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SOME NEW FILTERS AND THEIR PROPERTIES ON BL-ALGEBRAS

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Abstract. In this paper we introduce the notion of a DC-prime filter in BL-algebras and obtain some theorems in order to determine the relationships between this notion and some types of filters in a BL-algebra. By some examples, we show that each of these concepts is different from the other. Also, using the notion of DC-prime filters, a new class of BL-algebras are defined and studied.

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1. INTRODUCTION

BL-algebras are the algebraic structures for Hájek's basic logic. His motivations for introducing BL-algebras were of two kinds. The first one was to provide an algebraic counterpart of a propositional logic, called Basic Logic, which embodies a fragment common to some of the most important many-valued logics, namely Łukasiewicz Logic, Gödel Logic and Product Logic. This Basic Logic (BL for short) is proposed as "the most general" many-valued logic with truth values in $[0,1]$ and BL-algebras are the corresponding Lindenbaum-Tarski algebras. Hájek's second motivation was to provide an algebraic mean for the study of continuous t-norms (or triangular norms) on $[0,1]$. BL-algebras arise as Lindenbaum algebras from certain logical axioms in a similar manner that Boolean algebras or MV-algebras do from classical logic or Łukasiewicz logic, respectively [12]. Today this many valued logic has been developed into a fuzzy logic, which meets together with the motivations from the beginning of the twentieth century in one theory that connects quantum mechanics, mathematical logic, probability theory, computer science, algebra, soft computing, and many other important aspects of our modern world. Filters theory plays an important role in studying these logical algebras and ordered semigroups.

From logical point of view, various filters correspond to various sets of provable formulas. Based on these results, some types of filters on BL-algebras have been widely studied and some important results have been obtained. This was the motivation for the researchers of this study to introduce some new filters in BL-algebras.

The structure of the paper is as follows: Section 2 is a recall of some definitions and results about BL-algebras that are used in the paper. In Section 3, a new filter (DC-prime filter) in BL-algebras and a new class of BL-algebras (DCP-BL-algebra) are introduced and some of their basic properties are provided. Section 4 is the main section of the paper. In this section, many new properties of DC-prime filters in BL-algebras, are obtained. Also, the relationships between this new filter and other types of filters in BL-algebras are investigated. In Section 5, many different characterizations and many important properties of DC-prime filters in BL-algebras, are proved. Also, some new properties for prime filters, maximal filters and primary filters in BL-algebras were established.

2. PRELIMINARIES

Definition 1 ([9, Definition 2.3.3]). A BL-algebra is an algebra $(A,\wedge,\vee,*,\rightarrow,0,1)$ with four binary operations $\wedge,\vee,*,\rightarrow$ and two constants $0,1$ such that:

- (BL_1) $(A,\wedge,\vee,0,1)$ is a bounded lattice $L(A)$;
- (BL_2) $(A,*,1)$ is a commutative monoid;
- (BL_3) $*$ and $\rightarrow$ form an adjoint pair i.e, $c \leq a \rightarrow b$ if and only if $a*c \leq b$, for all $a,b,c \in A$;
- (BL_4) $a \wedge b = a*(a \rightarrow b)$;
- (BL_5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

For $a \in A$ and a natural number n we define $a^n = a*a*\cdots*a$, (n-times), and $a^- = a \rightarrow 0$.

Theorem 1. *In any* BL-*algebra A, the following properties hold for all* $x,y,z \in A$*:*

(1) $x \leq y$, *if and only if* $x \rightarrow y = 1$, *[9, Lemma 2.2.8-2.2.10];*
(2) $x \rightarrow (y \rightarrow z) = (x*y) \rightarrow z = y \rightarrow (x \rightarrow z)$, *[9, Lemma 2.2.8-2.2.10];*
(3) if $x \leq y$, *then* $y \rightarrow z \leq x \rightarrow z$ *and* $z \rightarrow x \leq z \rightarrow y$, *[9, Lemma 2.2.8-2.2.10];*
(4) $x*y \leq x,y$, *hence* $x*y \leq x \wedge y$ *and* $x*0 = 0$, *[9, Lemma 2.2.8-2.2.10];*
(5) $x \leq y$ *implies* $x*z \leq y*z$, *[9, Lemma 2.2.8-2.2.10];*
(6) $1 \rightarrow x = x$, $x \rightarrow x = 1$, $x \leq y \rightarrow x$, $x \rightarrow 1 = 1$, $0 \rightarrow x = 1$ *and* $x*x^- = 0$, *[9, Lemma 2.2.8-2.2.10];*
(7) $x*y = 0$, *if and only if* $x \leq y^-$ *and* $x \leq y$ *implies* $y^- \leq x^-$, *[20, Theorem 13,14];*
(8) $x \leq x^{--}$, $1^- = 0$, $0^- = 1$, $x^{---} = x^-$ *and* $x^{--} \leq x^- \rightarrow x$, *[20, Theorem 13,14];*
(9) if $x^{--} \leq x^{--} \rightarrow x$, *then* $x^{--} = x$, *[20, Theorem 13,14];*
(10) $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$, *[9, Lemma 2.2.8-2.2.10];*

(11) $x*(y\vee z)=(x*y)\vee(x*z)$, *[6];*
(12) $(x*y)^{-}=x\to y^{-}$, *[6];*
(13) $(x\wedge y)*(y\wedge z)\leq(x\wedge z)$, *[6];*
(14) $x\vee(y*z)\geq(x\vee y)*(x\vee z)$ *and we have* $x^m\vee y^n\geq(x\vee y)^{mn}$, *for any* $m,n\in\mathbb{N}$, *[7];*
(15) $(y\wedge z)\to x=(y\to x)\vee(z\to x)$, *[6].*

Definition 2. A BL-algebra A is called:

(1) an MV-algebra, if for all $x\in A$, $x^{--}=x$, [9, Definition 3.2.2];
(2) a Gödel algebra, if for all $x\in A$, $x^2=x$, [9, Definition 4.2.12];
(3) a Boolean algebra, if for all $x\in A$, $x^2=x$ and $x^{--}=x$, [9, Definition 1.5.9];
(4) a special BL-algebra, if for all $0\neq a,b\in A$, $(a\to b)^{-}=(b\to a)^{-}$, [13, Definition 3.1];
(5) an integral BL-algebra, if for any $x,y\in A$, $x*y=0$, implies $x=0$ or $y=0$, [5, Definition 3.1];
(6) a semi-integral BL-algebra, if for any $x,y\in A$, $x*y=0$, implies $x=0$ or $y^n=0$, for some $n\in\mathbb{N}$, [14, Definition 4].

From now, it is assumed that a BL-algebra A is non-trivial i.e. A has at least two members. In order to simplify the notation, a BL-algebra $(A,\wedge,\vee,*,\to,0,1)$ will be referred to its support set, A.

Recall that $D_s(A)=\{x\in A: x^{-}=0\}$ is the set of the dense elements of a BL-algebra A. Clearly $D_s(A)$ is a proper filter of A.

Recall that if $x^{--}=1$, for $x\in A$, then we say that x is an almost top element of A. Also, $N(A)=\{x\in A:\ x \text{ is an almost top element of } A\}$, [3, Definition 4.4].

Definition 3 ([9, Definition 2.3.13])**.** Let F be a non-empty subset of a BL-algebra A and $a,b\in A$. Then the following conditions are equivalent:

(i) F is a proper filter of A.
(ii) $a,b\in F$ imply $a*b\in F$ and if $a\in F$ and $a\leq b$ then $b\in F$.
(iii) $1\in F$, and if $a,a\to b\in F$ imply $b\in F$.

Definition 4 ([19, Page 82])**.** Let A be a BL-algebra and X be a non-empty subset of A. Then $<X>=\cap\{F\subseteq A: X\subseteq F,\ F \text{ is a filter of } A\}$. Clearly, $<X>$ is the smallest filter of A, containing X.

Theorem 2 ([19, Proposition 3.15, Remark 3.16, Lemma 3.18])**.** *Let* $x\in A$, $X\subseteq A$ *and* F,G *be filters of a* BL-*algebra* A*. Recall that*

(1) $\langle x\rangle=\{a\in A: a\geq x^n, \text{ for some } n\in\mathbb{N}\}$*;*
(2) $\langle X\rangle=\{a\in A: a\geq x_1*x_2*\cdots*x_n, \text{ for some } x_1,x_2,\ldots,x_n\in X\}$*;*
(3) $F\vee\langle x\rangle=\langle F\cup\langle x\rangle\rangle=\{a\in A: a\geq f*x^n, \text{ for some } f\in F \text{ and } n\in\mathbb{N}\}$*;*
(4) $\langle F\cup G\rangle=\{a\in A: a\geq f*g, \text{ for some } f\in F \text{ and } g\in G\}$*;*
(5) $F\vee G:=\langle F\cup G\rangle$ *and* $F\wedge G:=F\cap G$*;*
(6) $F\to G=\{a\in A: F\cap\langle a\rangle\subseteq G\}$.

Definition 5. A non-empty subset F of a BL-algebra A is called:

(1) a prime filter of A, if F is a proper filter and for all $x,y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$, [9, Definition 2.3.13];
(2) a maximal filter of A, if F is a proper filter and for all $x \in A$, if $x \notin F$, there exists $n \in \mathbb{N}$ such that $(x^n)^- \in F$, [9, Theorem 3.43];
(3) a Boolean filter of A, if F is a filter and for all $x \in A$, $x \vee x^- \in F$, [9, Definition 2.9];
(4) a primary filter of A, if F is a proper filter and for all $x,y \in A$, if $(x*y)^- \in F$ implies $(x^n)^- \in F$ or $(y^n)^- \in F$, for some $n \in \mathbb{N}$, [21, Definition 1];
(5) an implicative filter of A, if $1 \in F$ and for all $x,y,z \in A$, if $x \to (y \to z) \in F$, $x \to y \in F$ imply $x \to z \in F$, [10, Definition 3.1];
(6) a positive implicative filter of A, if $1 \in F$ and for all $x,y,z \in A$, if $x \to ((y \to z) \to y) \in F$, $x \in F$ imply $y \in F$, [10, Definition 3.9];
(7) a fantastic filter of A, if $1 \in F$ and for all $x,y,z \in A$, if $z \to (y \to x) \in F$ and $z \in F$ imply $((x \to y) \to y) \to x \in F$, [10, Definition 4.1];
(8) a normal filter of A, if F is a filter of A and if $1 \in F$ and for all $x,y,z \in A$, if $z \to ((y \to x) \to x) \in F$ and $z \in F$ imply $(x \to y) \to y \in F$, [2, Definition 9];
(9) an obstinate filter of A, if F is a proper filter of A and for all $x,y,z \in A$, if $x,y \notin F$ imply $x \to y \in F$ and $y \to x \in F$, [4, Definition 3.1];
(10) a semi maximal filter of A, if F is a filter of A and $\mathrm{Rad}(F) = F$, [18, Definition 4.1];
(11) a special filter of A, if F is a proper filter and $(x \to y)^- = (y \to x)^-$, for all $x,y \in F$, [13, Definition 4.1];
(12) an easy filter of A, if F is a filter and for all $x,y,z \in A$, if $x^{--} \to (y \to z) \in F$ and $x^{--} \to y \in F$ imply $x^{--} \to z \in F$, [8, Definition 13];
(13) an integral filter of A, if F is a proper filter and for all $x,y \in A$, if $(x*y)^- \in F$ imply $x^- \in F$ or $y^- \in F$, [5, Definition 4.1];
(14) a semi-integral filter of A, if F is a proper filter and for all $x,y \in A$, if $(x*y)^- \in F$ then there exists $n \in \mathbb{N}$ such taht $x^- \in F$ or $(y^n)^- \in F$, [14, Definition 5].

Definition 6 ([18, Definition 3.1])**.** If F is a proper filter of a BL-algebra A, the intersection of all maximal filters of L containing F is called the radical of F and it is denoted by $\mathrm{Rad}(F)$ and $\mathrm{Rad}(F) = \{a \in A : (a^n)^- \to a \in F, \text{for all } n \in \mathbb{N}\}$.

Definition 7 ([3, Definition 3.1])**.** If F is a filter of a BL-algebra A, the set of double complemented elements of F, is defined by $D(F) = \{x \in A : x^{--} \in F\}$.

Definition 8 ([9, Lemma 2.3.14])**.** Let F be a proper filter of a BL-algebra A. The relation $\sim_F$ defined on a BL-algebra A, by $(x,y) \in \sim_F$ if and only if $x \to y \in F$ and $y \to x \in F$, is a congruence relation on A. The quotient algebra $A/\sim_F$ denoted by A/F becomes a BL-algebra in a natural way, with the operations induced from those of A. So, the order relation on A/F is given by $x/F \le y/F$ if and only if $x \to y \in F$. Hence $x/F = 1/F$ if and only if $x \in F$ and $x/F = 0/F$ if and only if $x^- \in F$.

3. DC-PRIME FILTERS AND DCP-BL-ALGEBRAS

In this section, a new filter in BL-algebras and a new class of BL-algebras are introduced and also characterized.

Definition 9. A proper filter F of a BL-algebra A is called a "double complemented prime filter" or briefly " DC-prime filter" of A, if for any $x,y \in A$, $x \vee y \in F$ then $x^{--} \in F$ or $y^{--} \in F$.

A proper filter F of a BL-algebra A is a DC-prime filter if and only if for any $x,y \in A$, $x \vee y \in F$ implies $x \in D(F)$ or $y \in D(F)$.

Example 1. Let $A = \{0,a,b,c,d,1\}$, where $0 < b < c,d < 1$ and $0 < a < c < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Then $(A,\wedge,\vee,*,\rightarrow,0,1)$ is a BL-algebra ([16, Example 4.4]). Clearly $F = \{b,c,d,1\}$ is a DC-prime filter of A and $G = \{c,1\}$ is not a DC-prime filter (since $a \vee b = c \in G$ and $b^{--},a^{--} \notin G$).

The following theorem is an important property of DC-prime filters in BL-algebras.

Theorem 3. *Let F be a proper filter of a* BL-*algebra A. Then F is a* DC-*prime filter of A if and only if for all $x,y \in A$, $x^{--} \rightarrow y^{--} \in F$ or $y^{--} \rightarrow x^{--} \in F$.*

Proof. Let F be a DC-prime filter of A and $x,y \in A$. So according to $(x \rightarrow y) \vee (y \rightarrow x) = 1 \in F$, then $x^{--} \rightarrow y^{--} \in F$ or $y^{--} \rightarrow x^{--} \in F$. Now assume that for all $x,y \in A$, $x^{--} \rightarrow y^{--} \in F$ or $y^{--} \rightarrow x^{--} \in F$ and for $x,y \in A$, $x \vee y \in F$. Then as $x \vee y \leq (x \rightarrow y) \rightarrow y$ and $x \vee y \leq (y \rightarrow x) \rightarrow x$, so $(x \rightarrow y) \rightarrow y \in F$ and $(y \rightarrow x) \rightarrow x \in F$. Hence $(x^{--} \rightarrow y^{--}) \rightarrow y^{--} \in F$ and $(y^{--} \rightarrow x^{--}) \rightarrow x^{--} \in F$. So according to hypothesis it follows that $y^{--} \in F$ or $x^{--} \in F$. Therefore F is a DC-prime filter of A. □

We can deduce the following theorem according to Theorem 3:

Theorem 4 (Extension property for DC-prime filters)**.** *Let F be a* DC-*prime filter of a* BL-*algebra A and G be a proper filter of A containing F. Then G is a* DC-*prime filter of A.*

Remark 1. Let F be a filter of a BL-algebra A. It well known that $D(F)$ is a proper filter if and only if F is a proper filter if and only if $\mathrm{Rad}(F)$ is a proper filter.

According to Theorem 4, we have the following corollary.

Corollary 1. *Let F be a filter of a* BL*-algebra A. Then the following conclusions hold:*

(i) $\{1\}$ *is a* DC*-prime filter of A if and only if all proper filters of A are* DC*-prime.*
(ii) F *is a* DC*-prime filter of A if and only if $D(F)$ is a* DC*-prime filter of A.*
(iii) If F is a DC*-prime filter, then* $\mathrm{Rad}(F)$ *is a* DC*-prime filter of A.*

Definition 10. A BL-algebra A is called a DCP-BL-algebra if for any $x, y \in A$, $x \vee y = 1$ implies $x^{--} = 1$ or $y^{--} = 1$.

Example 2.

(i) Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([17, Example 3.4]) which is a DCP-BL-algebra.

(ii) Let $A = \overline{\mathbb{Z}} \cup \{-\infty\} \cup \{0, a, b, 1\}$, where $\overline{\mathbb{Z}}$ is the set of negative integer numbers and $-\infty < \cdots < -2 < -1 < 0 < a, b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	$-\infty$	$\ldots$	-3	-2	-1	0	a	b	1
$-\infty$	$-\infty$	$\ldots$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$	$-\infty$
$\vdots$	$\vdots$	$\ldots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\ldots$	-6	-5	-4	-3	-3	-3	-3
-2	$-\infty$	$\ldots$	-5	-4	-3	-2	-2	-2	-2
-1	$-\infty$	$\ldots$	-4	-3	-2	-1	-1	-1	-1
0	$-\infty$	$\ldots$	-3	-2	-1	0	0	0	0
a	$-\infty$	$\ldots$	-3	-2	-1	0	a	0	a
b	$-\infty$	$\ldots$	-3	-2	-1	0	0	b	b
1	$-\infty$	$\ldots$	-3	-2	-1	0	a	b	1

$\to$	$-\infty$	$\dots$	-3	-2	-1	0	a	b	1
$-\infty$	1	$\dots$	1	1	1	1	1	1	1
$\vdots$	$\vdots$	$\dots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
-3	$-\infty$	$\dots$	1	1	1	1	1	1	1
-2	$-\infty$	$\dots$	-1	1	1	1	1	1	1
-1	$-\infty$	$\dots$	-1	1	1	1	1	1	1
0	$-\infty$	$\dots$	-3	-2	-1	1	1	1	1
a	$-\infty$	$\dots$	-3	-2	-1	b	1	b	1
b	$-\infty$	$\dots$	-3	-2	-1	a	a	1	1
1	$-\infty$	$\dots$	-3	-2	-1	0	a	b	1

Then $(A,\wedge,\vee,*,\to,-\infty,1)$ is a BL-algebra ([11, Page 185]). All filters of A are $F_1=\{1\}$, $F_2=\{a,1\}$, $F_3=\{b,1\}$, $F_4=\{0,a,b,1\}$ and $F_5=\{\dots,-3,-2,-1,0,a,b,1\}-\{-\infty\}$. It is clear that F_1, F_2, F_3, F_4 and F_5 are DC-prime filters of A. So A is an infinite DCP-BL-algebra.

(iii) BL-algebra A in Example 1 is not a DCP-BL-algebra, since $a\vee d=1$ while $a^{--}\neq 1$ and $d^{--}\neq 1$.

Remark 2. It can be easily proved that a BL-algebra A is a DCP-BL-algebra, if and only if for any $x,y\in A$, $x\vee y=1$ implies $x\in D(\{1\})$ or $y\in D(\{1\})$.

Theorem 5. *The following conditions are equivalent:*

(i) A is a DCP-BL-*algebra.*

(ii) $\{1\}$ is a DC-*prime filter of a* BL-*algebra A.*

(iii) Every proper filter of a BL-*algebra A is a* DC-*prime filter.*

Proof.

$(i)\Leftrightarrow(ii)$**:** According to Definition 10, the proof is clear.

$(ii)\Leftrightarrow(iii)$**:** According to Theorem 4, the proof is easy.

□

Remark 3. Let F and G be proper filters of a BL-algebra A such that $F\subseteq G$. Then G is a DC-prime filter of A if and only if G/F is a DC-prime filter of a BL-algebra A/F.

Corollary 2. *Let F be a proper filter of a* BL-*algebra A. Then the following conditions are equivalent:*

(i) A/F is a DCP-BL-*algebra.*

(ii) F is a DC-*prime filter of A.*

(iii) Any proper filter of A/F is a DC-*prime filter.*

Proof.

$(i)\Leftrightarrow(iii)$**:** According to Theorem 5, the proof is clear.

$(ii) \Rightarrow (iii)$**:** Let F be a DC-prime filter of A. We need to show that $\{1/F\}$ is a DC-prime filter of A/F. Assume that $x/F \vee y/F \in \{1/F\}$, for $x/F, y/F \in A/F$. Hence $x \vee y \in F$ so based on hypothesis, $x^{--} \in F$ or $y^{--} \in F$. Thus $x^{--}/F \in \{1/F\}$ or $y^{--}/F \in \{1/F\}$ and so $\{1/F\}$ is a DC-prime filter of A/F. Therefore according to Theorem 5, any proper filter of A/F is a DC-prime filter.

$(iii) \Rightarrow (ii)$**:** Let any proper filter of A/F be a DC-prime filter so $\{1/F\}$ is a DC-prime filter. Also let $x \vee y \in F$, for $x, y \in A$. Hence $x/F \vee y/F \in \{1/F\}$. Thus $x^{--}/F \in \{1/F\}$ or $y^{--}/F \in \{1/F\}$ and so $x^{--} \in F$ or $y^{--} \in F$. Therefore F is a DC-prime filter of A.

□

For a BL-algebra A, take $A^{--} = \{x^{--} : x \in A\}$.

Proposition 1. *The following conditions are equivalent:*

(i) A is a DCP-BL-*algebra.*
(ii) A^{--} is linearly ordered.
(iii) $a \leq b^{--}$ or $b \leq a^{--}$, for all $a, b \in A$.

Proof.

$(i) \Leftrightarrow (ii)$**:** According to Theorem 5:

$$\begin{aligned} A \text{ is a DCP-BL-algebra} &\Leftrightarrow \{1\} \text{ is a DC-prime filter of } A, \\ &\Leftrightarrow a^{--} \to b^{--} = 1 \text{ or } b^{--} \to a^{--} = 1, \text{ for all } a, b \in A, \\ &\Leftrightarrow a^{--} \leq b^{--} \text{ or } b^{--} \leq a^{--}, \text{ for all } a, b \in A, \\ &\Leftrightarrow A^{--} \text{ is linearly oredered.} \end{aligned}$$

$(ii) \Rightarrow (iii)$**:** Let A^{--} be linearly ordered. So for all $a, b \in A$, $a^{--} \leq b^{--}$ or $b^{--} \leq a^{--}$. Hence for all $a, b \in A$, $a \leq a^{--} \leq b^{--}$ or $b \leq b^{--} \leq a^{--}$.

$(iii) \Rightarrow (ii)$**:** Let $a \leq b^{--}$ or $b \leq a^{--}$, for all $a, b \in A$. Hence $a^{--} \leq (b^{--})^{--} = b^{--}$ or $b^{--} \leq (a^{--})^{--} = a^{--}$, for all $a, b \in A$.

□

A node of a poset A is an element of A which is comparable with every element of A. The set of all node elements of A is denoted by $\text{nod}(A)$. A filter F of a BL-algebra A is called a nodal filter if it is a node of $F(A)$, [1, Definition 3.8]. We denote $DF(A) := \{D(F) : F \in F(A)\}$.

Lemma 1. *For any* BL-*algebra A, $DF(A)$ is a poset.*

Theorem 6. *A is a* DCP-BL-*algebra if and only if* $\text{nod}(DF(A)) = DF(A)$.

Proof. Let A be a DCP-BL-algebra and $DF(A) \nsubseteq \text{nod}(DF(A))$. Hence there exists $H \in DF(A)$ such that $H \notin \text{nod}(DF(A))$. So there exists $G \in DF(A)$ such that $H \nsubseteq G$ and $G \nsubseteq H$. As $G, H \in DF(A)$, there exist $G', H' \in F(A)$ such that $H = D(H')$ and

$G = D(G')$. Also there exists $g \in G - H$ and $h \in H - G$. According to Proposition 1, A^{--} is linearly ordered and so $g^{--} < h^{--}$ or $h^{--} < g^{--}$, (if $g^{--} = h^{--}$, then $h^{--} \in G$ and so $h \in D(G) = D(D(G')) = D(G') = G$, which is a contradiction). So $h \in D(H')$, hence $h^{--} \in H'$. Also $g \in D(G')$, hence $g^{--} \in G'$. On the other hand since $g \notin H = D(H')$ and $h \notin G = D(G')$, thus $h^{--} \notin G'$ and $g^{--} \notin H'$, which is a contradiction with $g^{--} < h^{--}$ or $h^{--} < g^{--}$. Therefore $DF(A) \subseteq \mathrm{nod}(DF(A))$, i.e. $\mathrm{nod}(DF(A)) = DF(A)$. Now assume that $\mathrm{nod}(DF(A)) = DF(A)$ and $a \vee b = 1$, for $a, b \in A$. Then $D(\langle a\rangle) \subseteq D(\langle b\rangle)$ or $D(\langle b\rangle) \subseteq D(\langle a\rangle)$. As $a \in \langle a\rangle \subseteq D(\langle a\rangle)$ and $b \in \langle b\rangle \subseteq D(\langle b\rangle)$, so $a \in D(\langle b\rangle)$ or $b \in D(\langle a\rangle)$. Thus $a^{--} \in \langle b\rangle$ or $b^{--} \in \langle a\rangle$. Hence $a^{--} \geq b^n$ or $b^{--} \geq a^m$, for some $n, m \in \mathbb{N}$. So $a^{--} \vee b^n = a^{--}$ or $b^{--} \vee a^m = b^{--}$, for some $n, m \in \mathbb{N}$. Since $a \vee b = 1$, then $a^n \vee b^n = 1$ and $a^m \vee b^m = 1$ for all $m, n \in \mathbb{N}$. On the other hand

$$a^{--} = a^{--} \vee b^n \geq a \vee b^n \geq a^n \vee b^n = 1 \text{ and}$$
$$b^{--} = b^{--} \vee a^m \geq a^m \vee b \geq a^m \vee b^m = 1.$$

Therefore $a^{--} = 1$ or $b^{--} = 1$ and so $\{1\}$ is a DC-prime filter. Hence based on Theorem 5, A is a DCP-BL-algebra. □

Corollary 3. *Let A be a* BL-*algebra and* $\mathrm{nod}(F(A)) = F(A)$. *Then A is a* DCP-BL-*algebra.*

Proposition 2. *Let A be a* BL-*algebra and for any $0 \neq a \in A$, $a^- = 0$. Then*

(i) A is a DCP-BL-*algebra.*
(ii) for any proper filter F of A, A/F is a DCP-BL-*algebra.*

Proof.

(i): Let for any $0 \neq a \in A$, $a^- = 0$ and $x \vee y = 1$, for $x, y \in A$. Then
- If $x = 0$ then $y = 1$, so $y^{--} = 1$.
- If $x \neq 0$ then $x^- = 0$. So $x^{--} = 1$.

Therefore A is a DCP-BL-algebra.

(ii): Let $0/F \neq a/F \in A/F$. Then $a \neq 0$ and so according to hypothesis $a^- = 0$. Hence $a^-/F = 0/F$. Therefore according to Part (i), A/F is a DCP-BL-algebra.

□

Remark 4. BL-algebra A in Example 2(i) is a DCP-BL-algebra while $a^- \neq 0$. Also $F = \{b, 1\}$ is a DC-prime filter hence A/F is a DCP-BL-algebra, while $a^- \neq 0$. Therefore the converse of Parts (i) and (ii) of Proposition 2 are not true in general.

The following corollary is a straightforward consequence of Proposition 2(i).

Corollary 4. *Let A be a* BL-*algebra. Then the following conclusions hold:*

(i) If for any $0 \neq a, b \in A$, $(a \to b)^- = (b \to a)^-$, then A is a DCP-BL-*algebra.*
(ii) If for any $0 \neq a, b \in A$, $a \to b^{--} = b \to a^{--}$, then A is a DCP-BL-*algebra.*

(iii) If A is a special BL-*algebra, then A is a* DCP-BL-*algebra.*
(iv) If $A = D_s(A) \cup \{0\}$ *then A is a* DCP-BL-*algebra.*

In the following example it is shown that the converse of the parts of Corollary 4 is not true in general.

Example 3. Let $A = \{0,a,b,c,d,1\}$ where $0 < d < c < a,b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	c	c	d	a
b	0	c	b	c	d	b
c	0	c	c	c	d	c
d	0	d	d	d	0	d
1	0	a	b	c	d	1

$\rightarrow$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([14, Example 1]), and $F_1 = \{1\}, F_2 = \{a,1\}, F_3 = \{b,1\}, F_4 = \{a,b,c,1\}$ are all filters of A. Clearly A is a DCP-BL-algebra, while $(b \rightarrow d)^- \neq (d \rightarrow b)^-$, $b \rightarrow d^{--} \neq d \rightarrow b^{--}$ and $d^- \neq 0$. Also, A is not a special BL-algebra and $A \neq D_s(A) \cup \{0\}$.

Lemma 2. *Let A be a Boolean algebra and* DCP-BL-*algebra. Then* $A = D_s(A) \cup \{0\}$.

Proof. Let A be a DCP-BL-algebra and $0 \neq x \in A$. Then all filters of A are DC-prime. As A is a Boolean algebra, $x \vee x^- = 1$, so $x^{--} = 1$ or $x^- = 1$. As $x \neq 0$ so $x^- \neq 1$, hence $x^{--} = 1$, i.e. $x \in D_s(A)$. □

Lemma 3. *Let A be a Boolean algebra. Then A is a* DCP-BL-*algebra if and only if* $A = \{0,1\}$.

Proof. Let A be a Boolean algebra and DCP-BL-algebra. Then according to hypothesis $x \vee x^- = 1$, for any $x \in A$. Hence $x^{--} = 1$ or $x^- = 1$, since A is a DCP-BL-algebra. So $x = 1$ or $x^- = 1$, i.e. $x = 1$ or $x = 0$. Therefore $A = \{0,1\}$. The converse is clear.

□

4. The relationships between DC-prime filters and other types of filters in BL-algebras

What follows is a discussion of the relations between DC-prime filters and some other filters in BL-algebras.

Theorem 7. *Every prime filter in* BL-*algebras, is a* DC-*prime filter.*

Proof. Let F be a prime filter of a BL-algebra A and $a,b \in A$. So $a \rightarrow b \in F$ or $b \rightarrow a \in F$. As $a \rightarrow b \leq (a \rightarrow b)^{--}$ and $b \rightarrow a \leq (b \rightarrow a)^{--}$, then $(a \rightarrow b)^{--} \in F$ or

$(b \to a)^{--} \in F$, for all $a, b \in A$. Therefore according to Theorem 3 F is a DC-prime filter of A. □

The following example is shown that the converse of Theorem 7 does not correct in general.

Example 4. Consider BL-algebra A in Example 3. Clearly $F = \{1\}$ is a DC-prime filter while is not a prime filter, since $a \to b \notin F$ and $b \to a \notin F$.

For a BL-algebra A, $MV(A) = \{x \in A : x^{--} = x\}$.
According to Theorems 7 and 5:

Corollary 5.

(i) If A is a linearly ordered BL-*algebra, then A is a* DCP-BL-*algebra.*
(ii) If A is a DCP-BL-*algebra and* $MV(A) = A$ *then A is a linearly ordered* BL-*algebra.*

Proof.

(i)**:** Let A be a linearly ordered BL-algebra. Then all proper filters of A are prime filter. Hence according to Theorem 7, all proper filters of A are DC-prime. Therefore based on Theorem 5, A is a DCP-BL-algebra.
(ii)**:** Let A be a DCP-BL-algebra and $MV(A) = A$. Then all proper filters of A are DC-prime filter and $x^{--} = x$, for $x \in A$. Therefore based on the definition of DC-prime filters, all proper filters of A are prime filter and so A is a linearly ordered BL-algebra.

□

Remark 5. The condition "$MV(A) = A$" is necessary in Corollary 5(ii) and the converse of Corollary 5(i) is not true in general. Consider Example 3. It is easy to check that A is a DCP-BL-algebra and $a^{--} \neq a$, while A is not a linearly ordered.

Corollary 6.

(i) Every maximal filter in BL-*algebras, is a* DC-*prime filter.*
(ii) Every obstinate filter in BL-*algebras, is a* DC-*prime filter.*

In the following example it is shown that the converse of Corollary 6 is not correct in general.

Example 5. Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Operations $*$ and $\to$ are defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(A,\wedge,\vee,*,\rightarrow,0,1)$ is a BL-algebra ([17, Example 5.2]). Clearly, $F=\{b,1\}$ is a DC-prime filter while is not a maximal filter nor an obstinate filter of A.

Theorem 8. *Let F be a proper filter of a* BL*-algebra A. Then the following conditions are equivalent:*

(i) F is a DC*-prime filter and Boolean filter of A.*
(ii) F is a maximal filter of A.
(iii) F is an obstinate filter of A.

Proof.

$(i)\Rightarrow(ii)$**:** Let F be a DC-prime filter and Boolean filter of A and $x\notin F$. As F is a Boolean filter, $x\vee x^{-}\in F$. Hence $x^{--}\in F$ or $x^{-}\in F$, since F is a DC-prime filter. If $x^{--}\in F$, since F is a Boolean filter, then $x\in F$, which is a contradiction. Therefore $x^{-}\in F$, i.e. F is a maximal filter of A.

$(ii)\Rightarrow(i)$**:** It is known that every maximal filter, in BL-algebras, is a Boolean filter. Hence according to Corollary 6(i), the proof is completed.

$(i)\Rightarrow(iii)$**:** Let F be a DC-prime filter and Boolean filter of A. Then according to $((i)\Leftrightarrow(ii))$, F is a maximal filter of A. Hence based on [4, Theorem 4.4], the proof is completed.

$(iii)\Rightarrow(i)$**:** Let F be an obstinate filter. Then based on [4, Theorem 4.1], F is a maximal filter. Then according to $((ii)\Leftrightarrow(i))$, the proof is completed.

□

Remark 6. The condition "Boolean filter" in Theorem 8(i) is necessary. Consider Example 5, clearly $F=\{b,1\}$ is a DC-prime filter while it is not a Boolean filter and nor a maximal filter.

A BL-algebra A is called degenerate if $0=1$ in A, otherwise A is non-degenerate.

The following lemma is a straightforward consequence of [20, Theorems 1,2,3] and Theorem 7.

Lemma 4. *Let A be a* BL*-algebra. Then the following conclusions hold:*

(i) A non-degenerate BL*-algebra contains a* DC*-prime filter.*
(ii) Any proper filter F of a non-degenerate BL*-algebra A can be extended to a* DC*-prime filter.*
(iii) In a non-degenerate BL*-algebra A, any proper filter can be extended to a maximal and* DC*-prime filter.*

Theorem 9. *Let F be a filter of a* BL*-algebra A. The the following conditions are equivalent.*

(i) F is a DC*-prime filter of A.*
(ii) $D(F)$ is a prime filter of A.
(iii) $D(F)$ is a DC*-prime filter of A.*
(iv) $A/D(F)$ is a linearly ordered BL*-algebra.*

(v) $A/D(F)$ *is a* DCP-BL-*algebra.*

Proof.

$(i) \Rightarrow (ii)$**:** Let F be a DC-prime filter of A. Then $x^{--} \to y^{--} \in F$ or $y^{--} \to x^{--} \in F$, for any $x, y \in A$. Hence $x \to y \in D(F)$ or $y \to x \in D(F)$, for any $x, y \in A$. Therefore $D(F)$ is a prime filter of A.

$(ii) \Rightarrow (iii)$**:** It is proved in Theorem 7.

$(iii) \Rightarrow (i)$**:** Let $D(F)$ be a DC-prime filter of A. Then $x^{--} \to y^{--} \in D(F)$ or $y^{--} \to x^{--} \in D(F)$, for any $x, y \in A$. Hence $x^{--} \to y^{--} \in F$ or $y^{--} \to x^{--} \in F$, for any $x, y \in A$. Therefore F is a DC-prime filter of A.

$(ii) \Leftrightarrow (iv), (iii) \Leftrightarrow (v)$**:** The proofs are obvious.

□

Corollary 7. *Let F be a proper filter of a* BL-*algebra A. F is a* DC-*prime filter of A if and only if* $\mathrm{nod}(F(A/D(F))) = F(A/D(F))$.

Proof. Let F be a DC-prime filter. Then based on Theorem 9, $D(F)$ is a prime filter and so $A/D(F)$ is a linearly ordered BL-algebra. Hence according to [15, Lemma 3.8], $\mathrm{nod}(F(A/D(F))) = F(A/D(F))$. Now let $\mathrm{nod}(F(A/D(F))) = F(A/D(F))$ and $a \vee b \in F$, for $a, b \in A$. So $a \vee b \in D(F)$, hence $a/D(F) \vee b/D(F) = 1/D(F) \in \{1/D(F)\}$. Based on [15, Proposition 3.10], $\{1/D(F)\}$ is a prime filter of $A/D(F)$ and so based on Theorem 7, $\{1/D(F)\}$ is a DC-prime filter of $A/D(F)$. Hence $a^{--}/D(F) \in \{1/D(F)\}$ or $b^{--}/D(F) \in \{1/D(F)\}$. So $a^{--} \in D(F)$ or $b^{--} \in D(F)$. Then $a^{--} \in F$ or $b^{--} \in F$. Therefore F is a DC-prime filter of A. □

Corollary 8. *(i) If $D(F)$ is a maximal filter of a* BL-*algebra A, then F is a* DC-*prime filter of A.*

(ii) If F is a DC-*prime filter of a* BL-*algebra A, then $D(F)$ is a primary filter of A.*

(iii) If F is a DC-*prime filter of a* BL-*algebra A, then* $\mathrm{Rad}(D(F))$ *is a maximal filter of A.*

Proof. According to Theorem 9 and [16, Theorem 3.3], the proofs are obvious.

□

Theorem 10. *Every* DC-*prime filter in* BL-*algebras is a primary filter.*

Proof. Let F be a DC-prime filter of a BL-algebra A and $(x*y)^- \in F$, for $x, y \in A$. Then according to hypothesis and Theorem 3, $x^{--} \to y^{--} \in F$ or $y^{--} \to x^{--} \in F$. Assume that $x^{--} \to y^{--} \in F$ so

$$\begin{aligned}(x^{--} \to y^{--}) * (x*y)^- &= (x^{--} \to y^{--}) * (y^{--} \to x^-)\\ &\leq x^{--} \to x^- = (x*x)^-\\ &= (x^2)^- \in F.\end{aligned}$$

Similarly, if $y^{--} \to x^{--} \in F$ and $(x*y)^- \in F$ then $(y^2)^- \in F$. Therefore F is a primary filter of A. □

Open Problem 1. *Is every primary filter in* BL*-algebras, a* DC*-prime filter?*

Corollary 9. *Let F be a* DC*-prime filter of a* BL*-algebra A. Then the following hold:*

(i) $\mathrm{Rad}(F)$ *is a primary filter of A.*
(ii) $\mathrm{Rad}(F)$ *is a maximal filter of A.*
(iii) $\mathrm{Rad}(F)$ *is a prime filter of A.*

Proof.

(i)**:** Let F be a DC-prime filter of A. Then based on Theorem 10, F is a primary filter of A. As $F \subseteq \mathrm{Rad}(F)$ so $\mathrm{Rad}(F)$ is a primary filter of A.

(ii)**:** Let F be a DC-prime filter of A. According to Theorem 10, F is a primary filter of A. Hence based on [16, Theorem 3.3], $\mathrm{Rad}(F)$ is a maximal filter.

(iii)**:** Based on Part (ii), the proof is obvious.

□

Corollary 10. *Let F be a* DC*-prime filter of a* BL*-algebra A such that it satisfies in at least one of the following conditions. Then F is a prime filter of A.*

(i) F is a normal filter of A.
(ii) F is a fantastic filter of A.
(iii) F is a Boolean filter of A.
(iv) F is a positive implicative filter of A.
(v) $D(F) \subseteq B(A)$.
(vi) $\mathrm{Rad}(F) \subseteq B(A)$.
(vii) Any almost top element of A/F is trivial.

Proof.

$(i),(ii),(iii),(iv),(v)$**:** Let F be a DC-prime filter of A such that satisfies in at least one of conditions (i), (ii), (iii), (iv) and (v). Hence according to [3, Theorems 3.25, 3.33] and [3, Corollary 3.28], $D(F) = F$. Based on Theorem 9, $D(F)$ is a prime filter, therefore F is a prime filter of A.

(vi)**:** Let F be a DC-prime filter of A such that it satisfies in condition (vi). Hence according to [18, Theorem 4.6] F is a semi-maximal filter of A, i.e. $\mathrm{Rad}(F) = F$. Based on Corollary 9, $\mathrm{Rad}(F)$ is a prime filter. Therefore F is a prime filter of A.

(vii)**:** Let F be a DC-prime filter of A. Then according to Theorem 9, $D(F)$ is a prime filter. Hence to close the proof, we need to show that $F = D(F)$. Assume that $x \in D(F)$. Then $x^{--} \in F$ and so $x^{--}/F = 1/F$, i.e. x/F is an almost top element of A/F. Hence based on hypothesis $x/F = 1/F$, so $x \in F$. Then $F = D(F)$. Therefore F is a prime filter of A.

□

The following examples show that the conditions in Corollary 10 are necessary.

Example 6.

(i) Let $A=\{0,a,b,c,1\}$, where $0<c<a,b<1$. Operations $*$ and $\to$ are defined as follows:

$*$	0	a	b	c	1
0	0	0	0	0	0
a	0	a	c	c	a
b	0	c	b	c	b
c	0	c	c	c	c
1	0	a	b	c	1

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	b	1
b	0	a	1	a	1
c	0	1	1	1	1
1	0	a	b	c	1

Then $(A,\wedge,\vee,*,\to,0,1)$ is a BL-algebra ([16, Example 3.2]). $F=\{1\}$ is a DC-prime filter while is not a fantastic filter, nor a normal filter and nor a prime filter.

(ii) Consider Example 3. It is easy to check that $F=\{1\}$ is a DC-prime filter while is not a prime filter (since $a\to b\notin F$ and $b\to a\notin F$) and is not a Boolean filter (since $a\vee a^-=a$) so is not a positive implicative filter. Also $D(F)=\mathrm{Rad}(F)=\{a,b,c,1\}$ and $B(A)=\{0,1\}$ so $\mathrm{Rad}(F)\nsubseteq B(A)$ and $D(F)\nsubseteq B(A)$. Also clearly $N(A/F)\neq\{1/F\}$.

Proposition 3. *Let F be a proper filter of a* BL*-algebra A and for any $x\in A$, $x^{--}\in F$ or $x^-\in F$. Then F is a* DC*-prime filter of A.*

Proof. Let $a\vee b\in F$, for $a,b\in A$. According to hypothesis, $a^{--}\in F$ or $a^-\in F$ and $b^{--}\in F$ or $b^-\in F$. Hence there are several possibilities:

(i)**:** $a^{--}\in F$ and $b^{--}\in F$.
(ii)**:** $a^{--}\in F$ and $b^-\in F$.
(iii)**:** $a^-\in F$ and $b^{--}\in F$.
(iv)**:** $a^-\in F$ and $b^-\in F$. Then $a^-\wedge b^-=(a\vee b)^-\in F$. So $a\vee b\notin F$, since F is a proper filter. That is a contradiction.

Therefore F is a DC-prime filter of A.

□

Remark 7. The converse of Proposition 3 is not true in general. Consider $F=\{b,1\}$ in Example 2. Clearly F is a DC-prime filter of A while $a^{--}\notin F$ and $a^-\notin F$.

Lemma 5. *Let F be a* DC*-prime and easy filter of a* BL*-algebra A. Then for all $x\in A$, $x^-\in F$ or $x^{--}\in F$.*

Proof. Let F be a DC-prime and easy filter and $x\in A$. It is known that $(x^{--}\to x^-)\vee(x^-\to x^{--})=1\in F$, hence $(x^{--}\to x^-)^{--}\in F$ or $(x^-\to x^{--})^{--}\in F$. So $x^{--}\to x^-\in F$ or $x^-\to x^{--}\in F$. If $x^{--}\to x^-\in F$, then $x^{--}\to x^-=x^{--}\to x^{---}=x^{--}\to(x^{--}\to 0)=(x^{--})^2\to 0\in F$. Thus $((x^{--})^2)^-\in F$, so according to hypothesis and [8, Proposition 51], $x^-\in F$. If $x^-\to x^{--}\in F$, then $x^{---}\to(x^-\to 0)\in F$. So $(x^-)^{--}\to(x^-\to 0)\in F$. As F is an easy filter and $(x^-)^{--}\to x^-=$

$x^- \to x^- = 1 \in F$, it is concluded that $(x^-)^{--} \to 0 = x^{--} \in F$. Therefore the proof is completed. □

Theorem 11. *Let F be a filter of a* BL*-algebra A. Then the following statements are equivalent:*

(i) F is a primary filter of A.
(ii) $\mathrm{Rad}(F)$ *is a* DC*-prime of A.*
(iii) $\mathrm{Rad}(F)$ *is a primary filter of A.*

Proof.

$(i) \Rightarrow (ii)$**:** Let F be a primary filter of A. Then based on [16, Theorem 3.3], $\mathrm{Rad}(F)$ is a maximal filter of A and according to Corollary 6(i), $\mathrm{Rad}(F)$ is a DC-prime filter of A.

$(ii) \Rightarrow (i)$**:** Let $\mathrm{Rad}(F)$ be a DC-prime filter of A. Then based on Corollary 9, $\mathrm{Rad}(\mathrm{Rad}(F)) = \mathrm{Rad}(F)$ is a maximal filter of A. Then based on [16, Theorem 3.3], F is a primary filter of A.

$(ii) \Rightarrow (iii)$**:** Let $\mathrm{Rad}(F)$ be a DC-prime filter of A. Based on $((ii) \Leftrightarrow (i))$, F is a primary filter of A. As $F \subseteq \mathrm{Rad}(F)$, $\mathrm{Rad}(F)$ is a primary filter of A.

$(iii) \Rightarrow (ii)$**:** Let $\mathrm{Rad}(F)$ be a primary filter of A. Then based on [16, Theorem 3.3], $\mathrm{Rad}(\mathrm{Rad}(F)) = \mathrm{Rad}(F)$ is a maximal filter of A and according to Corollary 6(i), $\mathrm{Rad}(F)$ is a DC-prime filter.

□

If F is a filter of a BL-algebra A, then $F \subseteq (F:x) = \{r \in A : r \vee x \in F\}$. It is known that $(F:x)$ is a proper filter of A, where $x \notin F$, [16, Proposition 4.2].

Lemma 6. *Let F be a* DC*-prime filter of a* BL*-algebra A. Then for any $x \in A - F$, the following hold:*

(i) $\mathrm{Rad}(F) = \mathrm{Rad}(F:x)$.
(ii) $(F:x) \subseteq \mathrm{Rad}(F)$.

Proof.

(i)**:** Let F be a DC-prime filter of A. Then based on Theorem 10, F is a primary filter and so according to [16, Theorem 4.5], $\mathrm{Rad}(F) = \mathrm{Rad}((F:x))$, where $x \in A - F$.

(ii)**:** As $(F:x) \subseteq \mathrm{Rad}(F:x)$, the proof gets by (i).

□

Remark 8. The inclusion of Lemma 6(ii) is not equal in general. Consider $F = \{a, 1\}$ in Example 6(i). Clearly $(F:b) = \{a, 1\}$ and $\mathrm{Rad}(F) = \{a, b, c, 1\}$.

Theorem 12. *Let F be a proper filter of a* BL*-algebra A. Then the following conditions are equivalent:*

(i) F is a DC*-prime filter of A.*

(ii) $D(F) = (D(F) : x)$, *where* $x \in A - D(F)$.
(iii) $D(F) = D((F : x))$, *where* $x \in A - D(F)$.
(iv) $(F : x) \subseteq D(F)$, *where* $x \in A - D(F)$.

Proof.

$(i) \Rightarrow (ii)$**:** Let F be a DC-prime filter of A and $y \in (D(F) : x)$, where $x \in A - D(F)$. Then $x \vee y \in D(F)$, hence $x^{--} \vee y^{--} \in F$. Since $x \notin D(F)$, so $x^{--} \notin F$. Thus $y^{--} \in F$, i.e. $y \in D(F)$. Hence $(D(F) : x) \subseteq D(F)$. Therefore $D(F) = (D(F) : x)$.

$(ii) \Rightarrow (i)$**:** Let for $x \in A - D(F)$, $D(F) = (D(F) : x)$ and also assume that $a \vee b \in F$ and $a^{--} \notin F$. So $a \notin D(F)$. Using Part (ii), $D(F) = (D(F) : a)$. As $a \vee b \in F \subseteq D(F)$ so $b \in (D(F) : a) = D(F)$. Therefore $b \in D(F)$, hence $b^{--} \in F$, i.e. F is a DC-prime filter of A.

$(i) \Rightarrow (iii)$**:** Let F be a DC-prime filter of A. Since $F \subseteq (F : x)$, then $D(F) \subseteq D((F : x))$. On the other hand if $y \in D((F : x))$ then $x^{--} \vee y^{--} \in F$. So $x \vee y \in D(F)$, i.e. $y \in (D(F) : x)$. Thus $D((F : x)) \subseteq (D(F) : x) = D(F)$, using $((i) \Leftrightarrow (ii))$.

$(iii) \Rightarrow (i)$**:** Let $D(F) = D((F : x))$, for any $x \in A - D(F)$ and $a \vee b \in F$, for $a, b \in A$. If $a^{--} \notin F$ $(a \notin D(F))$, then $D(F) = D((F : a))$. As $a \vee b \in F$, so $b \in (F : a) \subseteq D((F : a)) = D(F)$. Therefore $b \in D(F)$, i.e. F is a DC-prime filter of A.

$(i) \Rightarrow (iv)$**:** Let F be a DC-prime filter of A. It is known that $(F : x) \subseteq (D(F) : x)$. So $(F : x) \subseteq D(F)$, using $((i) \Leftrightarrow (ii))$.

$(iv) \Rightarrow (i)$**:** Let $(F : x) \subseteq D(F)$, where $x \in A - D(F)$ and $a \vee b \in F$ such that $a^{--} \notin F$ $(a \notin D(F))$. So $(F : a) \subseteq D(F)$. Then $b \in (F : a) \subseteq D(F)$, hence $b \in D(F)$. Therefore $b^{--} \in F$, i.e. F is a DC-prime filter of A.

□

According to Corollary 9(i) and [14, Proposition 8]:

Lemma 7. *If F is a* DC*-prime filter then there exists an unique maximal filter M of a* BL*-algebra A containing F.*

Remark 9. It can be easily proved that if BL-algebra A is an MV-algebra then F is a DC-prime filter of A if and only if F is a prime filter of A. So according to [18, Proposition 3.22]:

Corollary 11.

(i) *Let F and G be proper filters of a* BL*-algebra A such that $D_s(A) \subseteq F \subseteq G$. Then G/F is a prime filter of A/F if and only if G/F is a* DC*-prime filter of A/F.*
(ii) *Let F be a proper filter of a* BL*-algebra A and $D_s(A) = \{1\}$. Then F is a prime filter of A if and only if F is a* DC*-prime filter of A.*
(iii) *All* DC*-prime filters of a* BL*-algebra $A/D_s(A)$ are prime filters.*

Lemma 8. *Let F be a proper filter of a* BL*-algebra A. If F is a semi-integral and positive implicative filter then F is a* DC*-prime filter.*

Proof. Based on Corollary 6 and [14, Theorem 4], the proof is clear. □

Example 7. Let $A = \{0,a,b,c,d,e,f,g,1\}$, where $0 < a < b,d,e,g < 1$, $0 < b < e < 1$, $0 < c < d,e,g,f < 1$, $0 < d < e,g < 1$, $0 < f < g < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	c	d	e	f	g	1
0	0	0	0	0	0	0	0	0	0
a	0	a	a	0	a	a	0	a	a
b	0	a	b	0	a	b	0	a	b
c	0	0	0	c	c	c	c	c	c
d	0	a	a	c	d	d	c	d	d
e	0	a	b	c	d	e	c	d	e
f	0	0	0	c	c	c	f	f	f
g	0	a	a	c	d	d	f	g	g
1	0	a	b	c	d	e	f	g	1

$\rightarrow$	0	a	b	c	d	e	f	g	1
0	1	1	1	1	1	1	1	1	1
a	f	1	1	f	1	1	f	1	1
b	f	g	1	f	g	1	f	g	1
c	b	b	b	1	1	1	1	1	1
d	0	b	b	f	1	1	f	1	1
e	0	a	b	f	g	1	f	g	1
f	b	b	b	e	e	e	1	1	1
g	0	b	b	c	e	e	f	1	1
1	0	a	b	c	d	e	f	g	1

Then $(A,\wedge,\vee,*,\rightarrow,0,1)$ is a BL-algebra ([16, Example 4.3]). Clearly $F_1 = \{1\}$, $F_2 = \{e,1\}$, $F_3 = \{g,1\}$, $F_4 = \{b,e,1\}$, $F_5 = \{f,g,1\}$, $F_6 = \{d,e,g,1\}$, $F_7 = \{d,e,g,a,b,1\}$ and $F_8 = \{d,e,g,f,c,1\}$ are all filters of A. It is clear that F_4, F_5, F_7 and F_8 are DC-prime filters of A. Clearly, F_6 is a positive implicative filter while is not a DC-prime filter (since $a^{--} \rightarrow c^{--} \notin F_6$ and $c^{--} \rightarrow a^{--} \notin F_6$) also F_6 is not a semi-integral filter (since $(a*c)^- \in F_6$ and $(a^n)^-, (b^n)^- \notin F_6$, for all $n \in \mathbb{N}$).

Corollary 12. *Let F be a proper filter of a* BL*-algebra A. Then the following conclusions hold:*

(i) If F is an integral and positive implicative filter of A then F is a DC*-prime filter of A.*
(ii) If A is a Gödel algebra, F is a primary and positive implicative filter of A then F is a DC*-prime filter of A.*

(iii) If $D(F) = F$, F is an implicative and semi-integral filter then F is a DC*-prime filter of A.*
(iv) If A is a finite BL*-algebra, F is a positive implicative and perfect filter then F is a* DC*-prime filter of A.*
(v) If $F = A - \{0\}$, then F is a DC*-prime filter of A.*

Proof.

(i)**:** This part is a straightforward consequence of Lemma 8 and [14, Corollary 1].
(ii)**:** According to[14, Proposition 7] and Lemma 8, the proof is clear.
(iii)**:** Based on [3, Corollary 3.28] and Lemma 8, the proof is obvious.
(iv)**:** This part is a straightforward consequence of Lemma 8 and [14, Theorem 5].
(v)**:** Based on the definition of a DC-prime filter, the proof is obvious.

□

In the following, we show the necessity of the conditions in the Corollary 12.

Example 8.

(i) Let $A = \{0, a, b, 1\}$, where $0 < a, b < 1$. Operations $*$ and $\rightarrow$ are defined as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\rightarrow$	0	a	b	1
0	1	1	1	1
a	b	1	b	1
b	a	a	1	1
1	0	a	b	1

Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra ([11, Page163]). It is clear that $F = \{1\}$ is a positive implicative filter while is not an integral filter (since $(a * b)^- \in F$ and $a^-, b^- \notin F$), nor a DC-prime filter (since $a^{--} \rightarrow b^{--} \notin F$ and $b^{--} \rightarrow a^{--} \notin F$).

(ii) Consider Gödel algebra A in Example 1. Clearly $F = \{c, 1\}$ is a positive implicative filter while is not a DC-prime filter and nor a primary filter.

(iii) Consider Example 7. F_6 is an implicative filter, $D(F_6) = F_6$ while is not a semi-integral and nor a DC-prime filter (since $a^{--} \rightarrow c^{--} \notin F_6$ and $c^{--} \rightarrow a^{--} \notin F_6$).

(iv) Consider Example 7. F_6 is a positive implicative filter while is not a perfect filter (since it is not a semi-integrl filter) and nor a DC-prime filter (since $a^{--} \rightarrow c^{--} \notin F_6$ and $c^{--} \rightarrow a^{--} \notin F_6$).

(v) Consider Example 2(i). $F = \{b, 1\} \neq A - \{0\}$ is a DC-prime filter of A.

Proposition 4. *Let F be a proper filter of a* BL*-algebra A. If $A/D(F)$ is a special* BL*-algebra then F is a* DC*-prime filter of A.*

Proof. Let $A/D(F)$ be a special BL-algebra. Then based on Corollary 4(iii), $A/D(F)$ is a DCP-BL-algebra so $\{1/D(F)\}$ is a DC-prime filter of $A/D(F)$. Assume that $x \vee y \in F$, for $x, y \in A$. So $x \vee y \in D(F)$, hence $x \vee y/D(F) = 1/D(F)$. Thus $(x \vee y)^{--}/D(F) = 1/D(F)$. Hence $x^{--}/D(F) = 1/D(F)$ or $y^{--}/D(F) = 1/D(F)$. So $x^{--} \in D(F)$ or $y^{--} \in D(F)$ since $\{1/D(F)\}$ is a DC-prime filter of $A/D(F)$. Therefore $x^{--} \in F$ or $y^{--} \in F$, i.e. F is a DC-prime filter of A. □

Recall that if F is a filter of a BL-algebra A, $F^{-} = \{a \in A : a^{-} \in F\}$.

Corollary 13. *Let F be a* DC*-prime and Boolean filter of a* BL*-algebra A. Then*

(i) $F \cup F^{-} = A$.

(ii) *A/F has no zero divisor element.*

Proof.

- (i)**:** Let F be a DC-prime and Boolean filter of A and $x \in A$. Based on Lemma 5, $x^{--} \in F$ or $x^{-} \in F$. If $x^{-} \in F$, hence $x \in F^{-}$, and so $x \in F \cup F^{-}$. If $x^{--} \in F$, since F is Boolean filter, then $x \in F$. Hence $x \in F \cup F^{-}$. Therefore $F \cup F^{-} = A$.
- (ii)**:** Let $[x], [y] \in A/F$ and $[x] * [y] = [0]$. Then $(x * y)^{-} \in F$. As $(x * y)^{-} = x \to y^{-} \leq (x \to y^{-})^{--} = x^{--} \to y^{-}$ hence $x^{--} \to y^{-} \in F$. Based on Lemma 5, $x^{--} \in F$ or $x^{-} \in F$. If $x^{--} \in F$, then $y^{-} \in F$, so $[y] = [0]$. If $x^{-} \in F$, then $[x] = [0]$. Therefore A/F has no zero divisor element.

□

Example 9. Consider Example 6. $F = \{a, 1\}$ is a DC-prime filter while is not a Boolean filter (since $b \vee b^{-} \notin F$) and $F \cup F^{-} \neq A$ (since $F^{-} = \{0\}$).

The following examples show that there are no relationships between DC-prime filters and some other types of filters in BL-algebras.

Example 10.

- (i) Consider Example 7. Clearly, F_6 is a special filter, positive implicative filter (so is a Boolean filter, implicative filter, easy filter, fantastic filter and normal filter) while is not a DC-prime filter (since $a^{--} \to c^{--} \notin F_6$ and $c^{--} \to a^{--} \notin F_6$), nor a semi-integral filter (since $(a * c)^{-} \in F_6$ and $(a^{n})^{-}, (b^{n})^{-} \notin F_6$, for all $n \in \mathbb{N}$. $G = \{b, e, 1\}$ is a DC-prime filter while is not a special filter, nor a Boolean filter (since $D_s(G) = \{e, 1\} \neq G$ and $a \vee a^{-} \notin G$).
- (ii) Consider Example 6. $F = \{1\}$ is a DC-prime filter while F is not a fantastic filter, nor a normal filter and nor a positive implicative filter.
- (iii) Consider Example 1. Clearly $F = \{c, 1\}$ is a positive implicative filter while is not a DC-prime filter.
- (iv) Consider Example 5. $F = \{b, 1\}$ is a DC-prime filter while is not a semi-maximal filter, since $\mathrm{Rad}(F) = \{a, b, 1\}$.

(v) Let $A = \{0,a,b,c,d,1\}$, where $0 < a,b < c < 1$ and $0 < b < d < 1$. Operations $*$ and $\to$ are defined as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Then $(A,\wedge,\vee,*,\to,0,1)$ is a BL-algebra ([16, Example 4.3]). $F_1 = \{1\}$, $F_2 = \{d,1\}$ and $F_3 = \{a,c,1\}$ are all filters of A. Clearly F_1 is a nodal filter while is not a DC-prime filter, since $a^{--} \to b^{--} \notin F_1$ and $b^{--} \to a^{--} \notin F_1$.

In the following, we see that the intersection of two DC-prime filters is not a DC-prime filter in general.

Example 11. Let $A = \{0,a,b,c,d,1\}$, where $0 < b < a < 1$ and $0 < d < a,c < 1$. Define $*$ and $\to$ by the following

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

$\to$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

Then $(A,\wedge,\vee,*,\to,0,1)$ is a BL-algebra, [11, Example 4.3]. Clearly, $F_1 = \{a,b,1\}$, $F_2 = \{c,1\}$ are DC-prime filters while $F_1 \cap F_2 = \{1\}$ is not DC-prime. Since $a \vee c \in \{1\}$ but $a^{--}, c^{--} \notin \{1\}$.

Lemma 9. *Let F_1 and F_2 be* DC*-prime filters and F_2 be a maximal filter of an integral* BL*-algebra A. Then $F_1 \wedge F_2$ is a* DC*-prime filter of A.*

Proof. Let $a \vee b \in F_1 \wedge F_2$. As $F_1 \wedge F_2 \le F_1, F_2$ so $a \vee b \in F_1$ and $a \vee b \in F_2$. Hence $a^{--} \in F_1$ or $b^{--} \in F_1$ and $a^{--} \in F_2$ or $b^{--} \in F_2$. The following cases are considered:
Case (1): If $a^{--} \in F_1$ and $a^{--} \in F_2$, then $a^{--} \in F_1 \wedge F_2$.

Case (2): If $a^{--} \in F_1$ and $a^{--} \notin F_2$. As F_2 is a maximal filter, $F_2 \vee \langle a^{--} \rangle = A$. So $0 \in F_2 \vee \langle a^{--} \rangle$ hence there exists $f_2 \in F_2$ and $n \in \mathbb{N}$ such that $0 = f_2 * (a^{--})^n$. As A is an integral BL-algebra, $(a^{--})^n = 0$ or $f_2 = 0$. If $f_2 = 0$, then $0 \in F_2$, which is a contradiction. If $(a^{--})^n = 0$ then $0 \in F_1$, since $(a^{--})^n \in F_1$ which is a contradiction. So $a^{--} \in F_1$ and $a^{--} \in F_2$. Therefore $a^{--} \in F_1 \wedge F_2$.

Case (3): If $a^{--} \in F_2$ and $a^{--} \notin F_1$. It only needs to be proven that $b^{--} \in F_1 \wedge F_2$. Let $b^{--} \notin F_1 \wedge F_2$. As $b^{--} \in F_1$ so $b^{--} \notin F_2$. Hence $F_2 \vee \langle b^{--} \rangle = A$. So there exists $n \in \mathbb{N}$ and $f_3 \in F_2$ such that $0 = (b^{--})^n * f_3$. As A is an integral BL-algebra, $f_3 = 0$ or $(b^{--})^n = 0$. If $f_3 = 0$, then $0 \in F_2$. That is a contradiction. If $(b^{--})^n = 0$ then $0 \in F_1$, since $b^{--} \in F_1$. That is a contradiction. Therefore $b^{--} \in F_1 \wedge F_2$.

Case (4): If $b^{--} \in F_1$, $b^{--} \in F_2$, $a^{--} \notin F_1$ and $a^{--} \notin F_2$, then $b^{--} \in F_1 \wedge F_2$.

Therefore $F_1 \wedge F_2$ is a DC-prime filter of A. □

Lemma 10. *Let F and G be proper filters of a* BL*-algebra A such that $D(F) = D(G)$. Then F and G are* DC*-prime filters if and only if $F \wedge G$ is a* DC*-prime filter.*

Proof. Let $x \vee y \in F \wedge G$, for $x, y \in A$ and $x^{--} \notin F \wedge G$. Suppose that $x^{--} \notin F$. As F is a DC-prime so $y^{--} \in F$. Then $y \in D(F) = D(G) = D(F) \wedge D(G) = D(F \wedge G)$. Hence $y \in D(F \wedge G)$. Therefore $y^{--} \in F \wedge G$, i.e. $F \wedge G$ is a DC-prime filter of A. Now let $F \wedge G$ be a DC-prime filter and $D(F) = D(G)$. Then $D(F \wedge G) = D(F) \wedge D(G) = D(F) = D(G)$ is a DC-prime filter. Hence based on Theorem 9, F and G are DC-prime filters of A. □

Lemma 11. *Let $\{F_i : i \in I\}$ be a non-empty totally ordered subset of* DC*-prime filters in a* BL*-algebra A. Then $\cap_{i \in I} F_i$ and $\cup_{i \in I} F_i$ are* DC*-prime filters of A.*

Proof. Suppose that $x \vee y \in \cap_{i \in I} F_i$, for $x, y \in A$ and $x^{--} \notin \cap_{i \in I} F_i$. So there exists $j \in I$ such that $x^{--} \notin F_j$. As F_j is a DC-prime, $y^{--} \in F_j$. Now take $t \in I$. If $F_t \subseteq F_j$, then $x^{--} \notin F_t$ and so $y^{--} \in F_t$. If $F_j \subseteq F_t$, then $y^{--} \in F_t$. So for all $t \in I$, $y^{--} \in F_t$ and therefore $y^{--} \in \cap_{i \in I} F_i$ i.e. $\cap_{i \in I} F_i$ is a DC-prime filter. Clearly $\cup_{i \in I} F_i$ is a DC-prime filter. □

5. Characterizations of DC-prime filters in BL-algebras

In this section, the properties of a DC-prime filter in BL-algebras are studied, exactly. Also, some equivalent conditions are presented for it.

Theorem 13. *Let E_1 and E_2 be filters of a* BL*-algebra A and F be a proper filter of A. Then the following statements are equivalent:*

(i) If $F = E_1 \cap E_2$, then $D(F) = D(E_1)$ or $D(F) = D(E_2)$.

(ii) F is a DC*-prime filter.*

Proof.

$(i) \Rightarrow (ii)$**:** Let $a, b \in A$ and $a \vee b \in F$. Hence based on [19, Proposition 1.40], $(F \vee \langle a \rangle) \wedge (F \vee \langle b \rangle) = F$. So according to Part (i), $D(F \vee \langle a \rangle) = D(F)$ or

$D(F \vee \langle b \rangle) = D(F)$. As $a \in F \vee \langle a \rangle \subseteq D(F \vee \langle a \rangle)$ and $b \in F \vee \langle b \rangle \subseteq D(F \vee \langle b \rangle)$, so $a \in D(F)$ or $b \in D(F)$. Therefore $a^{--} \in F$ or $b^{--} \in F$, i.e. F is a DC-prime filter of A.

$(ii) \Rightarrow (i)$**:** Let F be a DC-prime filter and $E_1 \cap E_2 = F$. Assume that $D(F) \neq D(E_1)$ and $D(F) \neq D(E_2)$. Then there exists $a \in D(E_1)$ such that $a \notin D(F)$ and there exists $b \in D(E_2)$ such that $b \notin D(F)$. So $a^{--} \vee b^{--} \in E_1 \cap E_2 = F$. Then $a^{--} \in F$ or $b^{--} \in F$, since F is a DC-prime filter. Hence $a \in D(F)$ or $b \in D(F)$, which is a contradiction. Therefore the proof is completed.

□

According to Theorem 13:

Corollary 14. *Let E_i be filters of a* BL*-algebra A such that $i \in I$ and I be a finite set, and F be a proper filter of A. Then the following statements are equivalent:*

(i) If $F = \bigcap_{i \in I} E_i$, then there exists $i \in I$ such that $D(F) = D(E_i)$.

(ii) F is a DC*-prime filter.*

Theorem 14. *Let E_1 and E_2 be filters of a* BL*-algebra A and F be a* DC*-prime filter of A. If $E_1 \cap E_2 \subseteq F$, then $D(E_1) \subseteq D(F)$ or $D(E_2) \subseteq D(F)$.*

Proof. Let F be a DC-prime filter and $E_1 \cap E_2 \subseteq F$. Also assume that $D(E_1) \nsubseteq D(F)$ and $D(E_2) \nsubseteq D(F)$. Hence there exist $a \in D(E_1) - D(F)$ and $b \in D(E_2) - D(F)$. So $a^{--} \in E_1$ and $b^{--} \in E_2$ then $a^{--} \vee b^{--} \in E_1 \cap E_2 \subseteq F$. As F is a DC-prime filter, $a^{--} \in F$ or $b^{--} \in F$. Hence $a \in D(F)$ or $b \in D(F)$. Which is a contradiction. So the proof is completed. □

Example 12. Consider Example 6. Clearly, $F = \{a, 1\}$ is a DC-prime filter and $E_1 = \{b, 1\}$ and $E_2 = \{a, b, c, 1\}$ are filters of A. So the converse of above theorem, is not true in general.

Corollary 15. *Let F_1 and F_2 be filters of a* BL*-algebra A and F be a* DC*-prime filter of A. Then $D(F_1) \cap D(F_2) \subseteq F$ implies $D(F_1) \subseteq D(F)$ or $D(F_2) \subseteq D(F)$.*

Lemma 12. *Let A be a* DCP-BL*-algebra. Then for any proper filters F and G of A, $D(F) \subseteq D(G)$ or $D(G) \subseteq D(F)$.*

Proof. Let A be a DCP-BL-algebra. Then $D(F \cap G)$ is a DC-prime filter of A. As $D(F \cap G) = D(F) \cap D(G)$, according to Theorem 13, $D(D(F \cap G)) = D(D(F))$ or $D(D(F \cap G)) = D(D(G))$. Hence $D(F \cap G) = D(F)$ or $D(F \cap G) = D(G)$. Therefore as $D(F \cap G) = D(F) \cap D(G)$, $D(F) \subseteq D(G)$ or $D(G) \subseteq D(F)$, for any proper filters F and G of A. □

Example 13. Consider Example 1. $F_1 = \{1\}$, $F_2 = \{c, 1\}$ and $F_3 = \{b, c, d, 1\}$ are all filters of A. Also $D(F_1) = D(F_2) = F_2$ and $D(F_3) = F_3$. So $D(F_1) = D(F_2) \subseteq D(F_3)$ while F_1 is not a DC-prime filter (since $a^{--} \to b^{--} \notin F_1$ and $b^{--} \to a^{--} \notin F_1$). So A is not a DCP-BL-algebra.

Proposition 5. *Let F be a* DC*-prime filter of a* BL*-algebra A. Then the set $\{D(E) : F \subseteq E, E$ is a proper filter of $A\}$ is linearly ordered with respect to set theoretical inclusion.*

Proof. Let $D(E), D(G) \in T = \{D(E) : F \subseteq E, E$ is a proper filter of A$\}$. Assume $D(E) \nsubseteq D(G)$ and $D(G) \nsubseteq D(E)$. Hence there exist $a, b \in A$ such that $a \in D(E) - D(G)$ and $b \in D(G) - D(E)$. As F is a DC-prime filter, $a^{--} \to b^{--} \in F$ or $b^{--} \to a^{--} \in F$. If $a^{--} \to b^{--} \in F$ then $a^{--} \to b^{--} \in E \subseteq D(E)$. Then $b^{--} \in D(E)$, since $a \in D(E)$. Hence $b \in D(D(E)) = D(E)$ which is a contradiction since $b \notin D(E)$. If $b^{--} \to a^{--} \in F$ then $b^{--} \to a^{--} \in G \subseteq D(G)$. So $a^{--} \in D(G)$, since $b \in D(G)$. Hence $a \in D(D(G)) = D(G)$ which is a contradiction since $a \notin D(G)$. Therefore the proof is complete. □

Proposition 6.

(i) Let A be a DCP-BL*-algebra and F be a proper fantastic filter of A. Then for all $x, y \in A - F$, there exists $z \in A - F$ such that $x^{--} \leq z$ and $y^{--} \leq z$.*

(ii) Let F be a proper filter of a BL*-algebra A. If for all $x, y \in A - F$ there is $z \in A - F$ such that $x^{--} \leq z$ and $y^{--} \leq z$, then F is a* DC*-prime filter.*

Proof.

(i)**:** Let F be a fantastic filter of a DCP-BL-algebra A. According to Theorem 5, F is a DC-prime filter of A. Also, assume that there is $x, y \in A - F$ such that for all $z \in A - F$, $z < x^{--}$ or $z < y^{--}$. Since $x^{--} \leq x^{--} \vee y^{--}$ and $y^{--} \leq x^{--} \vee y^{--}$, so $x^{--} \vee y^{--} \in F$. As F is a DC-prime filter then $x^{--} \in F$ or $y^{--} \in F$. Hence $x \in F$ or $y \in F$, since F is a fantastic filter. That is a contradiction. Therefore the proof is completed.

(ii)**:** Let for all $x, y \in A - F$ there is $z \in A - F$ such that $x^{--} \leq z$ and $y^{--} \leq z$. Since $x \leq x^{--} \leq z$ and $y \leq y^{--} \leq z$, then $x \leq z$ and $y \leq z$. So according to [19, Proposition 1.45], F is a prime filter of A, hence based on Theorem 7, F is a DC-prime filter.

□

Proposition 7. *Let A be a* DCP-BL*-algebra and F be a proper filter of A. Then for any $x, y \in A - D(F)$, there exists $z \in A - D(F)$ such that $x \leq z$ and $y \leq z$.*

Proof. Let F be a proper filter of a DCP-BL-algebra A. According to Theorem 5, F is a DC-prime filter of A. Also, assume that there are $x, y \in A - D(F)$ such that for all $z \in A - D(F)$, $z < x$ or $z < y$. Since $x \leq x \vee y$ and $y \leq x \vee y$, so $x \vee y \in D(F)$. Then $x^{--} \vee y^{--} \in F$, so $x^{--} \in F$ or $y^{--} \in F$, since F is a DC-prime filter. Hence $x \in D(F)$ or $y \in D(F)$, which is a contradiction. □

Proposition 8. *Let F be a filter of a* BL*-algebra A and for any $x, y \in A - D(F)$, there exists $z \in A - D(F)$ such that $x \leq z$ and $y \leq z$. Then F is a DC-prime filter of A.*

Proof. According to Theorem 13, it only needs to be proven that if $F = E_1 \cap E_2$, then $D(F) = D(E_1)$ or $D(F) = D(E_2)$, for filters E_1 and E_2 of A. Now let $F = E_1 \cap E_2$ and $D(F) \neq D(E_1)$ and $D(F) \neq D(E_2)$. So there exists $x \in D(E_1)$ such that $x \notin D(F)$ and there exists $y \in D(E_2)$ such that $y \notin D(F)$. So according to hypothesis there exists $z \in A - D(F)$ such that $x \leq z$ and $y \leq z$. Since $x \in D(E_1)$ and $y \in D(E_2)$ so $z \in D(E_1) \cap D(E_2) = D(E_1 \cap E_2) = D(F)$. Therefore $z \in D(F)$ which is a contradiction. Thus the proof is completed. □

Corollary 16. *Let A be a* DCP-BL*-algebra and F be a proper filter of A. Then for any $x, y \in A/D(F)$ such that $x, y \neq 1/D(F)$, there exists $w \in A/D(F)$ such that $w \neq 1/D(F)$, $x \leq w$ and $y \leq w$.*

Proof. Let F be a proper filter of a DCP-BL-algebra A. According to Theorem 5, F is a DC-prime filter of A. Also, assume that $x, y \in A/D(F)$ such that $x, y \neq 1/D(F)$. Hence $x = a/D(F)$ and $y = b/D(F)$ for some $a, b \in A$, such that $a/D(F) \neq 1/D(F)$ and $b/D(F) \neq 1/D(F)$, and so $a, b \notin D(F)$. So according to Proposition 7, there exists $z \in A - D(F)$ such that $a \leq z$ and $b \leq z$. Hence $a \to z = 1$ and $b \to z = 1$. So $a/D(F) \leq z/D(F)$ and $b/D(F) \leq z/D(F)$. Therefore $x \leq w$ and $y \leq w$. □

Corollary 17. *Let F be a filter of a* BL*-algebra A and for any $x, y \in A/D(F)$ such that $x, y \neq 1/D(F)$, there exists $w \in A/D(F)$ such that $w \neq 1/D(F)$, $x \leq w$ and $y \leq w$. Then F is a DC-prime filter of A.*

Proof. According to Theorem 13, it only needs to be proven that if $F = E_1 \cap E_2$, then $D(F) = D(E_1)$ or $D(F) = D(E_2)$, for filters E_1 and E_2 of A. Now let $F = E_1 \cap E_2$ and $D(F) \neq D(E_1)$ and $D(F) \neq D(E_2)$. So there exists $a \in D(E_1)$ such that $a \notin D(F)$ and there exists $b \in D(E_2)$ such that $b \notin D(F)$. Thus $x = a/D(F) \neq 1/D(F)$ and $y = b/D(F) \neq 1/D(F)$. So according to hypothesis there exists $w = c/D(F) \in A/D(F)$ such that $w \neq 1/D(F)$, $x \leq w$ and $y \leq w$. Then $a/D(F) \leq c/D(F)$ and $b/D(F) \leq c/D(F)$. Hence $c/D(F) \in (D(E_1) \cap D(E_2))/D(F)$. As $F = E_1 \cap E_2$, $D(F) = D(E_1) \cap D(E_2)$. Therefore $w = c/D(F) = 1/D(F)$, which is a contradiction. Thus the proof is completed. □

Proposition 9. *Let A be a* DCP-BL*-algebra and F be a proper filter of A. Then $\langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq F$ implies $x^{--} \in F$ or $y^{--} \in F$, for all $x, y \in A$.*

Proof. According to Theorem 5, F is a DC-prime filter of A. Also, assume that $\langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq F$, for $x, y \in A$. Suppose that $x^{--} \notin F$ and $y^{--} \notin F$ so $x \notin D(F)$ and $y \notin D(F)$. Then based on Proposition 7, there is $z \in A - D(F)$ such that $x \leq z$ and $y \leq z$. Then $x^{--} \leq z^{--}$ and $y^{--} \leq z^{--}$. Hence $\langle z^{--} \rangle \subseteq \langle x^{--} \rangle$ and $\langle z^{--} \rangle \subseteq \langle y^{--} \rangle$. So $z^{--} \in \langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq F$ thus $z \in D(F)$, which is a contradiction. □

Proposition 10. *Let F be a filter of a* BL*-algebra A and for all $x, y \in A$, $\langle x^{--} \rangle \cap \langle y^{--} \rangle \subseteq F$ implies $x^{--} \in F$ or $y^{--} \in F$. Then F is a DC-prime filter of A.*

Proof. Assume that for any $x,y \in A$, $\langle x^{--}\rangle \cap \langle y^{--}\rangle \subseteq F$ implies $x^{--} \in F$ or $y^{--} \in F$. Also assume that $x \vee y \in F$. Hence $\langle x \vee y\rangle \subseteq F$ so $\langle x\rangle \cap \langle y\rangle \subseteq F$. As $\langle x^{--}\rangle \subseteq \langle x\rangle$ and $\langle y^{--}\rangle \subseteq \langle y\rangle$ thus $\langle x^{--}\rangle \cap \langle y^{--}\rangle \subseteq \langle x\rangle \cap \langle y\rangle \subseteq F$. According to hypothesis $x^{--} \in F$ or $y^{--} \in F$. Therefore F is a DC-prime filter of A. □

Proposition 11. *Let A be a* DCP-BL*-algebra and F be a proper filter of A. Then $\langle x\rangle \cap \langle y\rangle \subseteq F$ implies $x \in D(F)$ or $y \in D(F)$, for all $x,y \in A$.*

Proof. According to Theorem 5, F is a DC-prime filter of A. Also, assume that for any $x,y \in A$, $\langle x\rangle \cap \langle y\rangle \subseteq F$. So $\langle x^{--}\rangle \cap \langle y^{--}\rangle \subseteq F$. According to Proposition 9, $x \in D(F)$ or $y \in D(F)$. □

Proposition 12. *Let F be a filter of a* BL*-algebra A and for all $x,y \in A$, $\langle x\rangle \cap \langle y\rangle \subseteq F$ implies $x \in D(F)$ or $y \in D(F)$. Then F is a* DC*-prime filter of A.*

Proof. Let for any $x,y \in A$, if $\langle x\rangle \cap \langle y\rangle \subseteq F$ then $x \in D(F)$ or $y \in D(F)$ and assume that $x \vee y \in F$. So $\langle x \vee y\rangle \subseteq F$. Then $\langle x\rangle \cap \langle y\rangle \subseteq F$, since $\langle x\rangle \cap \langle y\rangle = \langle x \vee y\rangle$. Hence according to hypothesis $x \in D(F)$ or $y \in D(F)$. Therefore $x^{--} \in F$ or $y^{--} \in F$, i.e. F is a DC-prime filter of A. □

Recall that the set of all filters in a BL-algebra A is denoted by $F(A)$.

Corollary 18. *If $F(A)$ is a chain, then all proper filters of a* BL*-algebra A are* DC*-prime filters, i.e. A is a* DCP-BL*-algebra.*

Proof. Let $F(A)$ be a chain and F be a proper filter of A. Also assume that for any $x,y \in A$, $\langle x\rangle \cap \langle y\rangle \subseteq F$. Based on the condition of this corollary, there are two possibilities:

(i)**:** If $\langle x\rangle \subseteq \langle y\rangle$. Hence $x \in \langle x\rangle = \langle x\rangle \cap \langle y\rangle \subseteq F \subseteq D(F)$.

(ii)**:** If $\langle y\rangle \subseteq \langle x\rangle$. Hence $y \in \langle y\rangle = \langle x\rangle \cap \langle y\rangle \subseteq F \subseteq D(F)$.

Therefore according to Proposition 12, F is a DC-prime filter of A. □

Remark 10. In Example 2(ii), the converse of Corollary 18 is not true.

Recall that ${}^{\perp}X = \{a \in A : a \vee x = 1, \text{ for all } x \in X\}$ is a filter of a BL-algebra A, where X is a non-empty subset of A, [20, Definition 11]. Also recall that a filter F of A is called a linearly ordered filter if for any $x,y \in F$, either $x \leq y$ or $y \leq x$.

Lemma 13. *Let $F \neq \{1\}$ be a linearly ordered filter of A. Then ${}^{\perp}F$ is a* DC*-prime filter of A.*

Proof. According to the hypothesis $F \neq \{1\}$, hence ${}^{\perp}F \neq A$, i.e. ${}^{\perp}F$ is a proper filter. Now let $a \vee b \in^{\perp} F$, for $a,b \in A$, such that $a^{--} \notin^{\perp} F$ and $b^{--} \notin^{\perp} F$. So $a \notin^{\perp} F$, $b \notin^{\perp} F$. Then there exist $x,y \in F$ such that $a \vee x \neq 1$ and $b \vee y \neq 1$. Denote $z = x \wedge y$. Hence $z \in F$, $a \vee z \in F$ and $b \vee z \in F$, moreover $a \vee z \neq 1$ and $b \vee z \neq 1$ (if $a \vee z = 1$ then $a \vee x = 1$ and if $b \vee z = 1$ then $b \vee y = 1$, which are contradictions). As F is a linearly ordered filter, assume that $b \vee z \leq a \vee z$. Therefore as $z \in F$ and

$a\vee b\in^{\perp}F$, $1=(a\vee b)\vee z=a\vee(b\vee z)\leq a\vee(a\vee z)=a\vee z$, hence $a\vee z=1$, which is contradiction. Thus the proof is completed. □

A BL-algebra A is called a RS-BL-algebra, if for all non-empty subset X of A, $X_r=\{a\in A\colon x\to a=a$, for all $x\in X\}$ is a filter of A [17, Definition 3.1].

According to Lemma 13:

Corollary 19. *Let A be a RS-*BL*-algebra, $X\neq\{1\}$ be a non-empty linearly ordered upper subset of A. Then X_r is a* DC*-prime filter of A.*

Lemma 14. *Let x be an idempotent element of a* BL*-algebra A and $a,b\in A$. If $a\vee b=x$ implies $a=x$ or $b=x$ ($\vee$-irreducible), then $F_x=\{y\in A: x\leq y\}$ is a* DC*-prime filter of A.*

Proof. Claerly F_x is a proper filter of A. Let $y_1\vee y_2\in F_x$, for $y_1,y_2\in A$. So $x\leq y_1\vee y_2$. Then

$$\begin{aligned}x=x*x\leq x*(y_1\vee y_2)&\leq x*(y_1\vee y_2)^{--}\\&=x*(y_1^{--}\vee y_2^{--})\\&=(x*y_1^{--})\vee(x*y_2^{--})\\&\leq x\vee x=x.\end{aligned}$$

Then $x=(x*y_1^{--})\vee(x*y_2^{--})$. Since x is an $\vee$-irreducible so $x*y_1^{--}=x$ or $x*y_2^{--}=x$. It is known that $x*y_1^{--}\leq y_1^{--}$ and $x*y_2^{--}\leq y_2^{--}$, thus $x\leq y_1^{--}$ or $x\leq y_2^{--}$. Therefore $y_1^{--}\in F_x$ or $y_2^{--}\in F_x$ and so the proof is completed. □

Lemma 15. *Let F be a* DC*-prime filter of a* BL*-algebra A and G be a filter of A such that $G\subseteq F$. The set of all* DC*-prime filters F' of A such that $G\subseteq F'\subseteq F$ contains a minimal element.*

Proof. Take the set $\sum:=\{F'\colon F'$ is DC-prime filter such that $G\subseteq F'\subseteq F\}$. Clearly $F\in\sum$. The relation $\leq$, on $\sum$, is defined $F'\leq G'$ if and only if $G'\subseteq F'$, for all $F',G'\in\sum$. Clearly the relation $\leq$ is a partially ordered on $\sum$. Now let T be a chain of $\sum$. Take $d=\bigcap_{F'\in T}F'$. It is clear that d is a filter and for all $F'\in T$, $d\subseteq F'$. Then for all $F'\in T$, $F'\leq d$. We need to prove that d is a DC-prime filter. Now let for $x,y\in A$, $x\vee y\in d$ and $x^{--}\notin d$ and $y^{--}\notin d$. So there exist $F',G'\in T$ such that $x^{--}\notin F'$ and $y^{--}\notin G'$. Since T is a chain, so $G'\subseteq F'$ or $F'\subseteq G'$. If $F'\subseteq G'$, then $y^{--}\notin F'$. Hence based on $x\vee y\in d\subseteq F'$ and since F' is a DC-prime filter, $x^{--}\in F'$. That is a contradiction. Therefore d is a DC-prime filter. Also if $G'\subseteq F'$, the process is similarly. So d is an upper bound for T. Then by Zorn's lemma, $\sum$ contains a maximal element, hence it contains a minimal element. □

Proposition 13. *Let F be a proper filter of a* BL*-algebra A. Then for any $a\in A-D(F)$, there exists a* DC*-prime filter P such that $F\subseteq D(P)$ and $a\notin D(P)$.*

Proof. Consider the set $T := \{G : G$ is a filter of $A, F \subseteq D(G), a \notin D(G)\}$. Clearly $F \in T$. Now we define the relation $\leq$ on T as follows: for all $G, G' \in T$, $G \leq G'$ if and only if $G \subseteq G'$. It is easy to check that the relation $\leq$ is a partially ordered on T. Let L be a chain of T so $H = \bigcup_{G \in L} G$ is a proper filter and for all $G \in L$, $G \subseteq H$ i.e. $G \leq H$ and $F \subseteq D(G)$. So $F \subseteq \bigcup_{G \in L} D(G) \subseteq D(\bigcup_{G \in L} G) = D(H)$. Also for $G \in L$, $a \notin D(G)$. Hence for all $G \in L$, $a^{--} \notin G$, then $a^{--} \notin H = \bigcup_{G \in L} G$ thus $a \notin D(H)$, so $H \in T$. Therefore H is an upper bound for L. Then by Zorn's lemma the set T contains a maximal element like P, (clearly P is a proper filter of A). Now suppose that P is not a DC-prime filter. Thus there exist $x, y \in A$ such that $x \vee y \in P$ and $x^{--} \notin P$ and $y^{--} \notin P$. As $P \subsetneq P \vee \langle x^{--} \rangle$ and $P \subsetneq P \vee \langle y^{--} \rangle$ so $P \vee \langle x^{--} \rangle \notin T$ and $P \vee \langle y^{--} \rangle \notin T$, since P is maximal element of T. Hence as $F \subseteq D(P) \subseteq D(P \vee \langle x^{--} \rangle)$ and $F \subseteq D(P) \subseteq D(P \vee \langle y^{--} \rangle)$, then $a \in D(P \vee \langle x^{--} \rangle)$ and $a \in D(P \vee \langle y^{--} \rangle)$. Thus $a^{--} \in P \vee \langle x^{--} \rangle$ and $a^{--} \in P \vee \langle y^{--} \rangle$. So $a^{--} \geq \alpha * (x^{--})^n$ and $a^{--} \geq \beta * (y^{--})^m$, for some $\alpha, \beta \in P$ and for some $n, m \in \mathbb{N}$. According to rules on BL-algebras,

$$\begin{aligned} a^{--} &\geq (\alpha * (x^{--})^n) \vee (\beta * (y^{--})^m) \\ &\geq (\alpha \vee (\beta * (y^{--})^m)) * ((x^{--})^n \vee (\beta * (y^{--})^m)) \\ &\geq (\alpha \vee \beta) * (\alpha \vee (y^{--})^m) * ((x^{--})^n \vee \beta) * ((x^{--})^n \vee (y^{--})^m) \\ &\geq (\alpha \vee \beta) * (\alpha \vee (y^{--})^m) * ((x^{--})^n \vee \beta) * (x^{--} \vee y^{--})^{mn} \\ &\geq (\alpha \vee \beta) * \alpha * \beta * (x \vee y)^{mn}. \end{aligned}$$

It is obtained that $a^{--} \in P$. So $a \in D(P)$, which is a contradiction. Therefore P is a DC-prime filter of A. Hence the proof is completed. □

Corollary 20. *Let F be a proper filter of a* BL*-algebra A. Then*

$$D(F) = \bigcap_{F \subseteq D(G)} D(G),$$

where G is a DC*-prime filter of A.*

Proof. Let G be a DC-prime filter of A such that $F \subseteq D(G)$. So $D(F) \subseteq D(D(G)) = D(G)$. Hence $D(F) \subseteq \bigcap_{F \subseteq D(G)} D(G)$, where G is a DC-prime filter of A. Now assume that $x \in \bigcap_{F \subseteq D(G)} D(G)$ and $x \notin D(F)$, where G is a DC-prime filter of A. According to Proposition 13, there exists a DC-prime filter P of A such that $F \subseteq D(P)$ and $x \notin D(P)$. That is a contradiction, therefore the proof is completed.

□

Lemma 16. *Let F be a* DC*-prime filter of a* BL*-algebra A. Then for all $G \in F(A)$, $D(G \to D(F)) = D(F)$ or $D(G) \subseteq D(F)$.*

Proof. Based on [19, Corollary 1.37], $F(A)$ is a Heyting algebra. Hence based on the definition of Heyting algebra, $(G \to D(F)) \cap ((G \to D(F)) \to D(F)) = (G \to$

$D(F)) \cap D(F)$. As $D(F) \cap G \subseteq D(F)$, so according to [19, Lemma 1.35], $D(F) \subseteq G \to D(F)$. Then $D(F) \cap (G \to D(F)) = D(F)$, and so

$$(G \to D(F)) \cap ((G \to D(F)) \to D(F)) = D(F).$$

As F is a DC-prime filter, so $D(F)$ is a DC-prime filter, thus based on Theorem 13, $D(G \to D(F)) = D(D(F)) = D(F)$ or $D((G \to D(F)) \to D(F)) = D(D(F)) = D(F)$. On other hand by [19, Lemma 1.35], $D(F) = D((G \to D(F)) \to D(F)) = D(sup\{H : (G \to D(F)) \cap H \subseteq D(F)\}$. First prove that $G \in \{H : (G \to D(F)) \cap H \subseteq D(F)\}$. Assume that $a \in (G \to D(F)) \cap G$. As $a \in G$, so $\langle a \rangle \subseteq G$ and since $a \in (G \to D(F))$, so $G \cap \langle a \rangle \subseteq D(F)$. Thus $\langle a \rangle \subseteq D(F)$ hence $a \in D(F)$. Therefore $G \in \{H : (G \to D(F)) \cap H \subseteq D(F)\}$. Then $G \subseteq sup\{H : (G \to D(F)) \cap H \subseteq D(F)\} = D(F)$. Therefore $G \subseteq D(F)$ and so $D(G) \subseteq D(F)$. Thus the proof is completed. □

Lemma 17. *Let F be a* DC*-prime filter of a* BL*-algebra A. Then for all $G \in F(A)$, $D(G \to F) = D(F)$ or $D(G) \subseteq D(F)$.*

Proof. Based on [19, Corollary 1.37], $F(A)$ is a Heyting algebra. Hence $(G \to F) \cap ((G \to F) \to F) = F \cap (G \to F)$. Since $F \cap G \subseteq F$ so according to [19, Lemma 1.35], $F \subseteq G \to F$, then $F \cap (G \to F) = F$. So $(G \to F) \cap ((G \to F) \to F) = F$. Based on Theorem 13, $D(G \to F) = D(F)$ or $D((G \to F) \to F) = D(F)$. On other hand by [19, Lemma 1.35], $(G \to F) \to F = sup\{H : (G \to F) \cap H \subseteq F\}$. So $D((G \to F) \to F) = D(sup\{H : (G \to F) \cap H \subseteq F\} = D(F)$. First prove that $G \in \{H : (G \to F) \cap H \subseteq F\}$. Assume that $a \in (G \to F) \cap G$. As $a \in G$, so $\langle a \rangle \subseteq G$ and since $a \in (G \to F)$. Then $G \cap \langle a \rangle \subseteq F$, so $\langle a \rangle \subseteq F$. Therefore $a \in F$, hence $G \in \{H : (G \to F) \cap H \subseteq F\}$. Then $G \subseteq sup\{H : (G \to F) \cap H \subseteq F)\}$. So $D(G) \subseteq D(sup\{H : (G \to F) \cap H \subseteq F)\} = D(F)$. So $D(G) \subseteq D(F)$. Thus the proof is completed. □

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