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ŁUKASIEWICZ FUZZY SUB-HOOPS

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Abstract. With the purpose of applying the Łukasiewicz fuzzy set to sub-hoop in hoops, the concept of Łukasiewicz fuzzy sub-hoop is introduced, and its properties are investigated. The relationship between fuzzy sub-hoop and Łukasiewicz fuzzy sub-hoop are discussed. Conditions for Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy sub-hoop are provided, and characterizations of Łukasiewicz fuzzy sub-hoop are displayed. Conditions under which $\in$-set, q-set, and O-set can be sub-hoops are investigated.

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1. INTRODUCTION

Hoops, which are introduced by Bosbach and Bosbach (see [6, 7]), are a nice algebraic structure to research the many-valued logical system whose propositional value is given in a lattice. Several properties of hoops are displayed in [1–3, 5, 8, 12, 13]. Substructures in hoops are a very important factor in studying hoops, and studies have been conducted on various types of substructures. The sub-hoop is the most basic substructure of hoops. The fuzzy set acts as a bridge so that algebra theory can be applied to applied sciences. Various kinds of fuzzy sets have been used in the study of substructures such as ideals in BCK/BCI-algebras Borzooei et al. [4] studied fuzzy sub-hoops. Łukasiewicz logic is the logic of the Łukasiewicz t-norm, and it is a non-classical and many-valued logic. It was originally defined in the early 20th century by Jan Łukasiewicz as a three-valued logic. Using the idea of Łukasiewicz t-norm, Y. B. Jun constructed the concept of Łukasiewicz fuzzy sets based on a given fuzzy set and applied it to BCK-algebras and BCI-algebras (see [9, 11]).

For the purpose of applying the Łukasiewicz fuzzy set to the sub-hoop in hoops, we introduce the concept of Łukasiewicz fuzzy sub-hoop and study its properties. We discuss the relationship between Łukasiewicz fuzzy sub-hoop and fuzzy sub-hoop. We provide conditions for Łukasiewicz fuzzy set to be a Łukasiewicz fuzzy sub-hoop. We discuss characterizations of Łukasiewicz fuzzy sub-hoop, and explore the conditions under which $\in$-set, q-set, and O-set can be sub-hoops.

2. Preliminaries

2.1. *Basic concepts about hoops*

By a *hoop* we mean an algebra $(H,\odot,\to,1)$ in which $(H,\odot,1)$ is a commutative monoid and the following assertions are valid.

(H1) $(\forall a\in H)(a\to a=1)$,
(H2) $(\forall a,b\in H)(a\odot(a\to b)=b\odot(b\to a))$,
(H3) $(\forall a,b,c\in H)(a\to(b\to c)=(a\odot b)\to c)$.

On hoop H, we define $a\leq b$ if and only if $a\to b=1$. It is easy to see that $\leq$ is a partial order relation on H.

By a *sub-hoop* of a hoop H we mean a subset F of H which satisfies the condition:

$$(\forall a,b\in H)(a,b\in F\ \Rightarrow\ a\odot b\in F,\ a\to b\in F).$$

Note that every non-empty sub-hoop contains the element 1.

Every hoop H satisfies the following conditions (see [6,7]).

$$(\forall a,b\in H)(a\odot b\leq c\ \Leftrightarrow\ a\leq b\to c).$$
$$(\forall a,b\in H)(a\odot b\leq a,b).$$
$$(\forall a,b\in H)(a\leq b\to a).$$
$$(\forall a\in H)(a\to 1=1).$$
$$(\forall a\in H)(1\to a=a). \tag{2.1}$$

2.2. *Basic concepts about (Łukasiewicz) fuzzy set informations*

A fuzzy set λ in a set H of the form

$$\lambda(b):=\begin{cases} t\in(0,1] & \text{if } b=a,\\ 0 & \text{if } b\neq a,\end{cases}$$

is said to be a *fuzzy point* with support a and value t and is denoted by $\langle a/t\rangle$.

For a fuzzy set λ in a set H, we say that a fuzzy point $\langle a/t\rangle$ is

(i) *contained* in λ, denoted by $\langle a/t\rangle\in\lambda$, (see [10]) if $\lambda(a)\geq t$.
(ii) *quasi-coincident* with λ, denoted by $\langle a/t\rangle\,q\,\lambda$, (see [10]) if $\lambda(a)+t>1$.

If $\langle a/t\rangle\,\alpha\,\lambda$ is not established for $\alpha\in\{\in,q\}$, it is denoted by $\langle a/t\rangle\,\overline{\alpha}\,\lambda$.

Definition 1 ([4, Definition 1])**.** A fuzzy set λ in H is called a *fuzzy sub-hoop* of H (see [4]) if it satisfies:

$$(\forall a,b\in H)\left(\begin{cases}\lambda(a\odot b)\geq\min\{\lambda(a),\lambda(b)\}\\ \lambda(a\to b)\geq\min\{\lambda(a),\lambda(b)\}\end{cases}\right).$$

Definition 2 ([9, Definition 3.1])**.** Let λ be a fuzzy set in a set H and let $\delta \in (0,1)$. A function

$$Ł_\lambda^\delta : H \to [0,1],\ x \mapsto \max\{0, \lambda(x) + \delta - 1\}$$

is called the *Łukasiewicz fuzzy set* of λ in H.

Let λ be a fuzzy set in a set H. For the Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of λ in H and $t \in (0,1]$, consider the sets

$$(Ł_\lambda^\delta, t)_\in := \{x \in H \mid \langle x/t \rangle \in Ł_\lambda^\delta\} \text{ and } (Ł_\lambda^\delta, t)_q := \{x \in H \mid \langle x/t \rangle\, q\, Ł_\lambda^\delta\},$$

which are called the $\in$-*set* and q-*set*, respectively, of $Ł_\lambda^\delta$ (with value t). Also, consider a set:

$$O(Ł_\lambda^\delta) := \{x \in H \mid Ł_\lambda^\delta(x) > 0\},$$

which is called the O-*set* of $Ł_\lambda^\delta$. It is observed that

$$O(Ł_\lambda^\delta) = \{x \in H \mid \lambda(x) + \delta - 1 > 0\}.$$

3. ŁUKASIEWICZ FUZZY SUB-HOOPS

In what follows, let H denote a hoop unless otherwise specified.

Definition 3. Let λ be a fuzzy set in H and δ an element of $(0,1)$. Then its Łukasiewicz fuzzy set $Ł_\lambda^\delta$ in H is called a *Łukasiewicz fuzzy sub-hoop* of H if it satisfies:

$$(\forall x, y \in H)(\forall t_a, t_b \in (0,1]) \left(\begin{cases} \langle x/t_a \rangle \in Ł_\lambda^\delta,\ \langle y/t_b \rangle \in Ł_\lambda^\delta \\ \Rightarrow \begin{cases} \langle (x \odot y)/\min\{t_a, t_b\} \rangle \in Ł_\lambda^\delta \\ \langle (x \to y)/\min\{t_a, t_b\} \rangle \in Ł_\lambda^\delta \end{cases} \end{cases} \right). \tag{3.1}$$

Example 1. Let $H = \{0, a, b, c, d, 1\}$ be a set with binary operations "$\odot$" and "$\to$" in Table 1 and Table 2, respectively.

TABLE 1. Cayley table for the binary operation "$\odot$"

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	d	0	d	a
b	0	d	c	c	0	b
c	0	0	c	c	0	c
d	0	d	0	0	0	d
1	0	a	b	c	d	1

TABLE 2. Cayley table for the binary operation "$\rightarrow$"

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	c	1	b	c	b	1
b	d	a	1	b	a	1
c	a	a	1	1	a	1
d	b	1	1	b	1	1
1	0	a	b	c	d	1

Then $(H,\odot,\rightarrow,1)$ is a hoop. Define a fuzzy set λ in H as follows:

$$\lambda\colon H\rightarrow[0,1],\ x\mapsto\begin{cases}0.58 & \text{if } x=0,\\ 0.77 & \text{if } x=a,\\ 0.37 & \text{if } x=b,\\ 0.58 & \text{if } x=c,\\ 0.37 & \text{if } x=d,\\ 0.83 & \text{if } x=1.\end{cases}$$

Given $\delta:=0.58$, the Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of λ in H is given as follows:

$$Ł_\lambda^\delta\colon H\rightarrow[0,1],\ x\mapsto\begin{cases}0.16 & \text{if } x=0,\\ 0.35 & \text{if } x=a,\\ 0.00 & \text{if } x=b,\\ 0.16 & \text{if } x=c,\\ 0.00 & \text{if } x=d,\\ 0.41 & \text{if } x=1.\end{cases}$$

It is routine to verify that $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H.

Theorem 1. *If λ is a fuzzy sub-hoop of H, then its Łukasiewicz fuzzy set $Ł_\lambda^\delta$ in H is a Łukasiewicz fuzzy sub-hoop of H.*

Proof. Assume that λ is a fuzzy sub-hoop of H. Let $x,y\in H$ and $t_a,t_b\in(0,1]$ be such that $\langle x/t_a\rangle\in Ł_\lambda^\delta$ and $\langle y/t_b\rangle\in Ł_\lambda^\delta$. Then $Ł_\lambda^\delta(x)\geq t_a$ and $Ł_\lambda^\delta(y)\geq t_b$, so

$$\begin{aligned}Ł_\lambda^\delta(x\odot y)&=\max\{0,\lambda(x\odot y)+\delta-1\}\\ &\geq\max\{0,\min\{\lambda(x),\lambda(y)\}+\delta-1\}\\ &=\max\{0,\min\{\lambda(x)+\delta-1,\lambda(y)+\delta-1\}\}\\ &=\min\{\max\{0,\lambda(x)+\delta-1\},\max\{0,\lambda(y)+\delta-1\}\}\end{aligned}$$

$$= \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\} \geq \min\{t_a, t_b\}.$$

Hence $\langle (x \odot y)/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$. The similar way induces $\langle (x \to y)/\min\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$. Therefore, $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H. □

The following example shows that the converse of Theorem 1 may not be true.

Example 2. Consider the hoop $H = \{0, a, b, c, d, 1\}$ in Example 1 and define a fuzzy set λ in H as follows:

$$\lambda \colon H \to [0,1],\ x \mapsto \begin{cases} 0.22 & \text{if } x = 0, \\ 0.17 & \text{if } x = a, \\ 0.85 & \text{if } x = b, \\ 0.90 & \text{if } x = c, \\ 0.27 & \text{if } x = d, \\ 0.96 & \text{if } x = 1. \end{cases}$$

Given $\delta := 0.73$, the Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of λ in H is given as follows:

$$Ł_\lambda^\delta \colon H \to [0,1],\ x \mapsto \begin{cases} 0.69 & \text{if } x = 1, \\ 0.58 & \text{if } x = b, \\ 0.63 & \text{if } x = c, \\ 0.00 & \text{if } x \in \{0, a, d\}. \end{cases}$$

It is routine to verify that $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H. But λ is not a fuzzy sub-hoop of H since $\lambda(d \odot c) = \lambda(0) = 0.22 \ngeq 0.27 = \min\{\lambda(d), \lambda(c)\}$.

Lemma 1. *If λ is a fuzzy sub-hoop of H, then its Łukasiewicz fuzzy set $Ł_\lambda^\delta$ satisfies:*

$$(\forall x \in H)(\forall t \in (0,1]) \left(\langle x/t\rangle \in Ł_\lambda^\delta \ \Rightarrow\ \langle 1/t\rangle \in Ł_\lambda^\delta \right). \tag{3.2}$$

Proof. If λ is a fuzzy sub-hoop of H, then its Łukasiewicz fuzzy set $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H by Theorem 1. Hence the combination of (H1) and (3.1) induces (3.2). □

Remark 1. Let λ be a fuzzy sub-hoop of H, and assume that the condition (3.2) is valid. Since $\langle x/Ł_\lambda^\delta(x)\rangle \in Ł_\lambda^\delta$ for all $x \in H$, we have $\langle 1/Ł_\lambda^\delta(x)\rangle \in Ł_\lambda^\delta$ by (3.2). Hence $Ł_\lambda^\delta(1) \geq Ł_\lambda^\delta(x)$ for all $x \in H$. Conversely, suppose $Ł_\lambda^\delta$ satisfies $Ł_\lambda^\delta(1) \geq Ł_\lambda^\delta(x)$ for all $x \in H$. Let $x \in H$ and $t \in (0,1]$ be such that $\langle x/t\rangle \in Ł_\lambda^\delta$. Then $Ł_\lambda^\delta(1) \geq Ł_\lambda^\delta(x) \geq t$, that is, $\langle 1/t\rangle \in Ł_\lambda^\delta$. This shows that if λ is a fuzzy sub-hoop of H, then the condition (3.2) is equivalent to the condition below:

$$(\forall x \in H) \left(Ł_\lambda^\delta(1) \geq Ł_\lambda^\delta(x) \right).$$

Proposition 1. *For a Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of λ in H, if λ is a fuzzy sub-hoop of H, then the following assertions are equivalent.*

$$(\forall x \in H)\left(Ł_\lambda^\delta(x) = Ł_\lambda^\delta(1)\right). \tag{3.3}$$

$$(\forall x,y \in H)(\forall t \in (0,1])\left(\left\{\langle y/t\rangle \in Ł_\lambda^\delta \Rightarrow \begin{cases}\langle (y\odot x)/t\rangle \in Ł_\lambda^\delta \\ \langle (y\to x)/t\rangle \in Ł_\lambda^\delta\end{cases}\right.\right). \tag{3.4}$$

Proof. Let λ be a fuzzy sub-hoop of H. Then its Łukasiewicz fuzzy set $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H (see Theorem 1). Assume that (3.3) is valid. Let $x,y \in H$ and $t \in (0,1]$ be such that $\langle y/t\rangle \in Ł_\lambda^\delta$. Then $\langle 1/t\rangle \in Ł_\lambda^\delta$ by Lemma 1. Since $\langle x/Ł_\lambda^\delta(x)\rangle \in Ł_\lambda^\delta$ and $\langle y/Ł_\lambda^\delta(y)\rangle \in Ł_\lambda^\delta$ for all $x,y \in H$, we have

$$\langle (y\odot x)/\min\{Ł_\lambda^\delta(y), Ł_\lambda^\delta(x)\}\rangle \in Ł_\lambda^\delta, \qquad \langle (y\to x)/\min\{Ł_\lambda^\delta(y), Ł_\lambda^\delta(x)\}\rangle \in Ł_\lambda^\delta$$

by (3.1). It follows from Remark 1 and (3.3) that

$$Ł_\lambda^\delta(y\odot x) \geq \min\{Ł_\lambda^\delta(y), Ł_\lambda^\delta(x)\} = \min\{Ł_\lambda^\delta(1), Ł_\lambda^\delta(x)\} = Ł_\lambda^\delta(x) \geq t$$

and

$$Ł_\lambda^\delta(y\to x) \geq \min\{Ł_\lambda^\delta(y), Ł_\lambda^\delta(x)\} = \min\{Ł_\lambda^\delta(1), Ł_\lambda^\delta(x)\} = Ł_\lambda^\delta(x) \geq t.$$

Hence $\langle (y\odot x)/t\rangle \in Ł_\lambda^\delta$ and $\langle (y\to x)/t\rangle \in Ł_\lambda^\delta$ for all $x,y \in H$ and $t \in (0,1]$.

Conversely, suppose that (3.4) is valid. Since $\langle 1/Ł_\lambda^\delta(1)\rangle \in Ł_\lambda^\delta$, it follows from (2.1) and (3.4) that $\langle x/Ł_\lambda^\delta(1)\rangle = \langle (1\to x)/Ł_\lambda^\delta(1)\rangle \in Ł_\lambda^\delta$, that is, $Ł_\lambda^\delta(x) \geq Ł_\lambda^\delta(1)$ for all $x \in H$. Combining this with Remark 1 leads to $Ł_\lambda^\delta(x) = Ł_\lambda^\delta(1)$ for all $x \in H$. □

Theorem 2. *Let λ be a fuzzy set in H. If the Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of λ in H satisfies:*

$$(\forall x,y,z \in H)(\forall t_b,t_c \in (0,1])\left(\begin{cases} z\leq x,\ \langle y/t_b\rangle \in Ł_\lambda^\delta,\ \langle z/t_c\rangle \in Ł_\lambda^\delta \\ \Rightarrow \begin{cases}\langle (x\odot y)/\min\{t_b,t_c\}\rangle \in Ł_\lambda^\delta \\ \langle (x\to y)/\min\{t_b,t_c\}\rangle \in Ł_\lambda^\delta\end{cases}\end{cases}\right), \tag{3.5}$$

then $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H.

Proof. Let $x,y \in H$ and $t_a,t_b \in (0,1]$ be such that $\langle x/t_a\rangle \in Ł_\lambda^\delta$ and $\langle y/t_b\rangle \in Ł_\lambda^\delta$. Since $x \leq x$ for all $x \in H$, it follows from (3.5) that $\langle (x\odot y)/\min\{t_a,t_b\}\rangle \in Ł_\lambda^\delta$ and $\langle (x\to y)/\min\{t_a,t_b\}\rangle \in Ł_\lambda^\delta$. Therefore, $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H. □

Corollary 1. *Let λ be a fuzzy set in H. If the Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of λ in H satisfies:*

$$(\forall x,y,z \in H)(\forall t_b,t_c \in (0,1])\left(\begin{cases} z\leq x,\ \langle y/t_b\rangle \in Ł_\lambda^\delta,\ \langle z/t_c\rangle \in Ł_\lambda^\delta \\ \Rightarrow \begin{cases}\langle (x\odot y)/\max\{t_b,t_c\}\rangle \in Ł_\lambda^\delta \\ \langle (x\to y)/\max\{t_b,t_c\}\rangle \in Ł_\lambda^\delta\end{cases}\end{cases}\right),$$

then $Ł_\lambda^\delta$ *is a Łukasiewicz fuzzy sub-hoop of* H.

Theorem 3. *Let* λ *be a fuzzy set in* H. *If the Łukasiewicz fuzzy set* $Ł_\lambda^\delta$ *of* λ *in* H *satisfies:*

$$(\forall x,y,z \in H)(\forall t_b,t_c \in (0,0.5]) \left(\begin{cases} z \leq x,\ \langle y/t_b \rangle \in Ł_\lambda^\delta,\ \langle z/t_c \rangle \in Ł_\lambda^\delta \\ \quad \Rightarrow \begin{cases} \langle (x \odot y)/\min\{t_b,t_c\} \rangle\, q\, Ł_\lambda^\delta \\ \langle (x \rightarrow y)/\min\{t_b,t_c\} \rangle\, q\, Ł_\lambda^\delta \end{cases} \end{cases} \right), \tag{3.6}$$

then $Ł_\lambda^\delta$ *is a Łukasiewicz fuzzy sub-hoop of* H.

Proof. Let $x,y \in H$ and $t_a,t_b \in (0,0.5] \subseteq (0,1]$ be such that $\langle x/t_a \rangle \in Ł_\lambda^\delta$ and $\langle y/t_b \rangle \in Ł_\lambda^\delta$. Since $x \leq x$ for all $x \in H$, we have

$$\langle (x \odot y)/\min\{t_a,t_b\} \rangle\, q\, Ł_\lambda^\delta \qquad \langle (x \rightarrow y)/\min\{t_a,t_b\} \rangle\, q\, Ł_\lambda^\delta$$

by (3.6). Since $\min\{t_a,t_b\} \leq 0.5$, it follows that

$$Ł_\lambda^\delta(x \odot y) > 1 - \min\{t_a,t_b\} \geq \min\{t_a,t_b\}$$

and

$$Ł_\lambda^\delta(x \rightarrow y) > 1 - \min\{t_a,t_b\} \geq \min\{t_a,t_b\}.$$

This shows that $\langle (x \odot y)/\min\{t_a,t_b\} \rangle \in Ł_\lambda^\delta$ and $\langle (x \rightarrow y)/\min\{t_a,t_b\} \rangle \in Ł_\lambda^\delta$. Therefore, $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H. □

Corollary 2. *Let* λ *be a fuzzy set in* H. *If the Łukasiewicz fuzzy set* $Ł_\lambda^\delta$ *of* λ *in* H *satisfies:*

$$(\forall x,y,z \in H)(\forall t_b,t_c \in (0,0.5]) \left(\begin{cases} z \leq x,\ \langle y/t_b \rangle \in Ł_\lambda^\delta,\ \langle z/t_c \rangle \in Ł_\lambda^\delta \\ \quad \Rightarrow \begin{cases} \langle (x \odot y)/\max\{t_b,t_c\} \rangle\, q\, Ł_\lambda^\delta \\ \langle (x \rightarrow y)/\max\{t_b,t_c\} \rangle\, q\, Ł_\lambda^\delta \end{cases} \end{cases} \right),$$

then $Ł_\lambda^\delta$ *is a Łukasiewicz fuzzy sub-hoop of* H.

Theorem 4. *Let* λ *be a fuzzy set in* H. *If the Łukasiewicz fuzzy set* $Ł_\lambda^\delta$ *of* λ *in* H *satisfies:*

$$(\forall x,y,z \in H)(\forall t_b,t_c \in (0.5,1]) \left(\begin{cases} z \leq x,\ \langle y/t_b \rangle\, q\, Ł_\lambda^\delta,\ \langle z/t_c \rangle\, q\, Ł_\lambda^\delta \\ \quad \Rightarrow \begin{cases} \langle (x \odot y)/\min\{t_b,t_c\} \rangle \in Ł_\lambda^\delta \\ \langle (x \rightarrow y)/\min\{t_b,t_c\} \rangle \in Ł_\lambda^\delta \end{cases} \end{cases} \right), \tag{3.7}$$

then $Ł_\lambda^\delta$ *is a Łukasiewicz fuzzy sub-hoop of* H.

Proof. Let $x,y\in H$ and $t_a,t_b\in(0.5,1]\subseteq(0,1]$ be such that $\langle x/t_a\rangle\in Ł_\lambda^\delta$ and $\langle y/t_b\rangle\in Ł_\lambda^\delta$. Then $Ł_\lambda^\delta(x)\geq t_a>1-t_a$ and $Ł_\lambda^\delta(y)\geq t_b>1-t_b$, that is, $\langle x/t_a\rangle\, q\, Ł_\lambda^\delta$ and $\langle y/t_b\rangle\, q\, Ł_\lambda^\delta$. Since $x\leq x$ for all $x\in H$, it follows from (3.7) that $\langle(x\odot y)/\min\{t_a,t_b\}\rangle\in Ł_\lambda^\delta$ and $\langle(x\to y)/\min\{t_a,t_b\}\rangle\in Ł_\lambda^\delta$. Hence $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H. □

Corollary 3. *Let λ be a fuzzy set in H. If the Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of λ in H satisfies:*

$$(\forall x,y,z\in H)(\forall t_b,t_c\in(0.5,1])\left(\left\{\begin{array}{l} z\leq x,\ \langle y/t_b\rangle\, q\, Ł_\lambda^\delta,\ \langle z/t_c\rangle\, q\, Ł_\lambda^\delta \\ \Rightarrow\left\{\begin{array}{l}\langle(x\odot y)/\max\{t_b,t_c\}\rangle\in Ł_\lambda^\delta\\ \langle(x\to y)/\max\{t_b,t_c\}\rangle\in Ł_\lambda^\delta\end{array}\right.\end{array}\right.\right),$$

then $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H.

We explore the conditions under which the $\in$-set, q-set, and O-set of Łukasiewicz fuzzy set can be sub-hoops.

Theorem 5. *Let $Ł_\lambda^\delta$ be a Łukasiewicz fuzzy set of a fuzzy set λ in H. Then the $\in$-set $(Ł_\lambda^\delta,t)_\in$ of $Ł_\lambda^\delta$ is a sub-hoop of H for all $t\in(0.5,1]$ if and only if $\min\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}$ is less than or equal to*

$$\max\{Ł_\lambda^\delta(x\odot y),0.5\} \text{ and } \max\{Ł_\lambda^\delta(x\to y),0.5\}.$$

Proof. Assume that the $\in$-set $(Ł_\lambda^\delta,t)_\in$ of $Ł_\lambda^\delta$ is a sub-hoop of H for all $t\in(0.5,1]$. If $\min\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}>\max\{Ł_\lambda^\delta(x\odot y),0.5\}$ for some $x,y\in H$, then

$$t:=\min\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}\in(0.5,1],\qquad \langle x/t\rangle\in Ł_\lambda^\delta,\qquad \langle y/t\rangle\in Ł_\lambda^\delta.$$

Hence $x,y\in(Ł_\lambda^\delta,t)_\in$, and so $x\odot y\in(Ł_\lambda^\delta,t)_\in$. But $\langle(x\odot y)/t\rangle\,\overline{\in}\,Ł_\lambda^\delta$ implies $x\odot y\notin(Ł_\lambda^\delta,t)_\in$, a contradiction. Hence $\min\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}\leq\max\{Ł_\lambda^\delta(x\odot y),0.5\}$ for all $x,y\in H$. In the same way, we can verify $\min\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}\leq\max\{Ł_\lambda^\delta(x\to y),0.5\}$ for all $x,y\in H$.

Conversely suppose that

$$\begin{aligned}\min\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}&\leq\max\{Ł_\lambda^\delta(x\to y),0.5\},\\ \min\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}&\leq\max\{Ł_\lambda^\delta(x\odot y),0.5\}\end{aligned}\tag{3.8}$$

for all $x,y\in H$. Let $t\in(0.5,1]$ and $x,y\in H$ be such that $x\in(Ł_\lambda^\delta,t)_\in$ and $y\in(Ł_\lambda^\delta,t)_\in$. Then $Ł_\lambda^\delta(x)\geq t$ and $Ł_\lambda^\delta(y)\geq t$, which imply from (3.8) that

$$0.5<t\leq\min\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}\leq\max\{Ł_\lambda^\delta(x\odot y),0.5\}$$

and

$$0.5<t\leq\min\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}\leq\max\{Ł_\lambda^\delta(x\to y),0.5\}.$$

Hense $\langle (x\odot y)/t\rangle \in Ł_{\lambda}^{\delta}$ and $\langle (x\rightarrow y)/t\rangle \in Ł_{\lambda}^{\delta}$, i.e., $x\odot y\in (Ł_{\lambda}^{\delta},t)_{\in}$ and $x\rightarrow y\in (Ł_{\lambda}^{\delta},t)_{\in}$. Therefore $(Ł_{\lambda}^{\delta},t)_{\in}$ is a sub-hoop of H for $t\in(0.5,1]$. □

Theorem 6. *Let $Ł_{\lambda}^{\delta}$ be a Łukasiewicz fuzzy set of a fuzzy set λ in H. If λ is a fuzzy sub-hoop of H, then the q-set $(Ł_{\lambda}^{\delta},t)_q$ of $Ł_{\lambda}^{\delta}$ is a sub-hoop of H for all $t\in(0,1]$.*

Proof. If λ is a fuzzy sub-hoop of H, then $Ł_{\lambda}^{\delta}$ is a Łukasiewicz fuzzy sub-hoop of H (see Theorem 1). Let $t\in(0,1]$ and $x,y\in(Ł_{\lambda}^{\delta},t)_q$. Then $\langle x/t\rangle\, q\, Ł_{\lambda}^{\delta}$ and $\langle y/t\rangle\, q\, Ł_{\lambda}^{\delta}$, that is, $Ł_{\lambda}^{\delta}(x)+t>1$ and $Ł_{\lambda}^{\delta}(y)+t>1$. Since $\langle x/Ł_{\lambda}^{\delta}(x)\rangle\in Ł_{\lambda}^{\delta}$ and $\langle y/Ł_{\lambda}^{\delta}(y)\rangle\in Ł_{\lambda}^{\delta}$ for all $x,y\in H$, we have

$$\langle (x\odot y)/\min\{Ł_{\lambda}^{\delta}(x),Ł_{\lambda}^{\delta}(y)\}\rangle\in Ł_{\lambda}^{\delta},\qquad \langle (x\rightarrow y)/\min\{Ł_{\lambda}^{\delta}(x),Ł_{\lambda}^{\delta}(y)\}\rangle\in Ł_{\lambda}^{\delta}$$

by (3.1). It follows that

$$Ł_{\lambda}^{\delta}(x\odot y)+t\geq \min\{Ł_{\lambda}^{\delta}(x),Ł_{\lambda}^{\delta}(y)\}+t=\min\{Ł_{\lambda}^{\delta}(x)+t,Ł_{\lambda}^{\delta}(y)+t\}>1$$

and

$$Ł_{\lambda}^{\delta}(x\rightarrow y)+t\geq \min\{Ł_{\lambda}^{\delta}(x),Ł_{\lambda}^{\delta}(y)\}+t=\min\{Ł_{\lambda}^{\delta}(x)+t,Ł_{\lambda}^{\delta}(y)+t\}>1.$$

Hence $\langle (x\odot y)/t\rangle\, q\, Ł_{\lambda}^{\delta}$ and $\langle (x\rightarrow y)/t\rangle\, q\, Ł_{\lambda}^{\delta}$, that is, $x\odot y\in(Ł_{\lambda}^{\delta},t)_q$ and $x\rightarrow y\in(Ł_{\lambda}^{\delta},t)_q$. Thus, $(Ł_{\lambda}^{\delta},t)_q$ is a sub-hoop of H. □

Proposition 2. *Let $Ł_{\lambda}^{\delta}$ be a Łukasiewicz fuzzy set of a fuzzy set λ in H.*

(i) *If the q-set $(Ł_{\lambda}^{\delta},t)_q$ is a sub-hoop of H for all $t\in(0,0.5]$, then $Ł_{\lambda}^{\delta}$ satisfies:*

$$(\forall x,y\in H)(\forall t_a,t_b\in(0,0.5])\left(\begin{cases}\langle x/t_a\rangle\, q\, Ł_{\lambda}^{\delta},\ \langle y/t_b\rangle\, q\, Ł_{\lambda}^{\delta}\\ \Rightarrow\begin{cases}\langle (x\odot y)/\max\{t_a,t_b\}\rangle\in Ł_{\lambda}^{\delta}\\ \langle (x\rightarrow y)/\max\{t_a,t_b\}\rangle\in Ł_{\lambda}^{\delta}\end{cases}\end{cases}\right).$$

(ii) *If the $\in$-set $(Ł_{\lambda}^{\delta},t)_{\in}$ is a sub-hoop of H for all $t\in(0.5,1]$, then $Ł_{\lambda}^{\delta}$ satisfies:*

$$(\forall x,y\in H)(\forall t_a,t_b\in(0.5,1])\left(\begin{cases}\langle x/t_a\rangle\in Ł_{\lambda}^{\delta},\ \langle y/t_b\rangle\in Ł_{\lambda}^{\delta}\\ \Rightarrow\begin{cases}\langle (x\odot y)/\min\{t_a,t_b\}\rangle\, q\, Ł_{\lambda}^{\delta}\\ \langle (x\rightarrow y)/\min\{t_a,t_b\}\rangle\, q\, Ł_{\lambda}^{\delta}\end{cases}\end{cases}\right).$$

Proof.

(i) Assume that $(Ł_{\lambda}^{\delta},t)_q$ is a sub-hoop of H for all $t\in(0,0.5]$. Let $x,y\in H$ and $t_a,t_b\in(0,0.5]$ be such that $\langle x/t_a\rangle\, q\, Ł_{\lambda}^{\delta}$ and $\langle y/t_b\rangle\, q\, Ł_{\lambda}^{\delta}$. Then $x\in(Ł_{\lambda}^{\delta},t_a)_q\subseteq(Ł_{\lambda}^{\delta},\max\{t_a,t_b\})_q$ and $y\in(Ł_{\lambda}^{\delta},t_b)_q\subseteq(Ł_{\lambda}^{\delta},\max\{t_a,t_b\})_q$. Hence $x\odot y\in(Ł_{\lambda}^{\delta},\max\{t_a,t_b\})_q$ and $x\rightarrow y\in(Ł_{\lambda}^{\delta},\max\{t_a,t_b\})_q$. Since $\max\{t_a,t_b\}\leq 0.5$, it follows that

$$Ł_{\lambda}^{\delta}(x\odot y)>1-\max\{t_a,t_b\}\geq\max\{t_a,t_b\}$$

and

$$Ł_\lambda^\delta(x \to y) > 1 - \max\{t_a, t_b\} \geq \max\{t_a, t_b\}.$$

Therefore, $\langle (x \odot y)/\max\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$ and $\langle (x \to y)/\max\{t_a, t_b\}\rangle \in Ł_\lambda^\delta$.

(ii) Assume that $(Ł_\lambda^\delta, t)_\in$ is a sub-hoop of H for all $t \in (0.5, 1]$. Let $x, y \in H$ and $t_a, t_b \in (0.5, 1]$ be such that $\langle x/t_a\rangle \in Ł_\lambda^\delta$ and $\langle y/t_b\rangle \in Ł_\lambda^\delta$. Then $Ł_\lambda^\delta(x) \geq t_a \geq \min\{t_a, t_b\}$ and $Ł_\lambda^\delta(y) \geq t_b \geq \min\{t_a, t_b\}$, which imply that

$$x \in (Ł_\lambda^\delta, \min\{t_a, t_b\})_\in, \qquad y \in (Ł_\lambda^\delta, \min\{t_a, t_b\})_\in.$$

Hence

$$x \odot y \in (Ł_\lambda^\delta, \min\{t_a, t_b\})_\in, \qquad x \to y \in (Ł_\lambda^\delta, \min\{t_a, t_b\})_\in.$$

Since $\min\{t_a, t_b\} > 0.5$, it follows that

$$Ł_\lambda^\delta(x \odot y) \geq \min\{t_a, t_b\} > 1 - \min\{t_a, t_b\}$$

and

$$Ł_\lambda^\delta(x \to y) \geq \min\{t_a, t_b\} > 1 - \min\{t_a, t_b\}$$

Therefore, $\langle (x \odot y)/\min\{t_a, t_b\}\rangle\, q\, Ł_\lambda^\delta$ and $\langle (x \to y)/\min\{t_a, t_b\}\rangle\, q\, Ł_\lambda^\delta$.

□

Theorem 7. *If $Ł_\lambda^\delta$ satisfies:*

$$(\forall x, y \in H)(\forall t \in (0, 0.5]) \left(\begin{cases} \langle x/t\rangle \in Ł_\lambda^\delta,\ \langle y/t\rangle \in Ł_\lambda^\delta \\ \Rightarrow \begin{cases} \langle (x \odot y)/t\rangle\, q\, Ł_\lambda^\delta \\ \langle (x \to y)/t\rangle\, q\, Ł_\lambda^\delta \end{cases} \end{cases} \right), \tag{3.9}$$

then the $\in$-set $(Ł_\lambda^\delta, t)_\in$ is a sub-hoop of H for all $t \in (0, 0.5]$.

Proof. Let $t \in (0, 0.5]$ and assume that $Ł_\lambda^\delta$ satisfies the condition (3.9). If $x, y \in (Ł_\lambda^\delta, t)_\in$, then $\langle x/t\rangle \in Ł_\lambda^\delta$ and $\langle y/t\rangle \in Ł_\lambda^\delta$, and thus $\langle (x \odot y)/t\rangle\, q\, Ł_\lambda^\delta$ and $\langle (x \to y)/t\rangle\, q\, Ł_\lambda^\delta$ by (3.9). Since $t \leq 0.5$, it follows that $Ł_\lambda^\delta(x \odot y) > 1 - t \geq t$ and $Ł_\lambda^\delta(x \to y) > 1 - t \geq t$. Thus $x \odot y \in (Ł_\lambda^\delta, t)_\in$ and $x \to y \in (Ł_\lambda^\delta, t)_\in$ which shows that $(Ł_\lambda^\delta, t)_\in$ is a sub-hoop of H for all $t \in (0, 0.5]$. □

Theorem 8. *If $Ł_\lambda^\delta$ satisfies:*

$$(\forall x, y \in H)(\forall t \in (0.5, 1]) \left(\begin{cases} \langle x/t\rangle\, q\, Ł_\lambda^\delta,\ \langle y/t\rangle\, q\, Ł_\lambda^\delta \\ \Rightarrow \begin{cases} \langle (x \odot y)/t\rangle \in Ł_\lambda^\delta \\ \langle (x \to y)/t\rangle \in Ł_\lambda^\delta \end{cases} \end{cases} \right), \tag{3.10}$$

then the q-set $(Ł_\lambda^\delta, t)_q$ is a sub-hoop of H for all $t \in (0.5, 1]$.

Proof. Suppose that $Ł_\lambda^\delta$ satisfies the condition (3.10) and let $t \in (0.5, 1]$ and $x, y \in H$ be such that $x, y \in (Ł_\lambda^\delta, t)_q$. Then $\langle x/t \rangle \, q \, Ł_\lambda^\delta$ and $\langle y/t \rangle \, q \, Ł_\lambda^\delta$. Hence $\langle (x \odot y)/t \rangle \in Ł_\lambda^\delta$ and $\langle (x \rightarrow y)/t \rangle \in Ł_\lambda^\delta$ by (3.10). Since $t > 0.5$, it follows that $Ł_\lambda^\delta(x \odot y) \geq t > 1 - t$ and $Ł_\lambda^\delta(x \rightarrow y) \geq t > 1 - t$, that is, $x \odot y \in (Ł_\lambda^\delta, t)_q$ and $x \rightarrow y \in (Ł_\lambda^\delta, t)_q$. Therefore, $(Ł_\lambda^\delta, t)_q$ is a sub-hoop of H for all $t \in (0.5, 1]$. □

Theorem 9. *Let $Ł_\lambda^\delta$ be a Łukasiewicz fuzzy set of a fuzzy set λ in H. If λ is a fuzzy sub-hoop of H, then the O-set $O(Ł_\lambda^\delta)$ of $Ł_\lambda^\delta$ is a sub-hoop of H.*

Proof. Let $x, y \in O(Ł_\lambda^\delta)$. Then $\lambda(x) + \delta - 1 > 0$ and $\lambda(y) + \delta - 1 > 0$. If λ is a fuzzy sub-hoop of H, then $Ł_\lambda^\delta$ is a Łukasiewicz fuzzy sub-hoop of H (see Theoem 1). It follows that

$$Ł_\lambda^\delta(x \odot y) \geq \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\} = \min\{\lambda(x) + \delta - 1, \lambda(y) + \delta - 1\} > 0$$

and

$$Ł_\lambda^\delta(x \rightarrow y) \geq \min\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\} = \min\{\lambda(x) + \delta - 1, \lambda(y) + \delta - 1\} > 0.$$

Hence $x \odot y \in O(Ł_\lambda^\delta)$ and $x \rightarrow y \in O(Ł_\lambda^\delta)$, and therefore $O(Ł_\lambda^\delta)$ is a sub-hoop of H. □

Theorem 10. *Let λ be a fuzzy set in H. If a Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of λ in H satisfies:*

$$(\forall x, y \in H)(\forall t_a, t_b \in (0, 1]) \left(\begin{cases} \langle x/t_a \rangle \in Ł_\lambda^\delta, \ \langle y/t_b \rangle \in Ł_\lambda^\delta \\ \Rightarrow \begin{cases} \langle (x \odot y)/\max\{t_a, t_b\} \rangle \, q \, Ł_\lambda^\delta \\ \langle (x \rightarrow y)/\max\{t_a, t_b\} \rangle \, q \, Ł_\lambda^\delta \end{cases} \end{cases} \right). \tag{3.11}$$

then the O-set $O(Ł_\lambda^\delta)$ of $Ł_\lambda^\delta$ is a sub-hoop of H.

Proof. Assume that $Ł_\lambda^\delta$ satisfies the condition (3.11). If $x, y \in O(Ł_\lambda^\delta)$, then $\lambda(x) + \delta - 1 > 0$ and $\lambda(y) + \delta - 1 > 0$. Since $[x/Ł_\lambda^\delta(x)] \in Ł_\lambda^\delta$ and $[y/Ł_\lambda^\delta(y)] \in Ł_\lambda^\delta$. it follows from (3.11) that

$$\langle (x \odot y)/\max\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\} \rangle \, q \, Ł_\lambda^\delta \text{ and } \langle (x \rightarrow y)/\max\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\} \rangle \, q \, Ł_\lambda^\delta.$$

If $x \odot y \notin O(Ł_\lambda^\delta)$ or $x \rightarrow y \notin O(Ł_\lambda^\delta)$, then $Ł_\lambda^\delta(x \odot y) = 0$ or $Ł_\lambda^\delta(x \rightarrow y) = 0$. Hence

$$\begin{aligned} &Ł_\lambda^\delta(x \odot y) + \max\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\} = \max\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\} \\ &= \max\{\max\{0, \lambda(x) + \delta - 1\}, \max\{0, \lambda(y) + \delta - 1\}\} \\ &= \max\{\lambda(x) + \delta - 1, \lambda(y) + \delta - 1\} \\ &= \max\{\lambda(x), \lambda(y)\} + \delta - 1 \\ &\leq 1 + \delta - 1 = \delta \leq 1, \end{aligned}$$

or

$$Ł_\lambda^\delta(x \rightarrow y) + \max\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\} = \max\{Ł_\lambda^\delta(x), Ł_\lambda^\delta(y)\}$$

$$\begin{aligned}&= \max\{\max\{0,\lambda(x)+\delta-1\},\max\{0,\lambda(y)+\delta-1\}\}\\&= \max\{\lambda(x)+\delta-1,\lambda(y)+\delta-1\}\\&= \max\{\lambda(x),\lambda(y)\}+\delta-1\\&\leq 1+\delta-1=\delta\leq 1,\end{aligned}$$

that is, $\langle (x\odot y)/\max\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}\rangle\,\overline{q}\,Ł_\lambda^\delta$ or $\langle (x\to y)/\max\{Ł_\lambda^\delta(x),Ł_\lambda^\delta(y)\}\rangle\,\overline{q}\,Ł_\lambda^\delta$. This is a contradiction, and thus $x\odot y, x\to y\in O(Ł_\lambda^\delta)$. Therefore, $O(Ł_\lambda^\delta)$ is a sub-hoop of H. □

Theorem 11. *Let λ be a fuzzy set in H. If a Łukasiewicz fuzzy set $Ł_\lambda^\delta$ of λ in H satisfies:*

$$(\forall x,y\in H)(\forall t_a,t_b\in(0,1])\left(\begin{cases}\langle x/t_a\rangle\, q\, Ł_\lambda^\delta,\ \langle y/t_b\rangle\, q\, Ł_\lambda^\delta\\ \Rightarrow \begin{cases}\langle (x\odot y)/\max\{t_a,t_b\}\rangle\in Ł_\lambda^\delta\\ \langle (x\to y)/\max\{t_a,t_b\}\rangle\in Ł_\lambda^\delta\end{cases}\end{cases}\right), \tag{3.12}$$

then the O-set $O(Ł_\lambda^\delta)$ of $Ł_\lambda^\delta$ is a sub-hoop of H.

Proof. Assume that $Ł_\lambda^\delta$ satisfies the condition (3.12). Let $x,y\in O(Ł_\lambda^\delta)$. Then $\lambda(x)+\delta-1>0$ and $\lambda(y)+\delta-1>0$. Hence

$$Ł_\lambda^\delta(x)+1=\max\{0,\lambda(x)+\delta-1\}+1=\lambda(x)+\delta-1+1=\lambda(x)+\delta>1$$

and

$$Ł_\lambda^\delta(y)+1=\max\{0,\lambda(y)+\delta-1\}+1=\lambda(y)+\delta-1+1=\lambda(y)+\delta>1,$$

that is, $\langle x/1\rangle\, q\, Ł_\lambda^\delta$ and $\langle y/1\rangle\, q\, Ł_\lambda^\delta$. It follows from (3.12) that

$$\begin{aligned}\langle (x\odot y)/1\rangle &= \langle (x\odot y)/\max\{1,1\}\rangle\in Ł_\lambda^\delta,\\ \langle (x\to y)/1\rangle &= \langle (x\to y)/\max\{1,1\}\rangle\in Ł_\lambda^\delta.\end{aligned}\tag{3.13}$$

If $x\odot y\notin O(Ł_\lambda^\delta)$ or $x\to y\notin O(Ł_\lambda^\delta)$, then $Ł_\lambda^\delta(x\odot y)=0<1$ or $Ł_\lambda^\delta(x\to y)=0<1$, that is, $\langle (x\odot y)/1\rangle\,\overline{\in}\,Ł_\lambda^\delta$ or $\langle (x\to y)/1\rangle\,\overline{\in}\,Ł_\lambda^\delta$. This is a contradiction to (3.13), and thus $x\odot y$, $x\to y\in O(Ł_\lambda^\delta)$. Therefore, $O(Ł_\lambda^\delta)$ is a sub-hoop of H. □

Theorem 12. *If a Łukasiewicz fuzzy set $Ł_\lambda^\delta$ in H satisfies:*

$$(\forall x,y\in H)\left(\begin{cases}\langle x/\delta\rangle\, q\, Ł_\lambda^\delta,\ \langle y/\delta\rangle\, q\, Ł_\lambda^\delta\ \Rightarrow \begin{cases}\langle (x\odot y)/\delta\rangle\in Ł_\lambda^\delta\\ \langle (x\to y)/\delta\rangle\in Ł_\lambda^\delta\end{cases}\end{cases}\right). \tag{3.14}$$

then the O-set of $Ł_\lambda^\delta$ is a sub-hoop of H.

Proof. Let $x,y\in O(Ł_\lambda^\delta)$. Then $\lambda(x)+\delta>1$ and $\lambda(y)+\delta>1$, i.e., $\langle x/\delta\rangle\, q\,\lambda$ and $\langle y/\delta\rangle\, q\,\lambda$. It follows from (3.14) that $\langle (x\odot y)/\delta\rangle\in Ł_\lambda^\delta$ and $\langle (x\to y)/\delta\rangle\in Ł_\lambda^\delta$, which shows $Ł_{\lambda}^{\delta}(x \odot y) \geq \delta > 0$ and $Ł_{\lambda}^{\delta}(x \to y) \geq \delta > 0$. Hence $x \odot y, x \to y \in O(Ł_{\lambda}^{\delta})$, and therefore $O(Ł_{\lambda}^{\delta})$ is a sub-hoop of H. □

Theorem 13. *Let $Ł_{\lambda}^{\delta}$ be a Łukasiewicz fuzzy set in H that satisfies:*

$$(\forall x, y \in H)(\forall t_a, t_b \in [\delta, 1]) \left(\begin{cases} \langle x/t_a \rangle \, q \, \lambda, \, \langle y/t_b \rangle \, q \, \lambda \\ \Rightarrow x \odot y, x \to y \in (Ł_{\lambda}^{\delta}, \delta)_{\in} \end{cases} \right). \tag{3.15}$$

Then the O-set of $Ł_{\lambda}^{\delta}$ is a sub-hoop of H.

Proof. Assume that $Ł_{\lambda}^{\delta}$ satisfies (3.15). If $x, y \in O(Ł_{\lambda}^{\delta})$, then $\lambda(x) + t_a \geq \lambda(x) + \delta > 1$ and $\lambda(y) + t_b \geq \lambda(y) + \delta > 1$. Thus $\langle x/t_a \rangle \, q \, \lambda$ and $\langle y/t_b \rangle \, q \, \lambda$. Using (3.15) leads to $x \odot y, x \to y \in (Ł_{\lambda}^{\delta}, \delta)_{\in}$. Hence $Ł_{\lambda}^{\delta}(x \odot y) \geq \delta > 0$ and $Ł_{\lambda}^{\delta}(x \to y) \geq \delta > 0$, that is, $x \odot y, x \to y \in O(Ł_{\lambda}^{\delta})$. Consequently, $O(Ł_{\lambda}^{\delta})$ is a sub-hoop of H. □

Corollary 4. *Let $Ł_{\lambda}^{\delta}$ be a Łukasiewicz fuzzy set in H that satisfies:*

$$(\forall x, y \in H) \left(\begin{cases} \langle x/\delta \rangle \, q \, \lambda, \, \langle y/\delta \rangle \, q \, \lambda \\ \Rightarrow x \odot y, x \to y \in (Ł_{\lambda}^{\delta}, \delta)_{\in} \end{cases} \right).$$

Then the O-set of $Ł_{\lambda}^{\delta}$ is a sub-hoop of H.

REFERENCES

[1] W. J. Blok and I. M. A. Ferreirim, "Hoops and their implicational reducts," pp. 219–230, 1993.
[2] W. J. Blok and I. M. A. Ferreirim, "On the structure of hoops," *Algebra Univers.*, vol. 43, no. 2-3, pp. 233–257, 2000, doi: 10.1007/s000120050156.
[3] R. A. Borzooei and M. Aaly Kologani, "Filter theory of hoop-algebras." *J. Adv. Res. Pure. Math.*, vol. 6, no. 4, pp. 72–86, 2014.
[4] R. A. Borzooei, M. M. Takallo, M. A. Kologani, and Y. B. Jun, "Fuzzy sub-hoops based on fuzzy points," *Miskolc Math. Notes*, vol. 22, no. 1, pp. 91–106, 2021, doi: 10.18514/MMN.2021.2849.
[5] R. A. Borzooei, H. R. Varasteh, and K. Borna, "Fundamental hoop-algebras." *Ratio Mathematica*, vol. 29, no. 2, pp. 25–40, 2015, doi: 10.23755/rm.v29i1.20.
[6] B. Bosbach, "Komplementäre Halbgruppen. Axiomatik und Arithmetik," *Fundam. Math.*, vol. 64, pp. 257–287, 1969, doi: 10.4064/fm-64-3-257-287.
[7] B. Bosbach, "Komplementäre Halbgruppen. Kongruenzen und Quotienten," *Fundam. Math.*, vol. 69, pp. 1–14, 1970, doi: 10.4064/fm-69-1-1-14.
[8] S. Ghorbani, "Localization of hoop-algebras." *J. Adv. Res. Pure. Math.*, vol. 5, no. 3, pp. 1–13, 2013.
[9] Y. B. Jun, "Łukasiewicz fuzzy subalgebras in BCK-algebras and BCI-algebras." *Ann. fuzzy math. inform.*, vol. 23, no. 2, pp. 213–223, 2022, doi: 10.30948/afmi.2022.23.2.213.
[10] P.-M. Pu and Y.-M. Liu, "Fuzzy topology. I: Neighborhood structure of a fuzzy point and Moore-Smith convergence," *J. Math. Anal. Appl.*, vol. 76, pp. 571–599, 1980, doi: 10.1016/0022-247X(80)90048-7.
[11] S. Z. Song and Y. B. Jun, "Łukasiewicz fuzzy positive implicative ideals in BCK-algebras." *J. Algebr. Hyperstruct. Log. Algebr.*, vol. 3, no. 2, pp. 47–58, 2022, doi: 10.52547/HATEF.JAHLA.3.2.4.

[12] R. Tayebi Khorami and A. Borumand Saeid, "Some unitary operators on hoop-algebras." *Fuzzy Inf. Eng.*, vol. 9, no. 2, pp. 205–223, 2017, doi: 10.1016/j.fiae.2017.06.005.
[13] M. A. Zander, "Decomposability of the finitely generated free hoop residuation algebra," *Stud. Log.*, vol. 88, no. 2, pp. 233–246, 2008, doi: 10.1007/s11225-008-9103-z.

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