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ON DISCRETE PROPERTIES OF CONTINUOUS MONOTONE FUNCTIONS

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Abstract. We deal with strictly monotone continuous functions $h\colon I \to I$, where I is an interval of real numbers. The monounary algebra (I,h) contains at most 4 non-isomorphic components. We derive that there are 10 possibilities only how these components can be combined in (I,h). Two more options are cancelled in case that $I = \langle a,b\rangle$ for some real numbers $a,b,a<b$.

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1. INTRODUCTION

The notion of a function belongs to the fundamental notions in mathematics. If range of a function is a subset of its domain, then this function determines an algebraic structure which is called a monounary algebra, cf. e.g. [10]. Therefore these algebras naturally occur in many areas of mathematics and their properties are applied in many ways. A lot of results outside universal algebra can be formulated in notions of monounary algebras. For example, Sharkovski Theorem from the theory of dynamical systems is about monounary algebras which contain cycles of all natural lengths, cf. [12, 13]. The famous Łojasiewicz Theorem describes component partitions of all monounary algebras with a bijective operation such that relevant functional equations of iterative roots have a solution, cf. [1, 9].

Monounary algebras can be visualised in a natural way, as directed graphs with exactly one out-edge at each vertex. Many tasks in universal algebra can be simplified to monounary algebras. To investigate congruences means to go to unary operations,

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cf. [6]. To find a homomorphism is possible via a monounary algebras on enlarged sets, cf. [11].

Monounary algebras are elaborated from the point of view of universal algebra in great details, for overview see [2, 7]. This offers a tool for solving many questions e.g. in numerical analysis and functional equations, cf. [2, 3, 8]. Notice that methods to get the solution are constructive and they provide the ability to obtain algorithms which can be used by computer algebra systems.

Functions of $\mathbb{R}$ into $\mathbb{R}$ are called *real functions*. A function of I into I, where I is an interval of real numbers, is said to be an *interval function*. Sharkovski Theorem mentioned above is on continuous interval functions.

We deal with strictly monotone continuous interval functions. Strictly increasing (strictly decreasing) functions or sequences will be called increasing (decreasing) in short.

Monounary algebras with continuous increasing real functions were studied by O. Kopeček in [8]. We develop methods from this paper for monounary algebras with continuous increasing interval functions. We obtain a classification of all such functions according to types of components which occur in corresponding algebras. Then we apply this classification to classify continuous decreasing interval functions according to types of components which occur in corresponding algebras.

2. Preliminaries

The set of all positive integers is denoted by $\mathbb{N}$. If A is a set, then $\|A\|$ is the cardinality of this set.

Let A be a non-empty set and h be a function from A into A. We define by induction $h^n(x) = h(h^{n-1}(x))$ for every $x \in A$ and $n \in \mathbb{N} \setminus \{1\}$.

The pair (A,h) is called a *monounary algebra*. To (A,h) there corresponds an oriented graph with the vertex set A and the edge set $\{(a,h(a)) \mid a \in A\}$. We use the symbol $\cong$ for isomorphism between algebras. For monounary terminology see, e.g., [4, 5, 7, 10].

An algebra (A,h) is said to be *connected* if for every $x,y \in A$ there exist $m,n \in \mathbb{N}$ such that $h^m(x) = h^n(y)$.

Let (A,h) be a monounary algebra. An element $a \in A$ is called *cyclic* if there exists $k \in \mathbb{N}$ such that $h^k(a) = a$. If $h^k(a) = a$ for some $k \in \mathbb{N}$ and $h^l(a) \neq a$ for each $l < k, l \in \mathbb{N}$, then we say that a belongs to a k-element cycle. The algebra (A,h) is a *cycle* if it is connected and every element of A is cyclic. Then $\|A\| = i$ for some $i \in \mathbb{N}$ and (A,h) is said to be an i-element cycle.

Let $B \subseteq A$. We denote $h(B) = \{h(b), b \in B\}$ and $h|B$ the function that is the restriction of h onto B. If $h(B) \subseteq B$ and $(B,h|B)$ is a maximal connected subalgebra of (A,h), then we say that $(B,h|B)$ is a *component of* (A,h).

Suppose that h is injective and x is an element of range of h. Then we denote by $h^{-1}(x)$ the unique element y of A such that $h(y) = x$. Next, by induction, if $h^{-n}(x)$

is an element of range of h, then $h^{-(n+1)}(x)$ is the unique element $z \in A$ such that $h^{n+1}(z) = x$.

The next lemma is trivial.

Lemma 1. *Let $h\colon A \to A$ and $c \in A$. The following assertions are equivalent:*

(1) $h^2(c) = c$,

(2) *c belongs to a* 1*-element cycle in the algebra (A,h) or c belongs to a* 2*-element cycle in (A,h).*

We denote by N the monounary algebra that is defined on the set of all natural numbers with the successor function; analogously, Z is defined on the set of all integers. Further, let C_1, C_2 be fixed 1- or 2-element cycles, respectively.

Lemma 2. *Let $h\colon A \to A$ and $n \in \mathbb{N}$. Then*

(1) *An element $a \in A$ is cyclic in the algebra (A,h) if and only if it is cyclic in the algebra (A,h^n).*

(2) *h is injective if and only if h^n is injective.*

(3) *h is surjective if and only if h^n is surjective.*

(4) *$(A,h) \cong Z$ if and only if the algebra (A,h^n) consists of n components and each of them is isomorphic to Z.*

(5) *$(A,h) \cong N$ if and only if the algebra (A,h^n) consists of n components and each of them is isomorphic to N.*

Proof. Let $h^m(a) = a$. That means that the element a is cyclic in algebras (A,h) and (A,h^n).

Statements (2), (3) are obvious.

If $(A,h) \cong Z$, then it is easy to see that the algebra (A,h^n) consists of n components and each of them is isomorphic to Z.

Let (A,h^n) consist of n components and each of them is isomorphic to Z. In view of (2), (3) the operation h is bijective on A. Let B be a component of (A,h). Then B is a cycle or $B \cong Z$ since h is bijective. Further, B does not contain a cycle according to (1). Therefore $B \cong Z$. If (A,h) is not connected then (A,h^n) consists of more than n components.

If $(A,h) \cong N$, then the algebra (A,h^n) obviously consists of n components isomorphic to N.

Let (A,h^n) consist of n components and each of them is isomorphic to N. In view of (2) the operation h is injective on A and it is not surjective on A according to (3). That means that there exists a component B of the algebra (A,h) such that $(B,h) \cong N$. The algebra (B,h^n) consists of n components and thus $B = A$. □

If $a \in A \setminus h(A)$ then the element a is called a *source* of (A,h). If $(A,h) \cong N$, then there exists exactly one source of (A,h).

Let T be a set of connected monounary algebras. The algebra (A,h) will be called the *T-algebra*, if the following two conditions are satisfied:

(1) every component of (A,h) is isomorphic to some algebra from the set T;
(2) if $B \in T$, then there exists a component B' of (A,h) such that $B' \cong B$.

Lemma 3. *Let $A \subseteq \mathbb{R}$, $h\colon A \to A$ be increasing and $a \in A$.*

(1) *If $a < h(a)$, then $h^k(a) < h^{k+1}(a)$ for every $k \in \mathbb{N}$.*
(2) *If $h(a) < a$, then $h^{k+1}(a) < h^k(a)$ for every $k \in \mathbb{N}$.*

Proof. Let $h(a) < a$. Then $h^2(a) = h(h(a)) < h(a)$ since h is increasing. Suppose that $h^{k+1}(a) < h^k(a)$ for $k \in \mathbb{N}$. We use that h is increasing to obtain

$$h^{k+2}(a) = h(h^{k+1}(a)) < h(h^k(a)) = h^{k+1}(a).$$

□

Lemma 4. *If $A \subseteq \mathbb{R}$ and $h\colon A \to A$ is decreasing, then the function h^2 is increasing.*

Proof. It follows from definitions. □

Lemma 5. *Let h be decreasing and $a \in A$ be such that $h(a) = a$. If $b \in A \setminus \{a\}$, then $h(b) \neq b$.*

Proof. If $a < b$, then the assumption that h is decreasing yields that

$$h(b) < h(a) = a < b.$$

If $b < a$, then $b < h(b)$. □

Theorem 1. *Let $A \subseteq \mathbb{R}$ and $h\colon A \to A$.*

(1) *If h is an increasing function, then there exists $T \subseteq \{C_1, N, Z\}$ such that (A,h) is a T-algebra.*
(2) *If h is an decreasing function, then there exists $T \subseteq \{C_1, C_2, N, Z\}$ such that (A,h) is a T-algebra and (A,h) contains at most one 1-element cycle.*

Proof. Suppose h is increasing. Let $a \in A$ and $h(a) \neq a$. The sequence $\{h^n(a)\}_{n\in\mathbb{N}}$ consists of infinitely many elements according to Lemma 3. It means the component of (A,h) which contains the element a is isomorphic to N or Z since h is injective.

Now let h be decreasing. Then the function h^2 is increasing by Lemma 4 and components of (A,h^2) are algebras isomorphic to C_1, N or Z. That means components of (A,h) are algebras isomorphic to C_1, C_2, N or Z according to Lemmas 2 and 1. In view of Lemma 5 the function h has at most one fixed point. □

We denote

$$\mathrm{Fix}_h = \{a \in A\colon h(a) = a\}.$$

Let us remark that if $\mathrm{Fix}_h \neq \varnothing$, then the algebra $(\mathrm{Fix}_h, h|\mathrm{Fix}_h)$ is the greatest $\{C_1\}$-subalgebra of (A,h).

Let $a \in \mathbb{R}$. The interval $\langle a, a\rangle = \{a\}$ will be called *trivial*.

In the next two lemmas we suppose that $I \subseteq \mathbb{R}$ is an interval and $h\colon I \to I$.

Lemma 6. *Let h be continuous and $a,b \in \mathrm{Fix}_h, a < b$. If $c \in (a,b) \setminus \mathrm{Fix}_h$, then there exist $a',b' \in \mathrm{Fix}_h$ such that*

$$a \le a' < c < b' \le b \text{ and } (a',b') \cap \mathrm{Fix}_h = \varnothing.$$

Proof. Let $(a,b) \cap \mathrm{Fix}_h \neq \varnothing$. Consider $A = \langle a,c) \cap \mathrm{Fix}_h$ and $B = (c,b\rangle \cap \mathrm{Fix}_h$. The set A has a supremum $a' \in \langle a,c\rangle$. To see that $a' \in A$ suppose that $a' \notin A$. Then there is a sequence $\{a_i\}_{i=1}^{\infty}$ such that $a_i \in A$ and $\lim_{i\to\infty} a_i = a'$. Since h is continuous we obtain $a_i = h(a_i) \to h(a')$. Therefore $a' = h(a')$. Conclude $a' \in A$ according to $c \notin \mathrm{Fix}_h$, a contradiction.

Analogously the set B has a minimum $b', b' > c$. □

Lemma 7. *Let h be increasing, $m \in \mathbb{N}$ and $a \in I$.*

If $a < h(a)$, then $a < h^m(a)$ and $\langle a, h^m(a)\rangle \cap \mathrm{Fix}_h = \varnothing$.
If $a > h(a)$, then $a > h^m(a)$ and $\langle h^m(a), a\rangle \cap \mathrm{Fix}_h = \varnothing$.

Proof. Assume that $h(a) > a$. Then $h^m(a) > a$ according to Lemma 3. Consider $d \in (a, h(a))$. Then $h(a) < h(d)$ since h is increasing. Therefore $d < h(d)$, so $d \notin \mathrm{Fix}_h$. We obtained $\langle a, h(a)\rangle \cap \mathrm{Fix}_h = \varnothing$. That means the statement $\langle a, h^m(a)\rangle \cap \mathrm{Fix}_h = \varnothing$ is valid because the function h^m is increasing and $\mathrm{Fix}_h = \mathrm{Fix}_{h^m}$.

The second implication can be proved analogously. □

3. Continuous increasing interval functions

In this section we give a classification of algebras (I,h), where $I \subseteq \mathbb{R}$ is an interval and h is a continuous increasing function in the common sense. The algebras under consideration are T-algebras for some $T \subseteq \{C_1, N, Z\}$ (see Theorem 1). We will show that $T = \{N, Z\}$ is not possible. Moreover, $C_1 \in T$ in the case that $I = \langle a,b\rangle$ for some $a,b \in \mathbb{R}$.

Let $I \subseteq \mathbb{R}$ be an interval. The closure of the interval I will be denoted by $\bar{I}$.

We suppose that h is a function from I into I and it is a continuous increasing function in this section.

Lemma 8. *Let function $\bar{h}$ be defined on $\bar{I}$ by*

$$\bar{h}(x) = \begin{cases} h(x) & \text{if } x \in I, \\ \lim_{a\to x+} h(a) & \text{if } x \notin I \text{ and } x \le a \text{ for } a \in I, \\ \lim_{a\to x-} h(a) & \text{if } x \notin I \text{ and } x \ge a \text{ for } a \in I. \end{cases}$$

Then $\bar{h}\colon \bar{I} \to \bar{I}$ and it is continuous increasing. If $J \subset I$ is an interval such that $h(J) \subseteq J$, then $\overline{h|J} = \bar{h}|\bar{J}$.

Proof. Suppose $x \in \bar{I} \setminus I$. Then $x \le a$ for each $a \in I$ or $x \ge a$ for each $a \in I$. If $x \le a$ for $a \in I$, then $x \le h(a)$ since $h(I) \subseteq I$. Therefore $\lim_{a\to x+} h(a)$ exists because h is continuous. Analogously if $x \ge a$ for each $a \in I$, then $\lim_{a\to x-} h(a)$ exists. So, the function $\bar{h}$ is correctly defined. Obviously, it is continuous and increasing. The equality $\overline{h|J} = \bar{h}|\bar{J}$ is valid according to the definition. □

3.1. *On the set of fixed points*

Lemma 9. *Let $x \in I$ and $x < h(x)$. Then*

(1) *the sequence $\{h^n(x)\}_{n\in\mathbb{N}}$ is increasing and*
   (a) $\langle x, lim_{n\to\infty} h^n(x)) \cap \mathrm{Fix}_h = \varnothing$,
   (b) $lim_{n\to\infty} h^n(x) \in \mathrm{Fix}_{\overline{h}} \cup \{\infty\}$;

(2) *if $M = \{n \in \mathbb{N} \colon h^{-n}(x)$ is defined$\} \neq \varnothing$, then the sequence $\{h^{-n}(x)\}_{n\in M}$ is decreasing and if $M = \mathbb{N}$, then*
   (a) $(lim_{n\to\infty} h^{-n}(x), x\rangle \cap \mathrm{Fix}_h = \varnothing$,
   (b) $lim_{n\to\infty} h^{-n}(x) \in \mathrm{Fix}_{\overline{h}} \cup \{-\infty\}$.

Proof. The sequence $\{h^n(x)\}_{n\in\mathbb{N}}$ is increasing according to Lemma 3. Therefore it is convergent and its limit is ∞ or it is from $\overline{I}$ since $h^n(x) \in I$ for each $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} h^n(x) = a, a \in \overline{I}$. In view of $\overline{h}$ is continuous we obtain

$$\overline{h}(a) = \overline{h}(\lim_{n\to\infty} h^n(x)) = \lim_{n\to\infty} \overline{h}(h^n(x)) = \lim_{n\to\infty} h(h^n(x)) = a.$$

Put $y_n = h^{-n}(x)$ for each $n \in M$. Let $m \in \mathbb{N}$ be such that $y_{m+1} \geq y_m$. Then $y_{m+1} > y_m$. The assumption that h is increasing implies

$$x = h^{m+1}(y_{m+1}) > h^{m+1}(y_m) = h(x),$$

a contradiction. Thus $y_{n+1} < y_n$ for each $n \in M$. If $M = \mathbb{N}$, then $\{y_n\}_{n\in\mathbb{N}}$ is convergent and its limit is $-\infty$ or it is from $\overline{I}$ since $y_n \in I$ for each $n \in \mathbb{N}$. Suppose that $\lim_{n\to\infty} y_n = b, b \in \overline{I}$. In view of $\overline{h}$ is continuous we obtain

$$\overline{h}(b) = \overline{h}(\lim_{n\to\infty} y_n) = \lim_{n\to\infty} \overline{h}(y_n) = \lim_{n\to\infty} h(y_n) = \lim_{n\to\infty} y_{n-1} = b.$$

Statements (a) for $h^n(x)$ and $h^{-n}(x)$ follow from Lemma 7. □

Analogously we can prove

Lemma 10. *Let $x \in I$ and $x > h(x)$. Then*

(1) *the sequence $\{h^n(x)\}_{n\in\mathbb{N}}$ is decreasing and*
   (a) $(lim_{n\to\infty} h^n(x), x\rangle \cap \mathrm{Fix}_h = \varnothing$,
   (b) $lim_{n\to\infty} h^n(x) \in \mathrm{Fix}_{\overline{h}} \cup \{-\infty\}$;

(2) *if $M = \{n \in \mathbb{N} \colon h^{-n}(x)$ is defined$\} \neq \varnothing$, then the sequence $\{h^{-n}(x)\}_{n\in M}$ is increasing and if $M = \mathbb{N}$, then*
   (a) $\langle x, lim_{n\to\infty} h^{-n}(x)) \cap \mathrm{Fix}_h = \varnothing$,
   (b) $lim_{n\to\infty} h^{-n}(x) \in \mathrm{Fix}_{\overline{h}} \cup \{\infty\}$.

Lemma 11. *If $\overline{I} = \langle a, b\rangle$, $a, b \in \mathbb{R}$, then $\mathrm{Fix}_{\overline{h}} \neq \varnothing$.*

Proof. Consider the function $g \colon \overline{I} \to \overline{I}$ defined by $g(x) = \overline{h}(x) - x$. Then g is continuous and $g(a) \geq 0, g(b) \leq 0$. Thus Intermediate Value Theorem gives that there exists $c \in \langle a, b\rangle$ such that $g(c) = 0$. We have $c \in \mathrm{Fix}_{\overline{h}}$. □

Lemma 12. *Let $a \in I, (a - \varepsilon, a + \varepsilon) \subseteq I$ and $\varepsilon \in \mathbb{R}, \varepsilon > 0$.*

(1) *If* $(a-\varepsilon,a)\cap \mathrm{Fix}_h=\varnothing$ *and* $(a,a+\varepsilon)\subseteq \mathrm{Fix}_h$, *then* $a\in \mathrm{Fix}_h$.
(2) *If* $(a,a+\varepsilon)\cap \mathrm{Fix}_h=\varnothing$ *and* $(a-\varepsilon,a)\subseteq \mathrm{Fix}_h$, *then* $a\in \mathrm{Fix}_h$.

Proof. Let assumptions of (1) be satisfied. Then continuity of h implies

$$h(a)=\lim_{x\to a+}h(x)=\lim_{x\to a+}x=a.$$

□

Corollary 1. *Let* $\mathrm{Fix}_{\overline{h}}\neq\varnothing$. *Then the following conditions are equivalent:*

(1) $\mathrm{Fix}_{\overline{h}}$ *is an interval,*
(2) $\mathrm{Fix}_{\overline{h}}\in\{(-\infty,a\rangle,\langle a,b\rangle,\langle b,\infty),\mathbb{R}\}$ *for some* $a,b\in\mathbb{R},a\leq b$.
(3) $\langle c,d\rangle\cap\mathrm{Fix}_{\overline{h}}\neq\{c,d\}$ *for each* $c,d\in\overline{I}$ *such that* $c<d$.

Proof. The equivalence of (1) and (2) follows from the definition of $\overline{h}$ and Lemma 12. It is trivial that (2) implies (3).

Assume $\mathrm{Fix}_{\overline{h}}$ is not an interval. Then there exist $a,b\in\mathrm{Fix}_{\overline{h}},a<b$ such that $\langle a,b\rangle\not\subseteq\mathrm{Fix}_{\overline{h}}$. Let $c\in\langle a,b\rangle\setminus\mathrm{Fix}_{\overline{h}}$. Consider elements $a',b'\in\overline{I}$ such that a' is a supremum of $\{u\in\mathrm{Fix}_{\overline{h}}\colon a\leq u<c\}$ and b' is an infimum of $\{v\in\mathrm{Fix}_{\overline{h}}\colon c<v\leq b\}$. If $\{a_n\}_{n\in\mathbb{N}}$ is a sequence of elements of $\mathrm{Fix}_{\overline{h}}$ such that $a\leq a_n<c$ and $a_n\to a'$, then

$$\overline{h}(a')=h(\lim_{n\to\infty}a_n)=\lim_{n\to\infty}\overline{h}(a_n)=\lim_{n\to\infty}a_n=a'.$$

Therefore $a'\in\mathrm{Fix}_{\overline{h}}$. Analogously $b'\in\mathrm{Fix}_{\overline{h}}$. We obtained

$$\langle a',b'\rangle\cap\mathrm{Fix}_{\overline{h}}=\{a',b'\}.$$

□

3.2. $\{N\}$*-algebras*

Lemma 13. *Let h be not surjective and* $\mathrm{Fix}_{\overline{h}}=\varnothing$. *Then* (I,h) *is an* $\{N\}$*-algebra.*

Proof. In view of the Lemma 11 we have $I=\mathbb{R},\overline{I}=\langle a,\infty)$ or $\overline{I}=(-\infty,a\rangle$ for some $a\in\mathbb{R}$. Further, $\lim_{n\to\infty}h^n(x)\in\{-\infty,\infty\}$ for each $x\in I$ in view of Lemmas 9 and 10.

Let $x\in I$ and $\lim_{n\to\infty}h^n(x)=\infty$. Then the sequence $\{h^n(x)\}_{n=1}^{\infty}$ is increasing. If $h^{-k}(x)$ is defined for every $k\in\mathbb{N}$ then the sequence $\{h^{-k}(x)\}_{k=1}^{\infty}$ is decreasing. In view of Lemma 9 $\lim_{n\to\infty}h^n(x)=-\infty$. Therefore $I=\mathbb{R}$ and h is surjective, a contradiction. We conclude that $h^{-k}(x)$ is not defined for some $k\in\mathbb{N}$ and the component of x is isomorphic to N. □

Lemma 14. *Let* $\overline{I}=\langle a,b\rangle$, $a,b\in\mathbb{R}$. *If* $\mathrm{Fix}_h=\varnothing$ *and* $\|\mathrm{Fix}_{\overline{h}}\|=1$, *then* (I,h) *is an* $\{N\}$*-algebra.*

Proof. We have $\mathrm{Fix}_{\overline{h}}\subset\{a,b\}$ by Lemma 11. Let $\mathrm{Fix}_{\overline{h}}=\{a\}$. Take $x\in I$. Then $h(x)<x$ since $\mathrm{Fix}_h=\varnothing$. The sequence $\{h^n(x)\}_{n=1}^{\infty}$ is decreasing and its limit is equal to a according to Lemma 10. If $h^{-k}(x)$ is defined for every $k\in\mathbb{N}$, then in view of Lemma 10 the sequence $\{h^{-k}(x)\}_{k=1}^{\infty}$ is increasing and it has a limit from the set

$\{a,\infty\}$, what is not possible. Therefore $h^{-k}(x)$ is not defined for some $k \in \mathbb{N}$ and the component of x in (I,h) is isomorphic to N.

Analogously if $\mathrm{Fix}_{\overline{h}} = \{b\}$. □

Lemma 15. *Let $I = (a,\infty)$ or $I = (-\infty,a)$, $a \in \mathbb{R}$ and h be not surjective. If $\mathrm{Fix}_h = \varnothing$ and $\|\mathrm{Fix}_{\overline{h}}\| = 1$, then (I,h) is an $\{N\}$-algebra.*

Proof. We have $\mathrm{Fix}_{\overline{h}} = \{a\}$. Let $I = (a,\infty)$. Then h is bounded from above since h is not surjective. Take $x \in I$. Then $h(x) < x$ and the sequence $\{h^n(x)\}_{n=0}^{\infty}$ is decreasing with the limit equal to a according to Lemma 10. If $h^{-k}(x)$ is defined for every $k \in \mathbb{N}$, then the sequence $\{h^{-k}(x)\}_{k=1}^{\infty}$ is increasing. In view of Lemma 10 it has a limit equal to ∞, what is not possible since h is bounded. Therefore $h^{-k}(x)$ is not defined for some $k \in \mathbb{N}$ and the component of x in (I,h) is isomorphic to N.

For $I = (-\infty,a)$ the function h is bounded from below and the sequence $\{h^n(x)\}_{n=0}^{\infty}$ is increasing. □

3.3. *Main result*

Lemma 16. *Let $a \in \mathrm{Fix}_h$.*

(1) *If $\mathrm{Fix}_h \cap (-\infty,a) = \varnothing$, then $I \cap (-\infty,a)$ is a subalgebra of (I,h).*
(2) *If $\mathrm{Fix}_h \cap (a,\infty) = \varnothing$, then $I \cap (a,\infty)$ is a subalgebra of (I,h).*

Proof. Let $x \in I$ and $x < a$. If $h(x) < x$, then $h(x) < a$. If $h(x) > x$, then $h(x) < a$ in view of Lemma 9(1a).

Analogously we can see the second part of the assertion. □

Lemma 17. *Let h be not surjective, sets $\mathrm{Fix}_h, \mathrm{Fix}_{\overline{h}}$ be intervals and h is bounded on the set $I \setminus \mathrm{Fix}_h$. Then (I,h) is $\{N,C_1\}$-algebra.*

Proof. We have $\mathrm{Fix}_h \neq \varnothing$. The set $\overline{I} \setminus \mathrm{Fix}_{\overline{h}}$ is an interval or there exist intervals I_1, I_2 such that $\overline{I} \setminus \mathrm{Fix}_{\overline{h}} = I_1 \cup I_2$.

Let $\overline{I} \setminus \mathrm{Fix}_{\overline{h}}$ be an interval. Denote $A = I \setminus \mathrm{Fix}_h$. Then $\overline{h}|\overline{A} = \overline{h|A}$ according to Lemma 8. We have that A is a subalgebra of (I,h) according to Lemma 16. Further,

$$\mathrm{Fix}_{h|A} = \varnothing, \|\mathrm{Fix}_{\overline{h|A}}\| = 1$$

and h is not surjective on A. Therefore the algebra $(A, h|A)$ satisfies assumptions of Lemmas 14 or 15 and so it is an $\{N\}$-algebra.

Let $\overline{I} \setminus \mathrm{Fix}_{\overline{h}}$ be not an interval. Then $\mathrm{Fix}_{\overline{h}} = \mathrm{Fix}_h = \langle a,b \rangle$ for some $a,b \in \mathbb{R}, a \leq b$. Denote $J_1 = I_1 \cap I, J_2 = I_2 \cap I$. Intervals J_1, J_2 are non-empty and they create subalgebras of (I,h) according to Lemma 16. Further,

$$J_1 \cup \mathrm{Fix}_h \cup J_2 = I,\ \mathrm{Fix}_{h|J_1} = \mathrm{Fix}_{h|J_2} = \varnothing \text{ and } \|\mathrm{Fix}_{\overline{h|J_1}}\| = \|\mathrm{Fix}_{\overline{h|J_2}}\| = 1.$$

Moreover, $h(J_1) \neq J_1, h(J_2) \neq J_2$ since h is bounded. Thus algebras $(J_1, h|J_1)$, $(J_2, h|J_2)$ satisfy assumptions of Lemmas 14 or 15 and therefore they are $\{N\}$-algebras. □

Lemma 18. *Let $a, b \in \bar{I}$ be such that $a < b$. If $\langle a, b\rangle \cap \mathrm{Fix}_{\bar{h}} = \{a, b\}$, then (a, b) forms a subalgebra of (I, h) with all components isomorphic to Z.*

Proof. Let x be such that $a < x < b$. Then $x \in I$ and

$$a = \bar{h}(a) < h(x) = \bar{h}(x) < \bar{h}(b) = b$$

according to $\bar{h}$ is increasing. Therefore (a, b) forms a subalgebra of (I, h). The function h is bijective on (a, b) since it is continuous. Thus every component of $((a, b), h|(a, b))$ is isomorphic to Z or to C_1 according to h is increasing. Further,

$$(a, b) \cap \mathrm{Fix}_h = (a, b) \cap \mathrm{Fix}_{\bar{h}} = \varnothing$$

according to the assumption. □

Theorem 2. *Let I be an interval and $h\colon I \to I$ be continuous increasing function.*

(1) *Let h be surjective. Then the algebra (I, h) is one of the following 3 types:*
   (a) *$\{C_1\}$-algebra, if $\mathrm{Fix}_h = I$.*
   (b) *$\{Z\}$-algebra, if $\mathrm{Fix}_h = \varnothing$.*
   (c) *$\{C_1, Z\}$-algebra, if $\mathrm{Fix}_h \notin \{I, \varnothing\}$.*

(2) *Let h be not surjective. Then the algebra (I, h) is one of the following 3 types:*
   (a) *$\{N\}$-algebra, if $\mathrm{Fix}_h = \varnothing$.*
   (b) *$\{C_1, N\}$-algebra, if $\mathrm{Fix}_{\bar{h}}$ is an interval and h is bounded (above and below) on $I \setminus \mathrm{Fix}_h$.*
   (c) *$\{C_1, N, Z\}$-algebra, otherwise.*

Proof. The statement (1) is obvious.

Let h be not surjective. Suppose that $\mathrm{Fix}_h = \varnothing$. If $\mathrm{Fix}_{\bar{h}} = \varnothing$, then (I, h) is $\{N\}$-algebra according to Lemma 13. Suppose that $\mathrm{Fix}_{\bar{h}} \neq \varnothing$. We have $\bar{I} = \langle a, b\rangle$ or $I \in \{(a, \infty), (-\infty, a)\}$ for some $a, b \in \mathbb{R}, a < b$. If $I \in \{(a, \infty), (-\infty, a)\}$, then $\mathrm{Fix}_{\bar{h}} = \{a\}$ and (I, h) is $\{N\}$-algebra according to Lemma 15. If $\bar{I} = \langle a, b\rangle$, then $\mathrm{Fix}_{\bar{h}} = \{a\}$ or $\mathrm{Fix}_{\bar{h}} = \{b\}$ since h is continuous and not surjective. Therefore (I, h) is $\{N\}$-algebra according to Lemma 14.

The assertion (b) is proved in Lemma 17.

Let suppositions of (a),(b) be not valid. Then $\mathrm{Fix}_h \neq \varnothing$. Therefore the algebra (I, h) contains an 1-element cycle. Further, $h(I) \neq I$ since h is not surjective. Thus every $u \in I$ such that $u \notin h(I)$ is a source of (I, h). That means (I, h) contains components isomorphic to N according to Theorem 1.

Non-surjectivity of h means that h is bounded from above or h is bounded from below. Assume that $\mathrm{Fix}_{\bar{h}}$ is not an interval. Then there are $a, b \in \mathrm{Fix}_{\bar{h}}$ such that there exists $c \in (a, b) \setminus \mathrm{Fix}_{\bar{h}}$. In view of Lemma 6 there are $a', b' \in \mathrm{Fix}_{\bar{h}}$ such that

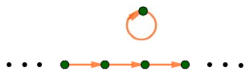
$$a \leq a' < c < b' \leq b \text{ and } (a', b') \cap \mathrm{Fix}_{\bar{h}} = \varnothing.$$

Thus by Lemma 18 the interval (a', b') determines a subalgebra of (I, h) and every component of $((a', b'), h|(a', b'))$ is isomorphic to Z.

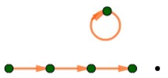
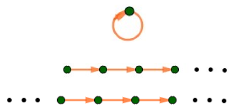
Now let $\mathrm{Fix}_{\overline{h}}$ be an interval. Then $\mathrm{Fix}_h \neq \varnothing$ by (a). That means Fix_h is an interval. Therefore h is not bounded from above on $I \setminus \mathrm{Fix}_h$ or h is not bounded from below on $I \setminus \mathrm{Fix}_h$ according to (b). Suppose that h is not bounded from above on $I \setminus \mathrm{Fix}_h$. Then $\overline{I} \in \{\mathbb{R}, \langle a', \infty)\}$ for some $a' \in \mathbb{R}$. Further, $\mathrm{Fix}_{\overline{h}} \in \{(-\infty, b\rangle, \langle a, b\rangle\}$ for some $a, b \in \overline{I}$ according to Corollary 1. Thus $(b,\infty) \cap \mathrm{Fix}_h = \varnothing$. We obtained the interval (b,∞) is a subalgebra of (I,h) and every component of the algebra $((b,\infty), h|(b,\infty))$ is isomorphic to Z according to Lemma 18. Analogously we proceed if h is not bounded from below on $I \setminus \mathrm{Fix}_h$. □

The following tables illustrate Theorem 2. They show all types of components that occur in the corresponding algebras.

(1) h surjective

$\mathrm{Fix}_h = \varnothing$	$\mathrm{Fix}_h = I$	otherwise
		

(2) h not surjective

$\mathrm{Fix}_h = \varnothing$	h bounded on $I \setminus \mathrm{Fix}_h$, $\mathrm{Fix}_{\overline{h}}$ interval	otherwise
		

Corollary 2. *Let I be an interval and $h\colon I \to I$ be continuous increasing function. Then the following assertions are equivalent:*

(i) *(I,h) is an $\{C_1, N, Z\}$-algebra.*

(ii) *$h(I) \neq I$, $\mathrm{Fix}_h \neq \varnothing$ and at least one of the following conditions is satisfied*

  (a) *$\mathrm{Fix}_{\overline{h}}$ is not an interval,*

  (b) *h is not bounded on $I \setminus \mathrm{Fix}_h$.*

Corollary 3. *Let I be an interval and $h\colon I \to I$ be continuous increasing function. Then the algebra (I,h) is not an $\{N,Z\}$-algebra.*

If $I = \langle a,b\rangle$, $a,b \in \mathbb{R}$, then (A,h) is an T-algebra such that

$$T \in \{\{C_1\}, \{C_1, Z\}, \{C_1, N\}, \{C_1, N, Z\}\}.$$

Proof. It follows from Theorem 2 and Lemma 11. □

Example 1. Let $h(x) = 2x$ for $x \in \mathbb{R}$. The algebra $(\mathbb{R},h)$ is $\{C_1, Z\}$-algebra. It contains exactly one 1-element cycle.

Example 2. Let $h(x) = e^x$ for $x \in \mathbb{R}$. The algebra $(\mathbb{R},h)$ is $\{N\}$-algebra.

Example 3. Let $h(x) = e^x - 1$ for $x \in \mathbb{R}$. The algebra $(\mathbb{R}, h)$ is $\{C_1, N, Z\}$-algebra with exactly one 1-element cycle. The interval $(-\infty, 0)$ forms the maximal $\{N\}$-subalgebra of $(\mathbb{R}, h)$. If $a \leq -1$, then intervals $\langle a, 0), (a, 0)$ form $\{N\}$-subalgebras of $(\mathbb{R}, h)$. The interval $(0, \infty)$ forms the maximal $\{Z\}$-subalgebra of $(\mathbb{R}, h)$ and there is no interval $I' \subset (0, \infty)$ such that I' forms a subalgebra of $(\mathbb{R}, h)$.

Example 4. Let

$$h(x) = \begin{cases} e^x - 1 & \text{if } x \in \langle -1, 0 \rangle, \\ x & \text{if } x \in (0, 1\rangle. \end{cases}$$

The algebra $(\langle -1, 1 \rangle, h)$ is $\{C_1, N\}$-algebra. The interval $\langle -1, 0)$ forms the maximal $\{N\}$-subalgebra of $(\langle -1, 1 \rangle, h)$. The interval $\langle 0, 1 \rangle$ the a maximal $\{C_1\}$-subalgebra of $(\langle -1, 1 \rangle, h)$.

4. Continuous decreasing interval functions

Let $I \subseteq \mathbb{R}$ be an interval. In this section we will suppose that h is a function from I into I and it is continuous and decreasing.

The algebra (I, h) is T-algebra with at most one 1-element cycle for some $T \subseteq \{C_1, C_2, N, Z\}$ by Theorem 1. We will show in this section that there are six possibilities for T only and that T is determined by the set Fix_{h^2}.

Lemma 19. *There exists $o \in I$ such that* $\mathrm{Fix}_h = \{o\}$.

Proof. In view of Lemma 5 we have $\|\mathrm{Fix}_h\| \leq 1$. Let $a \notin \mathrm{Fix}_h$. Without loss of generality let $h(a) < a$. Then $h^2(a) > h(a)$. Put $g(x) = h(x) - x$. We have $g(a) < 0 < g(h(a))$. The function g is continuous and thus there is

$$o \in (h(a), a) \text{ such that } g(o) = 0$$

according to Intermediate Value Theorem. We obtained $\mathrm{Fix}_h = \{o\}$. □

Corollary 4. *The algebra (I, h) contains exactly one component isomorphic to C_1.*

Lemma 20. *Let $a \in I \setminus \mathrm{Fix}_h$. Then the following properties are equivalent:*

(1) *$\{a, h(a)\}$ creates 2-element cycle of (I, h).*
(2) *there exists $c \in \mathbb{R}$ such that the line $y = -x + c$ contains points $[a, h(a)]$, $[h(a), h^2(a)]$.*
(3) *the point $[a, h(a)]$ is symmetric with the point $[h(a), h^2(a)]$ according to the line $x = y$.*

Proof. Let (1) be satisfied. Then $h^2(a) = a$. The property (3) is valid since points $[a, a], [h(a), a], [h(a), h(a)], [a, h(a)]$ are vertices of a square.

Let (3) be fulfilled. Therefore $h^2(a) = a$. The statement (2) follows from the fact that the line determined by points $[a, h(a)], [h(a), h^2(a)]$ is perpendicular to $x = y$.

If (2) is valid, then

$$h(a) = -a + c \text{ and } h^2(a) = -h(a) + c = -(-a+c) + c = a.$$

□

Lemma 21. *Let h be not surjective and $a \in \mathbb{R}$.*

(1) *If $\overline{h(I)} = (-\infty, a\rangle$, then*
   (a) $h(I) = (-\infty, a)$,
   (b) $I = \mathbb{R}$ *or there exists* $b \in \mathbb{R}$, $b > a$ *such that* $I = (-\infty, b)$,
   (c) $h^2(I) = (h(a), a)$.

(2) *If $\overline{h(I)} = \langle a, \infty)$, then*
   (a) $h(I) = (a, \infty)$,
   (b) $I = \mathbb{R}$ *or there exists* $b \in \mathbb{R}, b < a$ *such that* $I = (b, \infty)$,
   (c) $h^2(I) = (a, h(a))$.

Proof. Let $\overline{h(I)} = (-\infty, a\rangle$. We have $(-\infty, a) \subset I$ and $a \in I$ since h is not surjective. The equality $h(I) = (-\infty, a\rangle$ is not possible since if $c < a$, then $c \in I$ and $h(c) > h(a)$. Thus $h(I) = (-\infty, a)$. Assume that $I = (-\infty, b\rangle$ for some $b > a, b \in \mathbb{R}$. Then the number $h(b)$ is the minimum of the the function h on I since h is decreasing. Therefore the interval $h(I)$ is closed on the left side, a contradiction. We obtained $I = \mathbb{R}$ or $I = (-\infty, b)$. Further, $\lim_{x\to-\infty} h(x) = a$. Therefore

$$h^2(I) = h((-\infty, a)) = (h(a), \lim_{x\to-\infty} h(x)) = (h(a), a).$$

Similarly we argue if $h(I) = (a, \infty)$. □

Lemma 22. *Let h be not surjective, $a, b \in \mathbb{R}$ are such that $a < b$ and $\overline{h(I)} = \langle a, b\rangle$. Then $a \in I$ or $b \in I$ and*

(1) *if $a, b \in I$, then $h^2(I) \subseteq \langle h(b), h(a)\rangle$.*
(2) *if $b \notin I$, then $h^2(I) \subseteq (a, h(a)\rangle$.*
(3) *if $a \notin I$, then $h^2(I) \subseteq \langle h(b), b)$.*

Proof. There are $a \in I$ or $b \in I$ since h is not surjective. Assume that $a, b \in I$. Then

$$h^2(I) \subseteq h(\overline{h(I)}) = h(\langle a, b\rangle) = \langle h(b), h(a)\rangle.$$

Let $b \notin I$. Then $b \in \bar{I}$ since h is continuous and $h(I) \subset I$. Consider a sequence $\{b_n\}_{n\in\mathbb{N}}$ such that $b_n \in I$ and $\lim_{n\to\infty} b_n = b$. We obtain $b_n < b$ and $\lim_{n\to\infty} h(b_n) = a$ since h is decreasing. Therefore

$$h^2(I) \subseteq h(\overline{h(I)} \setminus \{b\}) = h(\langle a, b)) = (\lim_{n\to\infty} h(b_n), h(a)\rangle = (a, h(a)\rangle.$$

If $a \notin I$, then $h^2(I) \subseteq \langle h(b), b)$ by a similar way. □

Lemma 23. *Let h be not surjective. Then h^2 is bounded (above and below).*

Proof. The interval I is not trivial since h is not surjective. Suppose that $a, b \in \mathbb{R}$, $a < b$.

If $\overline{h(I)} = \langle a,b\rangle$, then h^2 is bounded according to Lemma 22. If $\overline{h(I)} = (-\infty,a\rangle$ or $\overline{h(I)} = \langle a,\infty)$, then h^2 is bounded in view of Lemma 21. □

The function h^2 is increasing according to Lemma 4. Thus the function $\overline{h^2}$ is defined by Lemma 8.

Lemma 24. *Let h be not surjective. Then* $\mathrm{Fix}_{h^2} = \mathrm{Fix}_{\overline{h^2}}$.

Proof. Suppose that $a,b \in \mathbb{R}$, $a < b$. To see that $\mathrm{Fix}_{\overline{h^2}} \subset I$ let $\overline{h(I)} = (-\infty,a\rangle$. Then $h(I) = (-\infty,a)$. We have $a \in I$ and $h^2(I) = (h(a),a)$ according to Lemma 21. Therefore $\mathrm{Fix}_{\overline{h^2}} \subseteq \langle h(a),a\rangle \subset I$. Analogously we can proceed if $\overline{h(I)} = \langle a,\infty)$. If $\overline{h(I)} = \langle a,b\rangle$, then use Lemma 22.

Let $x \in \mathrm{Fix}_{\overline{h^2}}$. Then $x \in I$ and $h^2(x) = \overline{h^2}(x) = x$. That means $x \in \mathrm{Fix}_{h^2}$. This yields that $\mathrm{Fix}_{h^2} = \mathrm{Fix}_{\overline{h^2}}$.

□

Theorem 3. *Let I be an interval and $h\colon I \to I$ be continuous decreasing function.*

(1) *Let h be surjective. Then the algebra (I,h) is one of the following 3 types:*
   (a) *$\{C_1,C_2\}$-algebra, if* $\mathrm{Fix}_{h^2} = I$*,*
   (b) *$\{C_1,C_2,Z\}$-algebra, if* $\mathrm{Fix}_{h^2} \neq I$ *and* $\|\mathrm{Fix}_{h^2}\| > 1$*,*
   (c) *$\{C_1,Z\}$-algebra, if* $\|\mathrm{Fix}_{h^2}\| = 1$.
(2) *Let h be not surjective. Then the algebra (I,h) is one of the following 3 types:*
   (a) *$\{C_1,N\}$-algebra, if* $\|\mathrm{Fix}_{h^2}\| = 1$*,*
   (b) *$\{C_1,C_2,N\}$-algebra, if* Fix_{h^2} *is a non-trivial interval,*
   (c) *$\{C_1,C_2,N,Z\}$-algebra, if* Fix_{h^2} *is not an interval.*

Proof. Suppose h is surjective. Then there is $T \subseteq \{C_1,C_2,Z\}$ such that (I,h) is an T-algebra according to Theorem 1. We have $C_1 \in T$ according to Lemma 19. If $\mathrm{Fix}_{h^2} \neq I$, then $Z \in T$ according to Lemma 1. If $\|\mathrm{Fix}_{h^2}\| > 1$, then $C_2 \in T$ according to Lemmas 1 and 19.

Suppose h is not surjective. The function h^2 is increasing according to Lemma 4 and it is not surjective according to Lemma 2(3). Moreover, it is bounded on both sides by Lemma 23. We will apply Theorem 2 to the algebra (I,h^2). The equality $\mathrm{Fix}_{h^2} = \mathrm{Fix}_{\overline{h^2}}$ is valid according to Lemma 24.

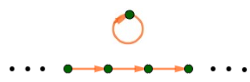
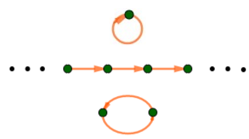
Let $\|\mathrm{Fix}_{h^2}\| = 1$. Then Fix_{h^2} is a trivial interval. The algebra (I,h^2) is an $\{C_1,N\}$-algebra according to Theorem 2(2b). Therefore the algebra (I,h) is an $\{C_1,N\}$-algebra according to Lemma 2(4).

Let Fix_{h^2} be a non-trivial interval. Then the algebra (I,h^2) is an $\{C_1,N\}$-algebra according to Theorem 2(2b). That means the algebra (I,h) is an $\{C_1,C_2,N\}$-algebra according to Lemma 2, Lemma 1 and Lemma 19.

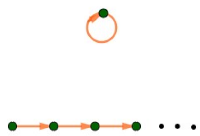
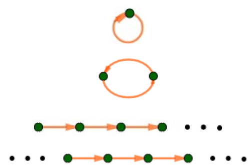
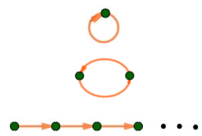
Now suppose Fix_{h^2} is not an interval. Then the set Fix_{h^2} contains at least 3 points according to Lemma 1 and Lemma 19. The algebra (I,h^2) is an $\{C_1,N,Z\}$-algebra according to Theorem 2(2c). In view of Lemma 2, Lemma 1 and Lemma 19 we obtain that (I,h) is an $\{C_1,C_2,N,Z\}$-algebra. □

The following tables illustrate Theorem 3. They show all types of components that occur in the corresponding algebras.

(1) $h\colon I \to I$ surjective

$\|\mathrm{Fix}_{h^2}\| = 1$	$\mathrm{Fix}_{h^2} = I$	otherwise
		

(2) $h\colon I \to I$ not surjective

$\|\mathrm{Fix}_{h^2}\| = 1$	Fix_{h^2} not interval	otherwise
		

Corollary 5. *The algebra (I,h) is not an $\{C_1,N,Z\}$-algebra.*

Example 5. Let

$$h(x) = \begin{cases} x^2 & \text{if } x \in \langle -1,0\rangle, \\ -x^2 & \text{if } x \in (0,1\rangle. \end{cases}$$

Then

$$h^2(x) = \begin{cases} -x^4 & \text{if } x \in \langle -1,0\rangle, \\ x^4 & \text{if } x \in (0,1\rangle. \end{cases}$$

The algebra $(\langle -1,1\rangle, h)$ is $\{C_1,C_2,Z\}$-algebra. Algebras $(\langle -1,1\rangle, h^2)$, $((-1,1),h)$ are $\{C_1,Z\}$-algebras.

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