & 1.3963 & 1.0324 & 1.8061 & 1.9805 & 1.4456 & 1.2211 \\ 0.7972 & 1.0332 & 0.6332 & 0.7645 & 0.8880 & 0.9636 & 0.6552 & 0.3169 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0.8449 & 0.6147 & 0.1231 & 0.6352 & 0.5358 & 0.8739 & 0.4170 & 0.1420 \\ 0.2094 & 0.3624 & 0.2055 & 0.2819 & 0.4452 & 0.2703 & 0.2060 & 0.1665 \\ 0.5523 & 0.0495 & 0.1465 & 0.5386 & 0.1239 & 0.2085 & 0.9479 & 0.6210 \\ 0.6299 & 0.4896 & 0.1891 & 0.6952 & 0.4904 & 0.5650 & 0.0821 & 0.5737 \\ 0.0320 & 0.1925 & 0.0427 & 0.4991 & 0.8530 & 0.6403 & 0.1057 & 0.0521 \end{bmatrix},$$

$$D_1 = \begin{bmatrix} 6.1406 & 8.0527 & 7.0589 & 5.8597 & 9.0681 & 8.7133 & 6.2116 & 7.1038 \\ 5.9687 & 6.8809 & 7.6271 & 6.0367 & 8.8409 & 8.1612 & 6.6011 & 7.8182 \\ 5.1132 & 6.4170 & 5.9975 & 5.0679 & 7.5120 & 7.2419 & 5.5252 & 6.0488 \\ 3.5206 & 3.7180 & 4.3838 & 3.6134 & 5.0063 & 4.7191 & 3.9893 & 4.7616 \\ 5.1204 & 6.1849 & 6.2281 & 5.2350 & 7.5842 & 7.1172 & 5.4687 & 6.3310 \\ 3.2391 & 4.1829 & 3.8441 & 3.4011 & 5.1168 & 4.6853 & 3.3821 & 4.2400 \\ 3.3819 & 3.8221 & 4.3239 & 3.6439 & 5.1230 & 4.6124 & 3.6780 & 4.5859 \end{bmatrix},$$

$$D_2 = \begin{bmatrix} 8.0148 & 8.6320 & 7.9558 & 8.7138 & 10.4261 & 10.2622 & 7.6799 & 8.7502 \\ 8.0950 & 8.6298 & 9.7167 & 8.8261 & 9.6116 & 9.5072 & 7.0522 & 8.1786 \\ 6.9905 & 7.4758 & 7.4314 & 7.6470 & 9.1041 & 8.3395 & 5.7783 & 6.9511 \\ 7.2252 & 8.3537 & 9.3238 & 7.8794 & 9.5693 & 9.1666 & 7.2035 & 7.9500 \\ 6.2462 & 6.5371 & 6.6738 & 6.9087 & 7.8271 & 7.9660 & 5.7225 & 6.5788 \\ 6.5421 & 7.5126 & 6.5417 & 6.4456 & 8.0466 & 8.4030 & 6.0659 & 6.3854 \\ 6.2975 & 7.3790 & 6.7705 & 6.7915 & 8.1127 & 8.4531 & 5.7726 & 5.9501 \end{bmatrix}.$$

It is easy to verify that the solvability conditions (4.3) are satisfied:

$$\|E_{A_1}D_1F_{B_1}\| = 2.1298e - 15, \quad \|E_{K_1}P_2^T JN_2F_{K_2}\| = 1.9240e - 15.$$

According to Algorithm 1, and taking the matrices V_{11} , V_{12} , V_{23} , W_3 , W_4 , R_1 and R_2 as zero matrices, we can obtain a pair of solution $X = X_1 + \epsilon X_2$ and $Y = Y_1 + \epsilon Y_2$ of Problem 3 as follows:

$$X_1 = \begin{bmatrix} 0.8679 & 1.5792 & 1.0708 & 1.1229 & 1.3338 & 1.4621 & 1.0616 & 1.2846 \\ -1.1069 & -2.0598 & -1.2459 & -0.5771 & -1.3804 & -1.5612 & -1.0286 & -0.8204 \\ 0.4778 & 0.1432 & -0.1177 & 0.4213 & 0.3441 & 0.4644 & 0.4361 & 0.5040 \\ 1.3293 & 1.9433 & 1.5302 & 1.7165 & 2.4117 & 1.9648 & 1.1896 & 1.7743 \\ 6.3263 & 8.7531 & 6.6704 & 5.4077 & 8.9556 & 8.9486 & 5.6859 & 6.2824 \\ -4.8304 & -7.3331 & -4.0270 & -5.1423 & -7.4170 & -7.3118 & -3.6731 & -4.9208 \end{bmatrix},$$

$$X_2 = \begin{bmatrix} 3.9555 & 5.5750 & 4.1899 & 3.9725 & 6.0625 & 5.9144 & 3.9358 & 3.9953 \\ 5.4489 & 8.3834 & 5.1434 & 5.5083 & 8.5723 & 8.3644 & 4.1819 & 4.6228 \\ 3.6170 & 5.4246 & 3.9892 & 3.6878 & 5.9108 & 5.5483 & 3.6692 & 3.7562 \\ 6.1274 & 8.2455 & 4.9003 & 5.4792 & 8.5528 & 9.4694 & 5.3390 & 5.3433 \\ 2.9899 & 2.4317 & 3.4133 & 2.4154 & 3.5625 & 4.2695 & 5.2274 & 4.4004 \\ -23.5531 & -33.3170 & -21.8236 & -21.3747 & -36.1257 & -37.4695 & -24.4449 & -23.8245 \end{bmatrix},$$

$$Y_1 = \begin{bmatrix} 0.2731 & 0.1583 & 0.2593 & 0.1392 & 0.1734 \\ -0.0619 & -0.1359 & -0.1246 & -0.1256 & -0.1735 \\ 0.1409 & 0.3755 & 0.3286 & 0.3476 & 0.4810 \\ -0.1599 & -0.1836 & -0.2151 & -0.1660 & -0.2172 \\ -0.2442 & 0.0414 & -0.0900 & 0.0411 & 0.0510 \\ 0.4746 & 0.0933 & 0.3429 & 0.0674 & 0.0352 \\ -0.1406 & -0.1897 & -0.2349 & -0.1644 & -0.1789 \end{bmatrix},$$

$$Y_2 = \begin{bmatrix} 0.5453 & -0.0005 & 0.4083 & -0.2108 & 0.3890 \\ -0.4241 & 0.0642 & -0.2844 & 0.2672 & -0.3632 \\ 1.1377 & -0.2030 & 0.7443 & -0.7519 & 0.9533 \\ -0.5297 & 0.0924 & -0.3426 & 0.3190 & -0.3464 \\ -0.3304 & -0.3287 & -0.4576 & -0.1886 & -0.6673 \\ 0.1814 & -0.0026 & 0.1125 & 0.0463 & -0.2845 \\ -0.0533 & 0.3677 & 0.2065 & 0.3118 & 0.6601 \end{bmatrix}.$$

The absolute errors are estimated by

$$\|A_1X_1 + Y_1B_1 - D_1\| = 4.0468e - 14,$$

$$\|A_2X_1 + A_1X_2 + Y_2B_1 + Y_1B_2 - D_2\| = 5.6005e - 14,$$

which implies that X and Y is a pair of solution of Problem 3.

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REFERENCES

- [1] J. K. Baksalary and R. Kala, "The matrix equation $AXB + CYD = E$," *Linear Algebra and its Applications*, vol. 30, pp. 141–147, 1980, doi: [10.1016/0024-3795\(80\)90189-5](https://doi.org/10.1016/0024-3795(80)90189-5).
- [2] K.-W. E. Chu, "Singular value and generalized singular value decompositions and the solution of linear matrix equations," *Linear Algebra and its Applications*, vol. 88/89, pp. 83–98, 1987, doi: [10.1016/0024-3795\(87\)90104-2](https://doi.org/10.1016/0024-3795(87)90104-2).
- [3] M. A. Clifford, "Preliminary sketch of biquaternions," *Proceedings of the London Mathematical Society*, vol. s1-4, pp. 381–395, 1873, doi: [10.1112/plms/s1-4.1.381](https://doi.org/10.1112/plms/s1-4.1.381).
- [4] A. Cohen and M. Shoham, "Application of hyper-dual numbers to rigid bodies equations of motion," *Mechanism and Machine Theory*, vol. 111, pp. 76–84, 2017.
- [5] D. Condurache and A. Burlacu, "Orthogonal dual tensor method for solving the $AX = XB$ sensor calibration problem," *Mechanism and Machine Theory*, vol. 104, pp. 382–404, 2016, doi: [10.1016/j.mechmachtheory.2016.06.002](https://doi.org/10.1016/j.mechmachtheory.2016.06.002).
- [6] D. S. Cvetković-Ilić, J. N. Radenković, and Q. Wang, "Algebraic conditions for the solvability to some systems of matrix equations," *Linear and Multilinear Algebra*, pp. 1–31, 2019, doi: [10.1080/03081087.2019.1633993](https://doi.org/10.1080/03081087.2019.1633993).
- [7] A. Dajić and J. J. Koliha, "Positive solutions to the equations $AX = C$ and $XB = D$ for Hilbert space operators," *Journal of Mathematical Analysis and Applications*, vol. 333, pp. 567–576, 2007, doi: [10.1016/j.jmaa.2006.11.016](https://doi.org/10.1016/j.jmaa.2006.11.016).
- [8] Y.-L. Gu and J. Y. S. Luh, "Dual-number transformation and its applications to robotics," *IEEE Journal of Robotics and Automation*, vol. 3, pp. 615–623, 1987, doi: [10.1109/JRA.1987.1087138](https://doi.org/10.1109/JRA.1987.1087138).
- [9] A. Li, L. Wang, and D. Wu, "Simultaneous robot-world and hand-eye calibration using dual-quaternions and kronecker product," *International Journal of the Physical Sciences*, vol. 5, pp. 1530–1536, 2010.
- [10] S. K. Mitra, "Common solutions to a pair of linear matrix equations $A_1XB_1 = C_1, A_2XB_2 = C_2$," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 74, pp. 213–216, 1973, doi: [10.1017/S030500410004799X](https://doi.org/10.1017/S030500410004799X).
- [11] S. K. Mitra, "The matrix equations $AX = C, XB = D$," *Linear Algebra and its Applications*, vol. 59, pp. 171–181, 1984, doi: [10.1016/0024-3795\(84\)90166-6](https://doi.org/10.1016/0024-3795(84)90166-6).
- [12] E. Pennestrì and R. Stefanelli, "Linear algebra and numerical algorithms using dual numbers," *Multibody System Dynamics*, vol. 18, pp. 323–344, 2007, doi: [10.1007/s11044-007-9088-9](https://doi.org/10.1007/s11044-007-9088-9).
- [13] E. Pennestrì and P. P. Valentini, "Linear dual algebra algorithms and their application to kinematics," *Computational Methods in Applied Sciences*, vol. 12, pp. 207–229, 2009.
- [14] E. Pennestrì, P. P. Valentini, G. Figliolini, and J. Angeles, "Dual Cayley-Klein parameters and mobius transform: theory and applications," *Mechanism and Machine Theory*, vol. 106, pp. 50–67, 2016.
- [15] R. Penrose, "A generalized inverse for matrices," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 51, pp. 406–413, 1955, doi: [10.1017/S0305004100030401](https://doi.org/10.1017/S0305004100030401).
- [16] A. Perez and J. M. McCarthy, "Bennett's linkage and the cylinder," *Mechanism and Machine Theory*, vol. 37, pp. 1245–1260, 2002, doi: [10.1016/s0094-114x\(02\)00055-1](https://doi.org/10.1016/s0094-114x(02)00055-1).

- [17] E. Study, *Geometrie der Dynamen*. Germany: Leipzig, 1903.
- [18] Y. Tian and H. Wang, "Relations between least-squares and least-rank solutions of the matrix equation $AXB = C$," *Applied Mathematics and Computation*, vol. 219, pp. 10 293–10 301, 2013, doi: [10.1016/j.amc.2013.03.137](https://doi.org/10.1016/j.amc.2013.03.137).
- [19] F. E. Udwardia, "Dual generalized inverses and their use in solving systems of linear dual equations," *Mechanism and Machine Theory*, vol. 156, 104158, 2021, doi: [10.1016/j.mechmachtheory.2020.104158](https://doi.org/10.1016/j.mechmachtheory.2020.104158).
- [20] F. E. Udwardia, E. Pennestri, and D. Falco, "Do all dual matrices have dual Moore-Penrose generalized inverses," *Mechanism and Machine Theory*, vol. 151, 103878, 2020, doi: [10.1016/j.mechmachtheory.2020.103878](https://doi.org/10.1016/j.mechmachtheory.2020.103878).
- [21] G. Wang, W. Li, C. Jiang, D. Zhu, H. Xie, X. Liu, and H. Ding, "Simultaneous calibration of multicoordinates for a dual-robot system by solving the $AXB = YCZ$ problem," *IEEE Transactions on Robotics*, vol. 37, 2021, doi: [10.1109/TRO.2020.3043688](https://doi.org/10.1109/TRO.2020.3043688).
- [22] Y. Yuan and H. Dai, "Generalized reflexive solutions of the matrix equation $AXB = D$ and an associated optimal approximation problem," *Computers and Mathematics with Applications*, vol. 56, pp. 1643–1649, 2008, doi: [10.1016/j.camwa.2008.03.015](https://doi.org/10.1016/j.camwa.2008.03.015).
- [23] Y. Yuan and K. Zuo, "The Re-nonnegative definite and Re-positive definite solutions to the matrix equation $AXB = D$," *Applied Mathematics and Computation*, vol. 256, pp. 905–912, 2015, doi: [10.1016/j.amc.2015.01.098](https://doi.org/10.1016/j.amc.2015.01.098).
- [24] K. Ziętak, "The l_p -solution of the linear matrix equation $AX + YB = C$," *Computing*, vol. 32, pp. 153–162, 1984.
- [25] K. Ziętak, "The Chebyshev solution of the linear matrix equation $AX + YB = C$," *Numerische Mathematik*, vol. 46, pp. 455–478, 1985, doi: [10.1007/BF01389497](https://doi.org/10.1007/BF01389497).

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