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# A POLYNOMIAL SYSTEM OF DEGREE FOUR WITH AN INVARIANT TRIANGLE CONTAINING AT LEAST FOUR SMALL AMPLITUDE LIMIT CYCLES

IVÁN SZÁNTÓ

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*Abstract.* In this work, the existence of a polynomial system of degree four with an invariant triangle containing at least four small-amplitude limit cycles is proved. This result improves the result obtained in [2].

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## 1. Introduction

In this paper, we discuss the particular problem for the systems that have three real invariant straight lines and they form a triangle, called invariant triangle. As the triangle is a graph, inside there is at least one singularity, we will assume that is a fine focus type singularity.

In 1954, N. N Bautin in [1], show that for quadratic system ($n = 2$), the system

$$\dot{x} = x(a + bx + cy), \quad \dot{y} = y(d + ex + fy),$$

with two invariant straight lines has no limit cycles since a Dulac function can be given. Center cases may occur. So for a quadratic polynomial system with an invariant triangle, the system has no limit cycles.

For $n = 3$, Ye Yanqian in [5] considers a kind of cubic system with an invariant triangle whose sides are invariant straight lines and whose vertices are saddles.

$$\begin{aligned}\dot{x} &= x(a_0(1-x^2) + a_1x(1-x) + a_2y + a_4xy + a_5y^2)\\ \dot{y} &= y(b_0(1-y^2) + b_1x + b_2y(1-y) + b_3x^2 + b_4xy)\end{aligned}$$

Under certain conditions on the coefficients, the relative positions of other critical points of the cubic system and its invariant straight lines with respect to the invariant triangle are determined. In all of the cases, no limit cycles are found.

Z.H.Liu et. al in [3] consider a class of cubic system $(n = 3)$ given by

$$X_\mu : \begin{cases} \dot{x} &= (-1+abx)(-cx-(a^2+ab-abc-b^2c)x^2+y+(2a^2 \\ & \quad -ab+a^2b-b^2+ab^2-abc)xy+(ab-2a^2b)y^2) \\ \dot{y} &= (-1+aby)(-x+(a^2+ab^2+b^3)xy-(a^2b+ab^2)y^2) \end{cases}$$

where $\mu = (a,b,c) \in \mathbb{R}^3$

This system have three real invariant straight lines forming an invariant triangle surrounding at least one limit cycle, where the small amplitude limit cycle are limit cycles which bifurcate out of a non hyperbolic focus.

Still remain an open problem of the existence of a cubic system with an invariant triangle containing more than one limit cycle.

For $n = 4$, Z.H.Liu et. al in [2] considers a class of systems of degree four with three real invariant straight lines forming an invariant triangle and at least three small amplitude limit cycles in the interior, the system is given by

$$X_\mu : \begin{cases} \dot{x} &= P(x,y)+bR(x,y) \\ \dot{y} &= Q(x,y)+bS(x,y) \end{cases}$$

where

$$P(x,y) = (1+x)(a^3\lambda x - a^3 y - 4xy + a^2xy + a^3\lambda xy - 4y^2 - 4ay^2 + a^2y^2)$$
$$Q(x,y) = (1+y)(a^3x - a^2x^2 - a^2\lambda x^2 - a^3\lambda x^2 + 4xy + 4axy - a^2xy - a^2\lambda xy + 4y^2)$$

and

$$R(x,y) = (1+x)y^2(1+y)$$
$$S(x,y) = (1+y)(ax^2 - x^3 - axy - y^2),$$

with $\mu = (a,\lambda,b) \in \mathbb{R}^3, a > 0$.

## 2. Main Results

In this work, the existence of a polynomial system of degree four with an invariant triangle containing at least four small-amplitude limit cycles is proved. This result improves the result obtained in [2]. Let us consider a class of systems of degree four $(n = 4)$

$$X_\mu : \begin{cases} \dot{x} &= (1+2x)(\lambda x-(1+2\lambda)x^2-(a-b+c)x^3 \\ & \quad -y-2(1+\lambda)xy-2cx^2y-2y^2-4bxy^2-8ay^3) \\ \dot{y} &= (1+2y)(x+2(a-b+c)x^3+3xy-(3a-3b-c)x^2y \\ & \quad +2y^2+2(3a+b)xy^2+4ay^3) \end{cases} \tag{2.1}$$

where $\mu = (a,b,c,\lambda) \in \mathbb{R}^4$.

**Lemma 1.** *The system (2.1) has three real invariant straight lines, namely,*

$$x = -\frac{1}{2}, \qquad y = -\frac{1}{2}, \qquad \textit{and } x + y = \frac{1}{2}.$$

*and the above straight lines forming a triangle surrounding the origin.*

*Proof.* It is clear that $x = -\frac{1}{2}$ and $y = -\frac{1}{2}$ are invariant straight lines, and for the line $x + y = \frac{1}{2}$, we have

$$\begin{aligned}\dot{x} + \dot{y} = (-1 + 2x + 2y)(&-x - \lambda x - x^2 - 2\lambda x^2 - ax^3 + bx^3 - cx^3 \\ &+ y - xy + 3ax^2y - 3bx^2y + cx^2y + 2y^2 - 6axy^2 + 2bxy^2 + 4ay^3).\end{aligned}$$

and the lines forming a triangle surrounding the origin and this proves the Lemma. □

**Theorem 1.** *If* $\lambda = 0, a = \frac{26551 + 11\sqrt{226537}}{21064}, b = -\frac{44187 + 31\sqrt{226537}}{21064}$ *and* $c = \frac{-3331 + \sqrt{226537}}{10532}$*, the system (2.1), at the singularity* $(0,0)$ *has a repelling fine focus of order four.*

*Proof.* The linear part of (2.1) at the singularity $(0,0)$ is

$$DX_{\mu}(0,0) = \begin{pmatrix} \lambda & -1 \\ 1 & 0 \end{pmatrix}$$

If $\lambda = 0$, we consider $\tilde{\mu} = (a,b,c,\lambda)$, and we have that $\mathrm{div} X_{\tilde{\mu}}(0,0) = 0$ and $\det DX_{\tilde{\mu}}(0,0) = 1$, then the critical point $(0,0)$ is a fine focus.

Using Mathematica Software [4], we are able to compute the Lyapunov constants $L_k, k = 0,1,2,3,4$ and then to determinate the topological type of the singular point at the origin.

If $\lambda = 0$ then

$$L_0 = 0, \qquad L_1 = \frac{b - c + 3a - 2}{4}.$$

If $b = c - 3a + 2$ we have

$$L_1 = 0, \qquad L_2 = \frac{6 - 2a + 11c}{6}.$$

If $a = \frac{6 + 11c}{2}$, we have

$$L_1 = 0, \qquad L_2 = 0, \qquad L_3 = -\frac{516 + 3331c + 5266c^2}{48}.$$

If $c = \dfrac{3331+\sqrt{226537}}{10532}$,

$$L_1 = 0, \qquad L_2 = 0, \qquad L_3 = 0, \qquad L_4 = \frac{98979}{1250}$$

and this prove that the system (2.1), at the singularity $(0,0)$ has a repelling fine focus of order four. □

**Theorem 2.** *In the parameters space $\mathbb{R}^4$, there exists an open set $\mathcal{N}$, such that for all $(a,b,c,\lambda) \in \mathcal{N}$, the system (2.1) has four small-amplitude limit cycles coexisting with an invariant triangle.*

*Proof.* We already know that the systems $X_\mu$ has a invariant triangle surrounding the origin. By Theorem 1, the system (2.1) has at the singularity $(0,0)$ a repelling weak focus of order four if $\lambda = 0$, $a = \frac{26551+11\sqrt{226537}}{21064}$, $b = -\frac{44187+31\sqrt{226537}}{21064}$ and $c = \frac{-3331+\sqrt{226537}}{10532}$. Perturbing the system in such way that the stability of the origin is reverse, a limit cycle is created (Hopf Bifurcation). Following the same process and finally Hyperbolizing the origin, four hyperbolic small amplitude limit cycles are created. □

## REFERENCES


[1] N. N. Bautin, "On periodic solutions of a system of differential equations," *Akad. Nauk SSSR. Prikl. Mat. Meh.*, vol. 18, p. 128, 1954.

[2] Z. H. Liu, E. Sáez, and I. Szántó, "A system of degree four with an invariant triangle and at least three small amplitude limit cycles," *Electron. J. Qual. Theory Differ. Equ.*, pp. No. 69, 7, 2010, doi: 10.14232/ejqtde.2010.1.69. [Online]. Available: https://doi-org.usm.idm.oclc.org/10.14232/ejqtde.2010.1.69

[3] Z. Liu, E. Sáez, and I. Szántó, "A cubic system with an invariant triangle surrounding at least one limit cycle," *Taiwanese J. Math.*, vol. 7, no. 2, pp. 275–281, 2003, doi: 10.11650/twjm/1500575064. [Online]. Available: https://doi-org.usm.idm.oclc.org/10.11650/twjm/1500575064

[4] S. Wolfram, "The mathematica book 5th edition; wolfram research," *Inc*, 2003.

[5] Y. Q. Ye, "Cubic system having three straight line integrals forming a triangle," *Ann. Differential Equations*, vol. 6, no. 3, pp. 365–372, 1990.



*Author's address*

**Iván Szántó**

Universidad Técnica Federico Santa María, Departamento de Matemática, Casilla 110-V, Valparaíso, Chile

*E-mail address:* ivan.szanto@usm.cl