Miskolc Mathematical Notes
Vol. 25 (2024), No. 1, pp. 141–151

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2024.4465

RIESZ POTENTIALS IN THE LOCAL VARIABLE MORREY-LORENTZ SPACES AND SOME APPLICATIONS

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Received 27 October, 2022

Abstract. In this paper, we prove the boundedness of the Riesz potential I_α in local variable Morrey-Lorentz spaces. Also we apply our results to particular operators such as fractional maximal operator, fractional Marcinkiewicz operator and fractional powers of some analytic semigroups in these spaces.

2010 *Mathematics Subject Classification:* 42B25; 42B35; 47G10

Keywords: local variable Morrey-Lorentz space, Riesz potential

1. INTRODUCTION

The Lorentz-Morrey space $\mathcal{L}_{p,q;\lambda}(\mathbb{R}^n)$ was first defined in [22] and also considered in [15, 24, 31]. Later, the local Morrey-Lorentz spaces $\mathcal{M}^{loc}_{p,q;\lambda}(\mathbb{R}^n)$ are introduced and the basic properties of these spaces are given in [1]. These spaces are a very natural generalization of the Lorentz spaces such that $\mathcal{M}^{loc}_{p,q;0}(\mathbb{R}^n) = L_{p,q}(\mathbb{R}^n)$. Recently, in [2, 13] and [14] the authors have studied the boundedness of the Hilbert transform, the Hardy-Littlewood maximal operator M and the Calderón-Zygmund operators T, and the Riesz potential I_α on the local Morrey-Lorentz spaces $\mathcal{M}^{loc}_{p,q;\lambda}$ by using related rearrangement inequalities, respectively.

The study of function spaces with variable exponent has been stimulated by problems of elasticity, fluid dynamics, calculus of variations and differential equations with non-standard growth conditions (see [6, 8, 25, 26, 33]). Various results on non-weighted and weighted boundedness in variable exponent Lebesgue spaces have been proved for maximal, singular and fractional type operators, we refer to surveying papers [9] and [27]. In [18], authors define the variable local Morrey-Lorentz spaces and proved the boundedness of maximal operator M and Calderon-Zygmund operators T in these spaces, also they apply their results to some operators of harmonic analysis such as Bochner-Riesz operator, Marcinkiewicz operator and fractional powers of some analytic semigroups. In [32], the authors give the definition of central Lorentz-Morrey space of variable exponent by the symmetric decreasing rearrangement. They prove the boundedness of maximal operator in these spaces and establish Sobolev's inequality for Riesz potentials.

Local variable Morrey-Lorentz spaces generalize variable exponent Lorentz spaces such that $\mathcal{M}^{loc}_{p(\cdot),q(\cdot);0} = L_{p(\cdot),q(\cdot)}$, when $\lambda = 0$. In [11] variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot)}$ are introduced and the boundedness of the singular integral and fractional type operators and corresponding ergodic operators are proved in these spaces. The inclusion theorems for variable Lorentz spaces were proved in [19]. We should point out that, due to their own fine structures, Lorentz spaces appear frequently in the study on various critical or endpoint analysis problems from many different research fields and there exist enormous literatures on this subject.

In this paper, we prove the boundedness of Riesz potential I_α from the local variable Morrey-Lorentz spaces $\mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$ to local variable Morrey-Lorentz spaces $\mathcal{M}^{loc}_{q(\cdot),s(\cdot),\lambda}(\mathbb{R}^n)$. We also give some applications of our main result.

The organization of this article is as follows. In Section 2, we give some notation and definitions. In Section 3, we prove the boundedness of the Riesz potential I_α from the local variable Morrey-Lorentz spaces $\mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$ to local variable Morrey-Lorentz spaces $\mathcal{M}^{loc}_{q(\cdot),s(\cdot),\lambda}(\mathbb{R}^n)$. In Section 4, as applications we get the boundedness of fractional maximal operator M_α, fractional Marcinkiewicz operator $\mu_{\Omega,\alpha}$ and fractional powers of some analytic semi groups $L^{-\alpha/2}$ from $\mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$ to $\mathcal{M}^{loc}_{q(\cdot),s(\cdot),\lambda}(\mathbb{R}^n)$.

Throughout the paper we use the letter C for a positive constant, independent of appropriate parameters and not necessary the same at each occurrence.

2. PRELIMINARIES

The present paper deals with the boundedness of the Riesz potential I_α defined by

$$I_\alpha f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad 0 < \alpha < n, \ f \in L_1^{loc}(\mathbb{R}^n),$$

in the local Morrey–Lorentz spaces $M^{loc}_{p,q;\lambda}(\mathbb{R}^n)$.

Further we apply this result to particular operators such as a fractional maximal operator, fractional Marcinkiewicz operator and fractional powers of some analytic semigroups.

For each measurable function f on $(0,\infty)$ and each $t > 0$, the following operator

$$(S_\alpha f)(t) = t^{\frac{\alpha}{n}-1} \int_0^t f(s)ds + \int_t^\infty s^{\frac{\alpha}{n}-1} f(s)ds$$

was defined by A. P. Calderón [5]. The importance of S_α is based on the fact that it dominates the Riesz potential I_α.

Theorem 1 ([23, 28]). *If the condition*

$$\left(S_\alpha f^*\right)(1) = \int_0^1 f^*(s)ds + \int_1^\infty s^{\frac{\alpha}{n}-1} f^*(s)ds < \infty \tag{2.1}$$

holds for $f \in L_1^{loc}(\mathbb{R}^n)$*, then the Riesz potential* $\left(I_\alpha f\right)(x)$*,* $x \in \mathbb{R}^n$*, exists almost everywhere. Furthermore, the inequality*

$$\left(I_\alpha f\right)^*(t) \le C S_\alpha\left(f^*\right)(t), \quad 0 < t < \infty, \tag{2.2}$$

is valid, where f^* *denotes the non-increasing rearrangement of* f *defined by*

$$f^*(t) = \inf\{\lambda > 0 : |\{y \in \mathbb{R}^n : |f(y)| > \lambda\}| \le t\} \quad \textit{for all } t \in (0,\infty)$$

and C *is a constant independent of* f *and* t.

Let $p(t)$ be a measurable function on $(0,\infty)$. We mainly suppose that

$$1 < p_- \le p(t) \le p_+ < \infty, \tag{2.3}$$

where

$$p_- := \inf_{0<t\le\infty} p(t), \quad p_+ := \sup_{0<t\le\infty} p(t).$$

We denote by $p'(\cdot) = \frac{p(t)}{p(t)-1}$. We will use the following decay conditions:

$$|p(t) - p(0)| \le \frac{A_0}{|lnt|}, \quad 0 < t \le \frac{1}{2} \tag{2.4}$$

$$|p(t) - p(\infty)| \le \frac{A_\infty}{lnt}, \quad t \ge 2, \tag{2.5}$$

where $A_0, A_\infty > 0$ do not depend on t.

By $p \in \mathcal{P}_{0,\infty}(0,\infty)$ we denote the set of bounded measurable functions (not necessarily with values in $[1,\infty)$), which satisfy the decay conditions (2.4) and (2.5). Also, by $L_{p(\cdot)}(0,\infty)$ we denote the variable exponent Lebesgue space of measurable functions φ on $(0,\infty)$ such that

$$\mathcal{I}_{p(\cdot)}(\varphi) = \int_0^\infty |\varphi(s)|^{p(s)} ds < \infty.$$

This is a Banach function space with respect to the norm (see e.g. [10])

$$\|\varphi\|_{L_{p(\cdot)}} = \inf\left\{\lambda > 0 : \mathcal{I}_{p(\cdot)}\left(\frac{\varphi}{\lambda}\right) \le 1\right\}.$$

Definition 1 ([21]). Let q satisfy the condition (2.3). We denote by $LM_{q(\cdot),\lambda} \equiv LM_{q(\cdot),\lambda}(0,\infty)$ the variable exponent local Morrey space with finite norm

$$\begin{aligned}
\|\varphi\|_{LM_{q(\cdot),\lambda}} &= \sup_{t>0} t^{-\frac{\lambda}{q_*(t)}} \|\varphi\|_{L_{q(\cdot)}(0,t)} \\
&= \sup_{t>0} \inf\left\{\eta > 0 : \int_0^t \left|\frac{\varphi(s)}{\eta t^{\frac{\lambda}{q_*(t)}}}\right|^{q(s)} ds \le 1\right\},
\end{aligned}$$

where $q_*(t) = q(0), 0 < t < 1$ and $q_*(t) = q(\infty), t \geq 1$.

Definition 2 ([11]). Let p, q satisfy the condition (2.3). We denote by $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ variable exponent Lorentz space, the space of functions f on $\mathbb{R}^n$ such that $t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^*(t) \in L_{q(\cdot)}(0,\infty)$, i.e.

$$\mathcal{J}_{p(\cdot),q(\cdot)}(f) = \int_0^\infty t^{\frac{q(t)}{p(t)}-1}(f^*(t))^{q(t)}dt < \infty$$

and we denote

$$\|f\|_{L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)} = \inf\left\{\sigma > 0 : \mathcal{J}_{p(\cdot),q(\cdot)}\left(\frac{f}{\sigma}\right) \leq 1\right\} = \left\|t^{\frac{1}{p(t)}-\frac{1}{q(t)}} f^*(t)\right\|_{L_{q(\cdot)}(0,\infty)},$$

where f^* denotes the non-increasing rearrangement of f such that

$$f^*(t) = \inf\{\lambda > 0 : |\{y \in \mathbb{R}^n : |f(y)| > \lambda\}| \leq t\}, \quad \forall t \in (0,\infty)$$

and

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds$$

(see [3]). More information about variable exponent Lorentz spaces can be found in [11, 16].

Now we give the definition of local variable Morrey-Lorentz spaces, which is defined in [18].

Definition 3. Let p, q satisfy the condition (2.3) and $0 \leq \lambda < 1$. We denote by $\mathcal{M}^{loc}_{p(\cdot),q(\cdot),\lambda} \equiv \mathcal{M}^{loc}_{p(\cdot),q(\cdot),\lambda}(\mathbb{R}^n)$ the local variable Morrey-Lorentz space, the space of all measurable functions with finite quasinorm

$$\|f\|_{\mathcal{M}^{loc}_{p(\cdot),q(\cdot),\lambda}} := \sup_{t>0} t^{-\frac{\lambda}{q_*(t)}} \|\tau^{\frac{1}{p(\tau)}-\frac{1}{q(\tau)}} f^*(\tau)\|_{L_{q(\cdot)}(0,t)}.$$

These spaces generalize variable exponent Lorentz spaces such that $\mathcal{M}^{loc}_{p(\cdot),q(\cdot);0} = L_{p(\cdot),q(\cdot)}$, when $\lambda = 0$ (see [11]). Also, if $\lambda = 0$ and $q(\cdot) = p(\cdot)$ then $\mathcal{M}^{loc}_{p(\cdot),p(\cdot);0} = L_{p(\cdot)}$ are variable exponent Lebesgue spaces (see [17]).

3. The boundedness of Riesz potential I_α in the local variable Morrey-Lorentz spaces

In this section, we prove the boundedness of Riesz potential I_α in the local variable Morrey-Lorentz spaces. We need the following two definitions about Hardy operators which are used in the proof of our main theorems. These operators are very important in analysis and have been widely studied.

Definition 4 ([11]). Let $\beta(t)$ and $\gamma(t)$ are measurable functions on $(0,\infty)$. The weighted Hardy operators $H^{\beta(\cdot)}_{\gamma(\cdot)}$ and $\mathcal{H}^{\beta(\cdot)}_{\gamma(\cdot)}$ with power weight acting on φ are defined by

$$H^{\beta(\cdot)}_{\gamma(\cdot)}\varphi(t) = t^{\beta(t)+\gamma(t)-1}\int_0^t \frac{\varphi(y)}{y^{\gamma(y)}}dy$$

and

$$\mathcal{H}^{\beta(\cdot)}_{\gamma(\cdot)}\varphi(t) = t^{\beta(t)+\gamma(t)}\int_t^\infty \frac{\varphi(y)}{y^{\gamma(y)+1}}dy.$$

The following lemma provides some minimal assumptions on the function $\tau^{\frac{\lambda}{q_*(\tau)}}$ under which the so-defined spaces contain "nice" functions.

Lemma 1 ([21]). *Let r satisfy the condition (2.3), $r \in \mathcal{P}_0(0,\infty)$ and $0 \le \lambda < 1$. Then the assumption*

$$\sup_{\tau>0} \frac{[min\{1,\tau\}]^{\frac{1}{r(0)}}}{\tau^{\frac{\lambda}{r_*(\tau)}}} < \infty \tag{3.1}$$

is sufficient for bounded functions f with compact support to belong to the local variable Morrey-Lorentz spaces $\mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$.

Lemma 2 ([21]). *Let r satisfy the condition (2.3), $r \in \mathcal{P}_0(0,\infty)$, $0 \le \lambda < 1$, $\lim_{t\to 0} t^{\gamma(t)}$ exists and finite and the condition (3.1) is satisfied. Suppose that the following conditions hold.*

(i) $t^{\gamma(t)-a}$ and $t^{\frac{\lambda}{r_*(t)}-\gamma(t)-a}$ are almost decreasing for some $a \in \mathbb{R}$, in the case of operator $H^{\beta(\cdot)}_{\gamma(\cdot)}$.

(ii) $t^{-\gamma(t)+b}$ and $t^{\frac{\lambda}{r_*(t)}-\gamma(t)+b}$ are almost increasing for some $b \in \mathbb{R}$, in the case of operator $\mathcal{H}^{\beta(\cdot)}_{\gamma(\cdot)}$.

Then the conditions

$$\gamma(t) < \frac{\lambda}{r_*(t)} + \frac{1}{r'(0)}, \qquad \gamma(t) > \frac{\lambda}{r_*(t)} - \frac{1}{r(\infty)}$$

are sufficient for the Hardy operators $H^{\beta(\cdot)}_{\gamma(\cdot)}$ and $\mathcal{H}^{\beta(\cdot)}_{\gamma(\cdot)}$, respectively, to be defined on the space $LM_{r(\cdot),\lambda}(\mathbb{R}^n)$.

Lemma 3 ([21]). *Let r satisfy the condition (2.3), $r \in \mathcal{P}_0(0,\infty)$, $0 \le \lambda < 1$. Suppose also that the conditions (3.1) and of Lemma 2 are satisfied. Then the operators $H^{\beta(\cdot)}_{\gamma(\cdot)}$ and $\mathcal{H}^{\beta(\cdot)}_{\gamma(\cdot)}$ are bounded from the space $LM_{r(\cdot),\lambda}(0,\infty)$ to the space $LM_{s(\cdot),\lambda}(0,\infty)$ if $\gamma(t) < \frac{\lambda}{r_*(t)} + \frac{1}{r'(t)}$, $\gamma(t) > \frac{\lambda}{r_*(t)} - \frac{1}{r(t)}$, respectively.*

The following theorem is the main result of our paper in which we give the boundedness of Riesz potential in the local variable Morrey-Lorentz spaces.

Theorem 2. *Let p,q,r,s satisfy the condition (2.3), $p,q,r,s \in \mathcal{P}_{0,\infty}(0,\infty)$, $0 \le \lambda < 1$ and $f \in \mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$. Suppose that the conditions (3.1) and of Lemma 2 are satisfied. If $\frac{r(t)}{r(t)+\lambda} < p(t) < \left(\frac{\lambda}{r(t)} + \frac{\alpha}{n}\right)^{-1}$, $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda\left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{n}$, then the Riesz potential I_α is bounded from the local variable Morrey-Lorentz spaces $\mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$ to $\mathcal{M}^{loc}_{q(\cdot),s(\cdot),\lambda}(\mathbb{R}^n)$.*

Proof. From the definition of local variable Morrey-Lorentz spaces and the inequality (2.2) we get

$$\begin{aligned}
\|I_\alpha f\|_{M^{loc}_{q(\cdot),s(\cdot),\lambda}} &= \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \|\tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)}} (I_\alpha f)^*(\tau)\|_{L_{s(\cdot)}(0,t)} \\
&\le C \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \left\| \tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)}} \left(\tau^{\frac{\alpha}{n}-1} \int_0^\tau f^*(y)dy + \int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y)dy \right) \right\|_{L_{s(\cdot)}(0,t)} \\
&\le C \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \left\| \tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)} + \frac{\alpha}{n} - 1} \int_0^\tau f^*(y)dy \right\|_{L_{s(\cdot)}(0,t)} \\
&+ C \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \left\| \tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)}} \int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y)dy \right\|_{L_{s(\cdot)}(0,t)} \\
&= I_1 + I_2.
\end{aligned}$$

Let us estimate I_1:

$$I_1 = C \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \left\| \tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)} + \frac{\alpha}{n} - 1} \int_0^\tau f^*(y)dy \right\|_{L_{s(\cdot)}(0,t)} = C \|H^{\beta(\cdot)}_{\gamma(\cdot)} g\|_{L_{s(\cdot),\lambda}(0,\infty)}.$$

We take $\gamma(t) = \frac{1}{p(t)} - \frac{1}{r(t)}$ and consider the Hardy operator $H^{\beta(\cdot)}_{\gamma(\cdot)}$ and $g(t) = t^{\frac{1}{p(t)} - \frac{1}{r(t)}} f^*(t)$. Therefore we get $\beta(t) = \frac{1}{q(t)} - \frac{1}{s(t)} + \frac{1}{r(t)} - \frac{1}{p(t)} + \frac{\alpha}{n}$. By Lemma 3, we have $\beta(t) = (1-\lambda)\left(\frac{1}{r(t)} - \frac{1}{s(t)}\right)$, then we obtain $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda\left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{n}$.

Hence the operator $H^{\beta(\cdot)}_{\gamma(\cdot)}$ is bounded from the Morrey space $L_{r(\cdot),\lambda}(0,\infty)$ to $L_{s(\cdot),\lambda}(0,\infty)$ under the condition $\gamma(t) = \frac{1}{p(t)} - \frac{1}{r(t)} < \frac{1}{r'(t)} + \frac{\lambda}{r_*(t)}$. Then we get

$$\begin{aligned}
I_1 \le C \|H^{\beta(\cdot)}_{\gamma(\cdot)} g\|_{L_{s(\cdot),\lambda}(0,\infty)} &\le C \|g\|_{L_{r(\cdot),\lambda}(0,\infty)} \\
&= C \sup_{t>0} t^{-\frac{\lambda}{r_*(\cdot)}} \|\tau^{\frac{1}{p(\tau)} - \frac{1}{r(\tau)}} f^*(\tau)\|_{L_{r(\cdot)}(0,t)} \\
&= C \|f\|_{M^{loc}_{p(\cdot),r(\cdot);\lambda}}. \qquad (3.2)
\end{aligned}$$

Now we consider I_2:

$$I_2 = C \sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \left\| \tau^{\frac{1}{q(\tau)} - \frac{1}{s(\tau)}} \int_\tau^\infty y^{\frac{\alpha}{n}-1} f^*(y)dy \right\|_{L_{s(\cdot)}(0,t)} = C \|\mathcal{H}^{\beta(\cdot)}_{\gamma(\cdot)} g\|_{L_{s(\cdot),\lambda}(0,\infty)}.$$

We take $\gamma(t) = \frac{1}{p(t)} - \frac{1}{r(t)} - \frac{\alpha}{n}$ in the Hardy operator $\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)}$ and $g(t) = t^{\frac{1}{p}-\frac{1}{r}} f^*(t)$. Therefore we get $\beta(t) = \frac{1}{q(t)} - \frac{1}{s(t)} + \frac{1}{r(t)} - \frac{1}{p(t)} + \frac{\alpha}{n}$. By Lemma 3, we have $\beta(t) = (1-\lambda)\left(\frac{1}{r(t)} - \frac{1}{s(t)}\right)$, then we obtain $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda\left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{n}$.

Hence the operator $\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)}$ is bounded from the Morrey space $L_{r,\lambda}(0,\infty)$ to $L_{s,\lambda}(0,\infty)$ under the condition $\frac{\lambda}{r_*(t)} - \frac{1}{r(t)} < \gamma(t) = \frac{1}{p(t)} - \frac{1}{r(t)} - \frac{\alpha}{n}$. Then we get

$$\begin{aligned}\sup_{t>0} t^{-\frac{\lambda}{s_*(t)}} \|\mathcal{H}_{\gamma(\cdot)}^{\beta(\cdot)}\|_{L_{s(\cdot)}(0,t)} \le C\|g\|_{L_{r(\cdot),\lambda}(0,\infty)} &= C\sup_{t>0} t^{-\frac{\lambda}{r_*(t)}} \|\tau^{\frac{1}{p(\tau)}-\frac{1}{r(\tau)}} f^*(\tau)\|_{L_{r(\cdot)}(0,t)} \\ &= C\|f\|_{M^{loc}_{p(\cdot),r(\cdot);\lambda}}. \end{aligned} \tag{3.3}$$

From inequalities (3.2) and (3.3) we obtain the boundedness of the operator I_α from $M^{loc}_{p(\cdot),r(\cdot);\lambda}$ to $M^{loc}_{q(\cdot),s(\cdot);\lambda}$.

□

Remark 1. In the case $\lambda = 0$, from Theorem 2 we get the boundedness of the operator I_α in the variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ which is proved in [11].

4. Some applications

Theorem 2 can be applied to various operators which are estimated from above by the Riesz potentials. In this section, we apply the theorem to the fractional maximal operator, fractional Marcinkiewicz operator and the fractional powers of some analytic semigroups.

4.1. *Fractional maximal operator*

For $0 \le \alpha < n$, we define the fractional maximal operator

$$M_\alpha f(x) = \sup_{t>0} |B(x,t)|^{\frac{\alpha}{n}-1} \int_{B(x,t)} |f(y)|dy,$$

where $B(x,t)$ is the open ball centered at x of radius t for $x \in \mathbb{R}^n$ and $|B(x,t)|$ is a Lebesgue measure of $B(x,t)$ such that $|B(x,t)| = \omega_n t^n$ in which ω_n denotes the volume of the unit ball in $\mathbb{R}^n$. The fractional maximal operator M_α is closely related to a Riesz potential operator such that

$$M_\alpha f(x) \le \omega_n^{\frac{\alpha}{n}-1} (I_\alpha |f|)(x). \tag{4.1}$$

Corollary 1. *Let p,q,r,s satisfy the condition (2.3), $p,q,r,s \in \mathcal{P}_{0,\infty}(0,\infty)$, $0 \le \lambda < 1$, and $f \in \mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$. Suppose that the conditions (3.1) and of Lemma 2 are satisfied. If $\frac{r(t)}{r(t)+\lambda} < p(t) < \left(\frac{\lambda}{r(t)} + \frac{\alpha}{n}\right)^{-1}$, $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda\left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{n}$, then the fractional maximal operator M_α is bounded from the local variable Morrey-Lorentz spaces $\mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$ to $\mathcal{M}^{loc}_{q(\cdot),s(\cdot),\lambda}(\mathbb{R}^n)$.*

Remark 2. In the case $\lambda = 0$, from Theorem 2 we get the boundedness of the operator M_α in the variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ which is proved in [11].

4.2. *Fractional Marcinkiewicz operator*

Let $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$ be the unit sphere in $\mathbb{R}^n$ equipped with the Lebesgue measure $d\sigma$. Suppose that Ω satisfies the following conditions.

(a) Ω is the homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, i.e.,

$$\Omega(tx) = \Omega(x) \ \text{ for any } \ t > 0, \ \ x \in \mathbb{R}^n \setminus \{0\}.$$

(b) Ω has mean zero on S^{n-1}, i.e.,

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0.$$

(c) $\Omega \in \mathrm{Lip}_\gamma(S^{n-1})$, $0 < \gamma \le 1$, that is, there exists a constant $C > 0$ such that

$$|\Omega(x') - \Omega(y')| \le C|x' - y'|^\gamma \ \text{ for any } \ x', y' \in S^{n-1}.$$

In 1958, Stein [29] defined the Marcinkiewicz integral of higher dimension μ_Ω as

$$\mu_{\Omega,\alpha}(f)(x) = \left(\int_0^\infty |F_{\Omega,\alpha,t}(f)(x)|^2 \frac{dt}{t^3} \right)^{1/2},$$

where

$$F_{\Omega,\alpha,t}(f)(x) = \int_{|x-y|\le t} \frac{\Omega(x-y)}{|x-y|^{n-1-\alpha}} f(y)dy.$$

The continuity of the Marcinkiewicz operator μ_Ω was extensively studied in [7, 12, 20, 30]. Let H be the space $H = \{h : \|h\| = (\int_0^\infty |h(t)|^2 dt/t^3)^{1/2} < \infty\}$. Then it is clear that $\mu_\Omega(f)(x) = \|F_{\Omega,t}(f)(x)\|$.

By Minkowski inequality and the conditions on Ω, we get

$$\begin{aligned} \mu_{\Omega,\alpha}(f)(x) &\le \int_{\mathbb{R}^n} \frac{|\Omega(x-y)|}{|x-y|^{n-1-\alpha}} |f(y)| \left(\int_{|x-y|}^\infty \frac{dt}{t^3} \right)^{1/2} dy \\ &\le C \int_{\mathbb{R}^n} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy = I_\alpha(|f|)(x). \end{aligned}$$

Then we have the following corollary.

Corollary 2. *Let p, q, r, s satisfy the condition (2.3), $p, q, r, s \in \mathcal{P}_{0,\infty}(0,\infty)$ and $f \in \mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$. Suppose that the conditions (3.1) and of Lemma 2 are satisfied. If $\frac{r(t)}{r(t)+\lambda} < p(t) < \left(\frac{\lambda}{r(t)} + \frac{\alpha}{n}\right)^{-1}$, $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda\left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{n}$, then the fractional Marcinkiewicz operator $\mu_{\Omega,\alpha}$ is bounded from the local variable Morrey-Lorentz spaces $\mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$ to $\mathcal{M}^{loc}_{q(\cdot),s(\cdot),\lambda}(\mathbb{R}^n)$.*

Remark 3. In the case $\lambda = 0$, from Theorem 2 we get the boundedness of the operator $\mu_{\Omega,\alpha}$ in the variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

4.3. *Fractional powers of some analytic semigroups*

Suppose that L is a linear operator on L_2 which generates an analytic semigoup e^{-tL} with the kernel $p_t(x,y)$ satisfying a Gaussian upper bound, i.e.,

$$|p_t(x,y)| \leq \frac{c_1}{t^{n/2}} e^{-c_2 \frac{|x-y|^2}{t}} \tag{4.2}$$

for $x,y \in \mathbb{R}^n$ and all $t > 0$, where $c_1, c_2 > 0$ are independent of x,y and t.

For $0 < \alpha < n$, the fractional powers $L^{-\alpha/2}$ of the operator L are defined by

$$L^{-\alpha/2} f(x) = \frac{1}{\Gamma(\alpha/2} \int_0^\infty e^{-tL} f(x) \frac{dt}{t^{-\alpha/2+1}}.$$

Note that if $L = -\Delta$ is the Laplacian on $\mathbb{R}^n$, then $L^{-\alpha/2}$ is the Riesz potential I_α. Property (4.2) is satisfied for large classes of differential operators. In [4], other examples of operators which are estimates from above by the Riesz potentials are given. Since the semigroup e^{-tL} has the kernel $p_t(x,y)$ which satisfies condition (4.2), it follows that

$$|L^{-\alpha/2} f(x)| \leq C I_\alpha(|f|)(x).$$

Hence we get the following corollary.

Corollary 3. *Let p,q,r,s satisfy the condition (2.3), $p,q,r,s \in \mathcal{P}_{0,\infty}(0,\infty)$, $0 \leq \lambda < 1$ and $f \in \mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$. Suppose that the conditions (3.1) and of Lemma 2 are satisfied. If $\frac{r(t)}{r(t)+\lambda} < p(t) < \left(\frac{\lambda}{r(t)} + \frac{\alpha}{n}\right)^{-1}$, $\frac{1}{p(t)} - \frac{1}{q(t)} = \lambda\left(\frac{1}{r(t)} - \frac{1}{s(t)}\right) + \frac{\alpha}{n}$, then the operator $L^{-\alpha/2}$ is bounded from the local variable Morrey-Lorentz spaces $\mathcal{M}^{loc}_{p(\cdot),r(\cdot),\lambda}(\mathbb{R}^n)$ to $\mathcal{M}^{loc}_{q(\cdot),s(\cdot),\lambda}(\mathbb{R}^n)$.*

Remark 4. In the case $\lambda = 0$, from Theorem 2 we get the boundedness of the operator $L^{-\alpha/2}$ in the variable exponent Lorentz spaces $L_{p(\cdot),q(\cdot)}(\mathbb{R}^n)$.

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