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ON COFINITELY FLAT QUADRATIC O_K-MODULES

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Abstract. In this paper, we considered quadratic O_K-modules over the number fields and we defined the new concepts *"cofinitely quadratic O_K-module"* and *"cofinitely flat quadratic O_K-module"* for the integral ring O_K of the quadratic number fields. We described tensor product for these modules, and we extended them to the cofinitely quadratic O_K-modules. Finally we provided a main theorem by using these definitions.

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1. INTRODUCTION

Quadratic modules play a ubiquitous role in real algebra (see [4,6–8] and the references therein). Many algebraic structures in real algebra and algebraic geometry are associated with quadratic modules. For instance, a quadratic module in a commutative ring A (with unit element) is a subset Q of A containing the unit element 1, which is closed under addition and under multiplication with squares and the ring A contains a smallest quadratic module, namely, the set $\sum A^2$ consisting of all sums of squares of elements of A. Quadratic modules of groups are algebraic models for homotopy connected 3-types introduced by Baues [3]. Baues in [3] constructed a functor from the category of simplicial groups to the category of quadratic modules. In [9], Lie algebra versions of quadratic modules was also defined, and the connections between 2-crossed modules, quadratic modules and simplicial Lie algebras were explored by using simplicial properties in [1].

In recent studies, quadratic modules defined on algebraic number fields has also gained importance in real algebra. For this purpose, in this work, the quadratic modules defined on such fields will be examined. Namely, quadratic modules belonging to integral rings of algebraic number fields which are finite extensions of $\mathbb{Q}$ will be

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discussed and new definitions and algebraic structions related to these modules will be given.

Let K be a number field over $\mathbb{Q}$ such that $[K:\mathbb{Q}]=n$. We shall use denotation O_K for the ring of integers of K where O_K is a finitely generated $\mathbb{Z}$-module.

In the theory of finite O_K-modules, these modules are identified in equipped with a quadratic form $q: O_K \to K/\sigma^{-1}$ where $\sigma^{-1}=\{x\in K \mid T_{r_{K/\mathbb{Q}}}(xy)\in\mathbb{Z}, \forall x,y\in O_K\}$ is called *different* of K and σ^{-1} is also a fractional ideal of O_K.

We notice that we have a function

$$T_r: K/\sigma^{-1}\to\mathbb{Q}/\mathbb{Z},\ T_r(a+\sigma^{-1})=t_r(a)+\mathbb{Z}.$$

Let $b\in a+\sigma^{-1}$. If we take $b=a+t$ for some $t\in\sigma^{-1}$ then $T_r(t)\in\mathbb{Z}$ holds.

2. FINITE QUADRATIC O_K-MODULES

Definition 1. Let M be a finite O_K-module, then a quadratic form on M is a function $q: M\to K/\sigma^{-1}$ which satisfies the following:

(i) $q(ax)=a^2q(x)$; $\forall a\in O_K, x\in M$, q is a non-degenerate quadratic form.

(ii) The form $\beta_q = M\times M\to K/\sigma^{-1}$ defined by

$$\beta_q(x,y)=\beta_q(x+y)-\beta_q(x)-\beta_q(y)$$

is O_K-bilinear and symmetric.

β_q is non-degenerate which means

$$\beta_q(x,y)=0 \quad \text{if and only if} \quad x=0 \quad \text{for all} \quad y\in M,$$
$$\beta_q(x,y)=0 \quad \text{if and only if} \quad y=0 \quad \text{for all} \quad x\in M,$$

where a finite quadratic O_K-module can be briefly shown as $\underline{M}=(M,q)$ with a pair (M,q).

It is clear that β_q is also an O_K-balanced form on M. Therefore, we can give the following definition for $\underline{M}$ and $\underline{N}$.

Definition 2. Let $\underline{M}=(M,q)$ and $\underline{N}=(N,q')$ be finite quadratic O_K-modules with associated bilinear forms β_q and $\gamma_{q'}$ respectively. The form $\beta_q\otimes\gamma_{q'}$ defined by

$$\beta_q\otimes\gamma_{q'}:(M\otimes N)\times(M\otimes N)\to K/\sigma^{-1}\times K/\sigma^{-1}$$

$(x\otimes x', y\otimes y')\mapsto\beta_q(x,y)\gamma_{q'}(x',y')$ is bilinear and symmetric. Since the form $\underline{M}\otimes\underline{N}=(M\otimes N, q\otimes q')$ is finite quadratic O_K-module associated with O_K-bilinear symmetric form $\beta_q\otimes\gamma_{q'}$ then $\underline{M}\otimes\underline{N}$ is called *"tensor product"* of the finite quadratic O_K-modules $\underline{M}=(M,q)$ and $\underline{N}=(N,q')$.

From the definitions which are given in [10] and [5, p. 44] for the modules over the commutative rings (with unit) we can give the following definition for the finite O_K-modules.

Definition 3. Let M be an O_K-module and U be a finite O_K-submodule of M. If M/U is finitely generated, then U is called *cofinite O_K-submodule* of M.

Definition 4. Let $\underline{M} = (M,q)$ be a finite quadratic O_K-module, U be a finite O_K-submodule of M and $0 \to U^{\#} \xrightarrow{f} M$ be an exact sequence of finite O_K-modules such that $M/f(U^{\#})$ is finitely generated. If the sequence $0 \to U \otimes U^{\#} \xrightarrow{I_U \otimes f} U \otimes M$ is exact, then U is called *cofinitely flat quadratic O_K-module*, where $U^{\#}$ is a dual group of U and it is defined by $U^{\#} = \{y \in \underline{M} \mid B_q(U,y) = 0\}$. We notice that $U^{\#}$ is also an O_K-submodule of M.

Proposition 1 ([2]). *$\underline{M} = (M,q)$ be a finite quadratic O_K-module associated with the bilinear form β_q and U be an O_K-submodule of $\underline{M}$. The application $x \mapsto \beta_q(x,\cdot)$ defines an exact sequence of O_K-modules:*

$$0 \to U^{\#} \to \underline{M} \to Hom(U,K/\sigma^{-1}) \to 0.$$

Here $Hom(U,K/\sigma^{-1})$ denotes the group of O_K-module homomorphism of U into K/σ^{-1}. In particular, one has $|U| \cdot |U^{\#}| = |M|$ and $(U^{\#})^{\#} = U$.

3. Main Theorem

Theorem 1. *Let $\underline{M}$ be a finite quadratic O_K-module and U be an O_K- submodule of $\underline{M}$. Let $i : T \to \underline{M}$ be an inclusion homomorphism where T is any cofinite O_K-submodule of $\underline{M}$. U is a cofinitely flat O_K-module if and only if $I_U \otimes i : U \otimes T \to U \otimes \underline{M}$ is injective.*

Proof. :⇒ Since U is a cofinitely flat O_K-module from the hypothesis of our theorem then it is clear that $U^{\#}$ is also a cofinitely flat O_K-module from Definition 4. Hence if we take $U^{\#}$ instead of T then the sequence $0 \to U \otimes U^{\#} \xrightarrow{I_U \otimes f} U \otimes \underline{M}$ is exact which gives the injectivity of $I_U \otimes i$.
⇐: Let U be a cofinitely flat O_K-module and $i : U^{\#} \to \underline{M}$ be inclusion map for $T = U^{\#}$ and let $I_U \otimes i : U \otimes U^{\#} \to U \otimes \underline{M}$ be injective. If we take the sequence $0 \to U \xrightarrow{f} \underline{M}$ of the finite O_K- modules such that $\underline{M}/f(U)$ is finitely generated then $\underline{M}/U^{\#}$ is finitely generated for $f(U) = U^{\#}$. If we get $h : U \to U^{\#}$, $y \mapsto h(y) = f(y)$ $(\forall y \in U)$ then it is clear that h is a homomorphism of finite O_K-modules.

Since $f = i \circ h$ then we can write $I_U \otimes f = (I_U \circ I_U) \otimes (i \circ h) = (I_U \otimes i) \circ (I_U \otimes h)$ where $I_U \otimes h$ is isomorphism of finite O_K-modules from properties (3) and (4) in [10, p. 92].

Since $(I_U \otimes i)$ and $(I_U \otimes h)$ are injective then $I_U \otimes f$ is injective and so the sequence $0 \to U \otimes U^{\#} \xrightarrow{I_U \otimes f} U \otimes \underline{M}$ is exact. Therefore, U is a cofinitely flat O_K-module. □

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