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# CONJOINED LORENZ-LIKE ATTRACTORS COINED

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*Abstract.* In contrast to most other periodically forced chaotic systems with infinitely many isolated coexisting strange attractors, little seems to be known about the ones that possess conjoined Lorenz-like attractors with potential existence of infinitely many pairs of wings/scrolls. To achieve this target, this note proposes a new periodically forced extended Lorenz-like system, which generates infinitely many singularly degenerate heteroclinic cycles or heteroclinic orbits to any two equilibria of a family of non-hyperbolic lines, the collapses of which create not only the desirable conjoined Lorenz-like attractors, but also infinitely many isolated coexisting ones. What is more, the state variable $x$ of that conjoined Lorenz-like attractor presents stochastic behaviors, confirming the links between long period and chaos. This also generalizes the classical concept of boundedness of chaos, i.e., the system orbits beginning from one sub-two-scroll of that conjoined Lorenz-like attractor might tend to the ones at infinity. Apart from those, the existence of an invariant surface and a family of infinitely many pairs of symmetrical heteroclinic orbits are proved by utilizing the Lyapunov function, the definitions of both $\alpha$-limit set and $\omega$-limit set.

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*Keywords:* new 3D periodically forced extended Lorenz-like system, conjoined Lorenz-like attractors, singularly degenerate heteroclinic cycle, heteroclinic orbit, Lyapunov function

## 1. INTRODUCTION

Since the introduction of Lorenz system, scholars made many attempts to shed some light on the forming mechanism of various strange attractors, including self-excited and hidden ones, etc. Combining contraction map and boundary problem [8],

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Shilnikov et al. formulated an effective method to determine the existence of homoclinic and heteroclinic orbits and further classified chaos occurring in 3D quadratic autonomous differential systems into the following four cases:

(1) chaos of the Shilnikov homoclinic-orbit type;
(2) chaos of the Shilnikov heteroclinic-orbit type;
(3) chaos of the hybrid type with both Shilnikov homoclinic and heteroclinic orbits;
(4) chaos of other types.

After the appearance of powerful computational tools, the computer-assisted proof for the Lorenz attractor became a reality [9, 12, 13]. Kokubu and Roussarie firstly found that the broken of singularly degenerate heteroclinic cycles (i.e., each of which consists of invariant sets formed by a non-hyperbolic (i.e., at least one of eigenvalues is null) line of equilibria together with heteroclinic orbits connecting two of the equilibria) created strange attractors [3]. Liao put forward two sufficient conditions to guarantee a continuous system to exhibit chaotic motions, i.e., the existence of at least one positive Lyapunov exponent and one ultimate bound set [7]. Specifically, the first one repels the trajectories of studied systems from inside to outside, while the second one attracts the trajectories from outside to inside. Kuznetsov et al. searched a new hidden Lorenz-like attractor by applying homotopy and numerical continuation to synthesise a new transition scenario among the well-known Glukhovsky-Dolghansky, the Lorenz system and the Rabinovich system, and analytical/numerical methods based on the continuation and perpetual points [5]. Further, Leonov and Kuznetsov generalized the second part of the celebrated Hilbert's 16th problem [2]: the number and mutual disposition of attractors and repellers in the chaotic multidimensional dynamical systems, and, in particular, their dependence on the degree of polynomials in the model [4]. Zhang and Chen then also formulated a chaotic analog when searching for infinitely many chaotic attractors in lower-dimensional autonomous systems, and with an extension to the fractional-order setting [19]. Meanwhile, they also found some new phenomena that generalized the classical concept of sensitive dependence on the initial conditions, i.e., the system orbits beginning from two very close initial points might either converge and then stay on one attractor or diverge to two different attractors. Along the same line, by introducing sine functions, Yang and Yang proposed a new 3D autonomous system with infinitely many coexisting chaotic attractors and infinitely many coexisting periodic attractors in the following three cases:

(i) no equilibria,
(ii) only infinitely many nonhyperbolic double-zero equilibria,
(iii) both infinitely many hyperbolic saddles and nonhyperbolic pure-imaginary equilibria [17].

Based on the guess that the decrease of powers of some variable states may widen the ranges of some parameters for which hidden attractors exist, Wang et al. numerically found multitudinous potential hidden attractors in a sub-quadratic Lorenz-like system [15].

Remarkably, all of aforementioned attractors are isolated ones. To this end, by carrying out numerical simulation on a kind of complex Lorenz-type systems, Zhang proposed an open question: There might exist an infinitely many-scroll attractor for that system, where the attractor oscillates as well as it goes to infinity along the $x_1$-axis. Since the well-known chaotic Lorenz systems have sensitive dependence on initial conditions, we cannot simulate the systems for a long time, and the numerical experiments might not reflect the true orbits. This is a very strange and interesting phenomenon for future work [18, Question 4.1, p.2150101-21].

As the question itself stated, it may be hard to answer at present. Therefore, the existence of conjoined Lorenz-like attractors may prove to be the next best thing. Now, how to generate a conjoined Lorenz-like attractor is not only theoretically significant but also practically important, motivating the work to be presented in this paper.

The new introduced system should satisfy the following three principles:

(1) The system should be a Lorenz-like one with infinitely many lines of non-hyperbolic equilibria.
(2) The system should generate singularly degenerate heteroclinic cycles or heteroclinic orbits with nearby chaotic attractors.
(3) The system should be sensitive dependence on the initial conditions as the one in [19], which guarantees potential existence of strong connection among different isolated chaotic attractors.

On the basis of these three simple tips, one tries to construct a new 3D periodically forced extended Lorenz-like system and find the parameters of conjoined Lorenz-like attractors by a trial-and-error process.

As far as we know, very few research on the conjoined Lorenz-like attractor is available in the published literatures. Accordingly, it is a demanding work to reveal the forming mechanism of it. The main contributions are as follows:

(1) Proposing a new 3D periodically forced extended Lorenz-like system family which generates singularly degenerate heteroclinic cycles or heteroclinic orbits with nearby conjoined Lorenz-like attractors.
(2) Illustrating another new kind of extreme sensitivity to initial conditions, i.e., the system's orbits starting from two very close initial points of any one line of non-hyperbolic equilibria might converge to another equilibria of either the same line of non-hyperbolic equilibria and then create singularly degenerate heteroclinic cycles, or the neighbouring ones and thus forms heteroclinic orbits.
(3) Proving the existence of an invariant surface and infinitely many pairs of heteroclinic orbits.

The rest of the paper is structured as follows. In Section 2, some basic concepts are introduced. Section 3 formulates a novel periodically forced extended Lorenz-like system and presents conjoined Lorenz-like attractors. The basic dynamics, such as stability, Hopf bifurcation and invariant surface, are analyzed in Section 4. Section 5 illustrates singularly degenerate heteroclinic cycles and heteroclinic orbits with nearby infinitely many isolated coexisting Lorenz-like attractors. In Section 6, one gives a rigorous proof for the existence of infinitely many pairs of symmetric heteroclinic orbits.

## 2. PRELIMINARY

Consider the differential system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\xi)$, where $\mathbf{x} \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^m$ are vectors representing phase variables and control parameters respectively. Assume that $\mathbf{f}$ is of class $C^\infty$ in $\mathbb{R}^n \times \mathbb{R}^m$. Suppose that system has an equilibrium point $\mathbf{x} = \mathbf{x_0}$ at $\xi = \xi_\mathbf{0}$. If at least one eigenvalue of the Jacobian matrix associated with linearized vector field about $x_0$ is zero or has a zero real part, then $x_0$ is said to be non-hyperbolic.

Let $\mathbb{R}[x,y,z]$ be the ring of real polynomials in the variables $x$, $y$ and $z$ . We say that $Q(x,y,z) \in \mathbb{R}[x,y,z]$ is a Darboux polynomial of a three dimensional ODEs system if the time derivative of it satisfies

$$\frac{dQ(x,y,z)}{dt} = \frac{\partial Q(x,y,z)}{\partial x}\dot{x} + \frac{\partial Q(x,y,z)}{\partial y}\dot{y} + \frac{\partial Q(x,y,z)}{\partial z}\dot{z} = k(x,y,z)Q(x,y,z),$$

where $k(x,y,z)$ is a real polynomial called the cofactor of $Q(x,y,z)$. If $Q(x,y,z)$ is a Darboux polynomial, then the surface $Q(x,y,z) = 0$ is known as invariant algebraic surface. If $Q(x,y,z)$ contains trigonometric functions, then we call it invariant surface.

Let the set of points: $S$ (either connected or disconnected ) be equilibria of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\xi)$ and $D \subset \mathbb{R}^n$ to be a domain containing $S$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $V(S) = 0$ and $V(x) > 0$ in $D\backslash S$, $\dot{V}(x) \leq 0$ in $D$. The derivative of $V(x)$ along the trajectories of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x},\xi)$, denoted by $\dot{V}(x)$, is given by $\dot{V}(x) = \Sigma_{i=1}^n \frac{\partial V}{\partial x_i}\dot{x}_i = \Sigma_{i=1}^n \frac{\partial V}{\partial x_i} f_i(x)$. Then, $S$ is stable. If $\dot{V}(x) < 0$ in $D\backslash S$, then $S$ is asymptotically stable. Moreover, if $D = \mathbb{R}^n$, then $S$ is globally asymptotically stable.

## 3. CONJOINED LORENZ-LIKE ATTRACTORS

In light of Lorenz-like systems that easily generate singularly degenerate heteroclinic cycles with nearby bifurcated strange attractors, we firstly replace the state variable $x$ of the following 3D extended Lorenz system [6]:

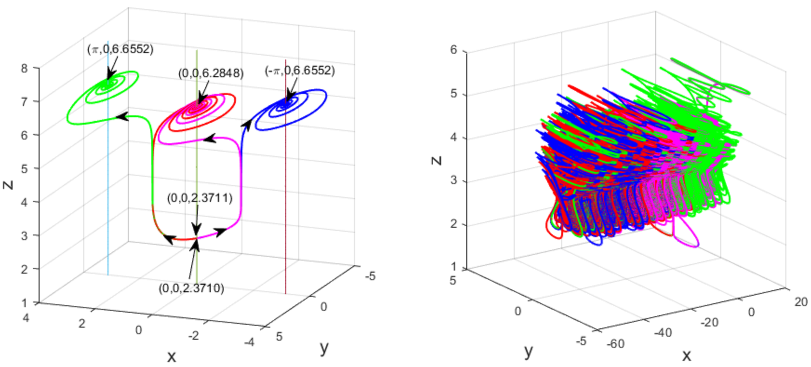
$$\begin{cases} \dot{x} = y, \\ \dot{y} = mx - ny - mxz - px^3, \\ \dot{z} = -az + bx^2, \end{cases}$$

with $\sin(2x)$ and $\sin(x)$, and formulates the following new periodically forced analogue:

$$\begin{cases} \dot{x} = y, \\ \dot{y} = \sin(2x)\left[a + c_1 z + c_2 \sin^2(x)\right] - by, \\ \dot{z} = -d_1 z + d_2 \sin^2(x), \end{cases} \tag{3.1}$$

where $a, b, c_1, c_2, d_1, d_2 \in \mathbb{R}$.

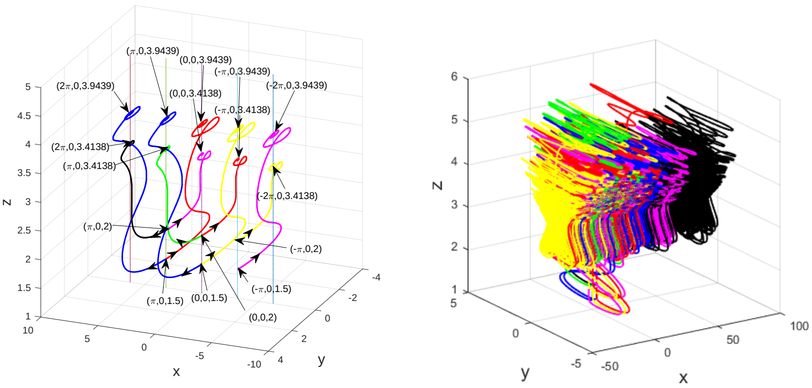
Secondly, based on the dynamics of $E_z$ in Proposition 3 in Section 4, we choose the parameters $(a, b, c_1, c_2, d_1, d_2) = (12, 1.0, -4, -1, 0, 1)$ and initial values $(x_0^{1,2}, y_0^{1,2}, z_0^{\prime,\prime\prime}) = (\pm 1.3, \pm 13, \pm 16) \times 10^{-8} + (0, 0, 2.3710)$ and $(x_0^{1,2}, y_0^{1,2}, z_0^{\prime\prime\prime,\prime\prime\prime\prime}) = (\pm 1.3, \pm 13, \pm 16) \times 10^{-8} + (0, 0, 2.3711)$. Fig. 1 shows heteroclinic orbits and singularly degenerate heteroclinic cycles and nearby conjoined Lorenz-like attractors of system (3.1), which extends the classical concept of sensitive dependence on the initial conditions, i.e., the trajectories from two very close initial points $(0, 0, 2.3710)$ and $(0, 0, 2.3711)$ might converge to $(\pm\pi, 0, 6.6552)$ and then create heteroclinic orbits, or $(0, 0, 6.2848)$ and thus forms singularly degenerate heteroclinic cycles.




(a) $d_1 = 0$ (b) $d_1 = 0.05$

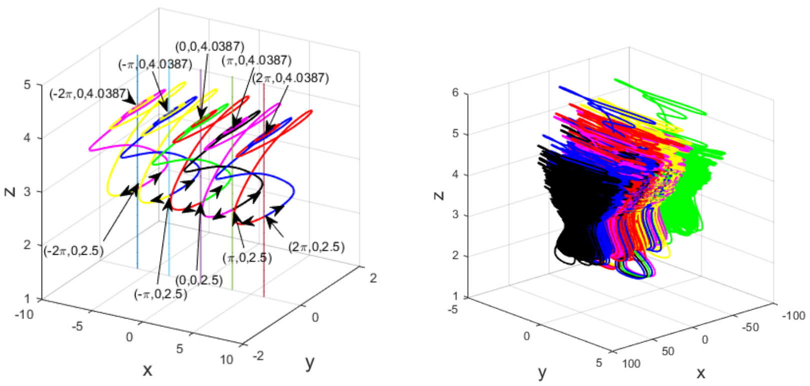
FIG. 1. (a) Heteroclinic orbits to $(0, 0, 2.3710)$ and $(\pi, 0, 6.6552)$, $(0, 0, 2.3710)$ and $(-\pi, 0, 6.6552)$; singularly degenerate heteroclinic cycles consisting of $(0, 0, 2.3711)$ and $(0, 0, 6.2848)$ of system (3.1), and (b) nearby conjoined Lorenz-like attractors when $(a, b, c_1, c_2, d_2) = (12, 1.0, -4, -1, 1)$, generalizing the classical concept of sensitive dependence on the initial conditions.

Choose other initial conditions $(\pm\pi, 0, 1.5)$, $(0, 0, 1.5)$, $(\pm\pi, 0, 2)$, $(0, 0, 2)$, $(\pm 2\pi, 0, 2.5)$, $(0, 0, 2.5)$ and $(\pm\pi, 0, 2.5)$. Figs. 2-4 also depict ten heteroclinic orbits and five pairs of singularly degenerate heteroclinic cycles with nearby bifurcated conjoined Lorenz-like attractors. Fig. 5 illustrates time series of the variable $x$ of conjoined Lorenz-like attractors with initial conditions $(0, 0, 2)$ and $(0, 0, 1000\pi)$, verifying the complex structures of conjoined Lorenz-like attractors having the characteristics of random walk. Moreover, conjoined Lorenz-like attractors may extend the classical concept of boundedness of chaos, i.e., the system orbits beginning from one sub-two-scroll of that conjoined Lorenz-like attractor might tend to infinity.




(a) $d_1 = 0$ (b) $d_1 = 0.05$

FIG. 2. (a) Heteroclinic orbits to $(\pi,0,1.5)$ and $(2\pi,0,3.9439)$, $(\pi,0,2)$ and $(2\pi,0,3.3148)$, $(\pi,0,1.5)$ and $(0,0,3.9439)$, $(\pi,0,2)$ and $(2\pi,0,3.3148)$, $(0,0,1.5)$ and $(\pi,0,3.9439)$, $(0,0,2)$ and $(\pi,0,3.3148)$, $(0,0,1.5)$ and $(-\pi,0,3.9439)$, $(0,0,2)$ and $(2\pi,0,3.3148)$, $(-\pi,0,1.5)$ and $(-2\pi,0,3.9439)$, $(-\pi,0,2)$ and $(-2\pi,0,3.3148)$ of system (3.1), and (b) nearby conjoined Lorenz-like attractors when $(a,b,c_1,c_2,d_2) = (12,1.0,-4,-1,1)$




(a) $d_1 = 0$ (b) $d_1 = 0.05$

FIG. 3. (a) Singularly degenerate heteroclinic cycles consisting of $(-2\pi,0,2.5)$ and $(-2\pi,0,4.0378)$, $(-\pi,0,2.5)$ and $(-\pi,0,4.0378)$, $(0,0,2.5)$ and $(0,0,4.0378)$, $(\pi,0,2.5)$ and $(\pi,0,4.0378)$, $(2\pi,0,2.5)$ and $(2\pi,0,4.0378)$ of system (3.1), and (b) nearby conjoined Lorenz-like attractors when $(a,b,c_1,c_2,d_2) = (12,1.0,-4,-1,1)$.

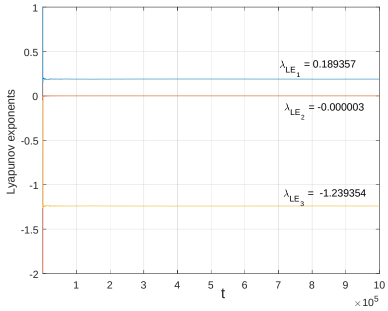



FIG. 4. Lyapunov exponents of system (3.1) with $(a,b,c_1,c_2,d_1,d_2) = (12,1.0,-4,-1,0.05,1)$ and $(x_0^1,y_0^1,z_0^1) = (1.3,1.3,16)\times 10^{-8}$.

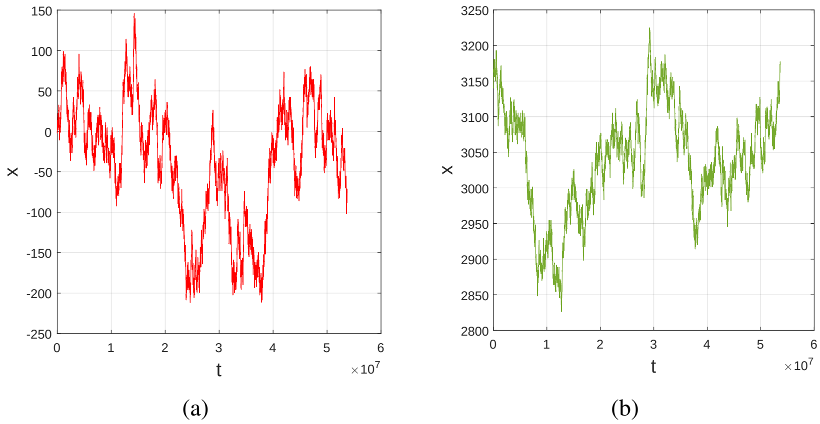



FIG. 5. Time series of the variable $x$ of system (3.1) when $(a,b,c_1,c_2,d_1,d_2) = (12,1.0,-4,-1,0.05,1)$, time interval $T = [0,2\times 10^7]$, $(y_0^2,z_0^2) = (1.3\times 10^{-8}, 2+16\times 10^{-8})$ and (a) $x_0^2 = 2+16\times 10^{-8}$, (b) $x_0^3 = 1000\pi + 1.3\times 10^{-8}$, which suggest stochastic behaviors of conjoined Lorenz-like attractors.

## 4. BASIC BEHAVIORS

In this section, we mainly give some results on some basic dynamics of system (3.1), i.e., the distribution of equilibrium points, stability, Hopf bifurcation, etc.

First of all, the distribution of equilibrium points easily follows from the algebraic structure of system (3.1) and is listed in the following proposition.

**Proposition 1.**

*(1) When $d_1 = 0$, system (3.1) has a family of parallel lines of non-hyperbolic equilibria $E_z = \{(k\pi, 0, z) | k \in \mathbb{N}, z \in \mathbb{R}\}$.*

(2) *When* $a=0$, $c_1=d_1\neq 0$ *and* $c_2=-d_2\neq 0$, *or* $c_1=-d_1\neq 0$ *and* $c_2=d_2\neq 0$, $E_p=\{(x,0,\frac{d_2\sin^2 x}{d_1})|x\in\mathbb{R}\}$ *is a family of curves of non-hyperbolic equilibria.*

(3) *When* $d_1\neq 0$, $E_0=\{(k\pi,0,0)|k\in\mathbb{N}\}$ *is an infinite set of isolated equilibria in system* (3.1) *for* $\frac{-ad_1}{c_1d_2+c_2d_1}>1$ *or* $\frac{-ad_1}{c_1d_2+c_2d_1}<0$*; while* $0<\frac{-ad_1}{c_1d_2+c_2d_1}\leq 1$, $E_\pm=\{(k\pi\pm\frac{1}{2}\arcsin\sqrt{\frac{-ad_1}{c_1d_2+c_2d_1}},0,-\frac{ad_2}{c_1d_2+c_2d_1})|k\in\mathbb{N}\}$ *are infinitely many pairs of symmetrical isolated equilibria.*

*Remark* 1. When $d_1\neq 0$, $a$ passes through the null value and $0<\frac{-ad_1}{c_1d_2+c_2d_1}\leq 1$, system (3.1) undergoes a generic pitchfork bifurcation at points of $E_0$. While $a\neq 0$, $d_1$ crosses the null value and $0<\frac{-ad_1}{c_1d_2+c_2d_1}\leq 1$, system (3.1) undergoes a degenerate pitchfork bifurcation at points of $E_z$.

Next, in order to determine the stability and bifurcation of equilibria, we have to calculate Jacobian matrix associated vector field of system (3.1):

$$J=\begin{pmatrix} 0 & 1 & 0\\ 2\cos 2x(a+c_1z+c_2\sin^2 x)+c_2\sin^2 2x & -b & c_1\sin 2x\\ d_2\sin 2x & 0 & -d_1\end{pmatrix}.$$

One can easily calculate the characteristic equations of points of $E_z$, $E_0$, $E_p$ and $E_\pm$:

(1) The one of each of $E_z$ is
$$\lambda\left[\lambda^2+b\lambda-2(a+c_1z)\right]=0,$$
with $\lambda_1=0$, $\lambda_{2,3}=\frac{-b\pm\sqrt{b^2+8(a+c_1z)}}{2}$.

(2) The one of each of $E_0$ is
$$(\lambda+d_1)\left[\lambda^2+b\lambda-2a\right]=0,$$
with $\lambda_1=-d_1$, $\lambda_{2,3}=\frac{-b\pm\sqrt{b^2+8a}}{2}$.

(3) For $c_1=d_1$ and $c_2=-d_2$ (resp. $c_1=-d_1$ and $c_2=d_2$), the one of each of $E_p$ is
$$\lambda\left[\lambda^2+(b+d_1)\lambda+bd_1+d_2\sin^2 2x\right]=0,$$
with $\lambda_1=0$, $\lambda_{2,3}=\frac{-(b+d_1)\pm\sqrt{(b+d_1)^2-4(bd_1+d_2\sin^2 2x)}}{2}$ (resp.
$$\lambda\left[\lambda^2+(b+d_1)\lambda+bd_1-d_2\sin^2 2x\right]=0,$$
with $\lambda_1=0$, $\lambda_{2,3}=\frac{-(b+d_1)\pm\sqrt{(b+d_1)^2-4(bd_1-d_2\sin^2 2x)}}{2}$).

(4) The one of points of $E_\pm$ is:
$$\begin{aligned}&\lambda^3+(b+d_1)\lambda^2+d_1\left[b+\frac{4ac_2}{c_1d_2+c_2d_1}\left(1+\frac{ad_1}{c_1d_2+c_2d_1}\right)\right]\lambda\\ &+4ad_1\left(1+\frac{ad_1}{c_1d_2+c_2d_1}\right)=0.\end{aligned}\tag{4.1}$$

TABLE 1. The dynamical behaviors of points of $E_z$.

| $b$ | $a+c_1z$ | Property of $E_z$ |
|---|---|---|
| $>0$ | $<0$ | A 2D $W^s_{loc}$ and a 1D $W^c_{loc}$ |
| | $=0$ | A 1D $W^s_{loc}$ and a 2D $W^c_{loc}$ |
| | $>0$ | A 1D $W^s_{loc}$, a 1D $W^c_{loc}$ and a 1D $W^u_{loc}$ |
| $=0$ | $<0$ | A 3D $W^c_{loc}$ |
| | $=0$ | A 3D $W^c_{loc}$ |
| | $>0$ | A 1D $W^s_{loc}$, a 1D $W^c_{loc}$ and a 1D $W^u_{loc}$ |
| $<0$ | $<0$ | A 1D $W^c_{loc}$ and a 2D $W^u_{loc}$ |
| | $=0$ | A 2D $W^c_{loc}$ and a 1D $W^u_{loc}$ |
| | $>0$ | A 1D $W^s_{loc}$, a 1D $W^c_{loc}$ and a 1D $W^u_{loc}$ |

TABLE 2. The dynamical behaviors of points of $E_p$.

| $b+d_1$ | $-(bd_1 \pm d_2 \sin^2 2x)$ | Property of $E_p$ |
|---|---|---|
| $>0$ | $<0$ | A 2D $W^s_{loc}$ and a 1D $W^c_{loc}$ |
| | $=0$ | A 1D $W^s_{loc}$ and a 2D $W^c_{loc}$ |
| | $>0$ | A 1D $W^s_{loc}$, a 1D $W^c_{loc}$ and a 1D $W^u_{loc}$ |
| $=0$ | $<0$ | A 3D $W^c_{loc}$ |
| | $=0$ | A 3D $W^c_{loc}$ |
| | $>0$ | A 1D $W^s_{loc}$, a 1D $W^c_{loc}$ and a 1D $W^u_{loc}$ |
| $<0$ | $<0$ | A 1D $W^c_{loc}$ and a 2D $W^u_{loc}$ |
| | $=0$ | A 2D $W^c_{loc}$ and a 1D $W^u_{loc}$ |
| | $>0$ | A 1D $W^s_{loc}$, a 1D $W^c_{loc}$ and a 1D $W^u_{loc}$ |

TABLE 3. The dynamical behaviors of points of $E_0$.

| $b$ | $a$ | Property of $E_0$ |
|---|---|---|
| $>0$ | $<0$ | A 2D $W^s_{loc}$ and a 1D $W^c_{loc}$ |
| | $=0$ | A 1D $W^s_{loc}$ and a 2D $W^c_{loc}$ |
| | $>0$ | A 1D $W^s_{loc}$, a 1D $W^c_{loc}$ and a 1D $W^u_{loc}$ |
| $=0$ | $<0$ | A 3D $W^c_{loc}$ |
| | $=0$ | A 3D $W^c_{loc}$ |
| | $>0$ | A 1D $W^s_{loc}$, a 1D $W^c_{loc}$ and a 1D $W^u_{loc}$ |
| $<0$ | $<0$ | A 1D $W^c_{loc}$ and a 2D $W^u_{loc}$ |
| | $=0$ | A 2D $W^c_{loc}$ and a 1D $W^u_{loc}$ |
| | $>0$ | A 1D $W^s_{loc}$, a 1D $W^c_{loc}$ and a 1D $W^u_{loc}$ |

**Proposition 2.**

*(1) When $a, b, c_1, c_2, d_1, d_2 \in \mathbb{R}$, the local dynamical behaviors of points of $E_z$ are totally summarized in Table 1. While $a = 0$, $c_1 = d_1 \neq 0$, $c_2 = -d_2 \neq 0$ (resp.*

*$c_1 = -d_1$ and $c_2 = d_2$), $-(bd_1 + d_2 \sin^2 2x) \in \mathbb{R}$ (resp. $-(bd_1 - d_2 \sin^2 2x) \in \mathbb{R}$), Table 2 lists the local dynamics of each of $E_p$.*

*(2) Moreover, for $a = 0$, $b > 0$, $d_1 = -c_1 > 0$ and $d_2 = c_2 < 0$, each of $E_p$ is globally asymptotically stable. In particular, when*

$$b = d_1, \qquad Q = \frac{1}{2}\left[-d_2 y^2 + (-d_1 z + d_2 \sin^2 x)^2\right]$$

*is an invariant surface with cofactor $-2b$.*

*Proof.*

(1) Firstly, the local stability of points of $E_z$ and $E_p$ easily follow from the linear analysis and are omitted here.

(2) Secondly, we discuss the global stability of points of $E_p$, i.e., each point of $E_p$ is globally stable. For $a = 0$, $b > 0$, $d_1 = -c_1 > 0$ and $d_2 = c_2 < 0$, set the following Lyapunov function:

$$Q = \frac{1}{2}\left[-d_2 y^2 + (-d_1 z + d_2 \sin^2 x)^2\right],$$

with

$$\frac{dQ}{dt}\Big|_{(3.1)} = bd_2 y^2 - d_1(-d_1 z + d_2 \sin^2 x)^2,$$

which yields

$$\dot{Q} = 0 \quad \Leftrightarrow \quad y = -d_1 z + d_2 \sin^2 x = 0,$$

which implies the stability of points of $E_p$. According to LaSalle theorem [1], each of $E_p$ is also globally asymptotically stable.

In particular, for $b = d_1$ and $Q_0 = \frac{1}{2}\left[-d_2 y_0^2 + (-d_1 z_0 + d_2 \sin^2 x_0)^2\right]$, we arrive at

$$\frac{dQ}{dt}\Big|_{(3.1)} = -b\left[-d_2 y^2 + (-d_1 z + d_2 \sin^2 x)^2\right] = -2bQ,$$

which leads to

$$0 \le Q = Q_0 e^{-2b(t-t_0)} \to 0, \quad t \to +\infty.$$

Namely, $Q$ is an invariant surface with cofactor $-2b$. The proof is finished. □

**Proposition 3.**

*(1) If $d_1 < 0$, then each of $E_0$ is unstable.*

*(2) If $d_1 = 0$, then the dynamics of points of $E_0$ are the same to the ones of points of $E_z$ and listed in Table 1.*

*(3) If $d_1 > 0$, then the dynamics of points of $E_0$ are presented in Table 3.*

*Proof.* The local stability of points of $E_0$ easily follows from the linear analysis and are omitted here. □

**Proposition 4.** *Set* $W = \{(a,b,c_1,c_2,d_1,d_2) \in \mathbb{R}^6 | 0 < \frac{-ad_1}{c_1d_2+c_2d_1} \leq 1\}$, $W_2 = W \backslash W_1$

$$W_1 = \{(a,b,c_1,c_2,d_1,d_2) \in W | b+d_1 > 0,$$
$$d_1\left[b + \frac{4ac_2}{c_1d_2+c_2d_1}\left(1+\frac{ad_1}{c_1d_2+c_2d_1}\right)\right] > 0,$$
$$4ad_1\left(1+\frac{ad_1}{c_1d_2+c_2d_1}\right) > 0\}$$

*and*

$$W_1^1 = \{(a,b,c_1,c_2,d_1,d_2) \in W_1 : \Sigma < 0\},$$
$$W_1^2 = \{(a,b,c_1,c_2,d_1,d_2) \in W_1 : \Sigma = 0\},$$
$$W_1^3 = \{(a,b,c_1,c_2,d_1,d_2) \in W_1 : \Sigma > 0\},$$

*where* $\Sigma = d_1\left[(b+d_1)(b+\frac{4ac_2}{c_1d_2+c_2d_1}(1+\frac{ad_1}{c_1d_2+c_2d_1})) - 4a(1+\frac{ad_1}{c_1d_2+c_2d_1})\right]$.
*The following two assertions hold.*

*(1) Points of $E_\pm$ are unstable when $(a,b,c_1,c_2,d_1,d_2) \in W_1^1 \cup W_2$ whereas points of $E_\pm$ are asymptotically stable when $(a,b,c_1,c_2,d_1,d_2) \in W_1^3$.*
*(2) While $(a,b,c_1,c_2,d_1,d_2) \in W_1^2$, system (3.1) simultaneously undergoes Hopf bifurcation at points of $E_\pm$.*

*Proof.*

(1) The proof of stability of points of $E_\pm$ easily follows from Routh-Hurwitz criterion and is omitted here.
(2) Assume $\lambda_1 = -(b_*+d_1) < 0$, $\lambda_{2,3} = \pm\omega i$ with

$$\omega = \sqrt{d_1\left[b_* + \frac{4ac_2}{c_1d_2+c_2d_1}\left(1+\frac{ad_1}{c_1d_2+c_2d_1}\right)\right]},$$

where $b_*$ satisfies

$$d_1\left[(b_*+d_1)\left(b_* + \frac{4ac_2}{c_1d_2+c_2d_1}\left(1+\frac{ad_1}{c_1d_2+c_2d_1}\right)\right)\right.$$
$$\left. -4a\left(1+\frac{ad_1}{c_1d_2+c_2d_1}\right)\right] = 0.$$

Calculating the derivatives on both sides of Eq. (4.1) with respect to $b$ and substituting $\lambda$ with $\omega i$ lead to

$$\left.\frac{dRe(\lambda_2)}{db}\right|_{b=b_*} = -\frac{\omega^2 + d_1(b_*+d_1)}{2\left[\omega^2 + (b_*+d_1)^2\right]} \neq 0,$$

which validates the condition of the transversality.

Consequently, Hopf bifurcation simultaneously happens at points of $E_\pm$, as shown in Fig. 6. The proof is completed.

□

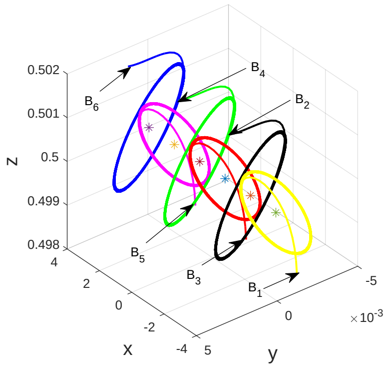



FIG. 6. Hopf bifurcation of system (3.1) when $(a,b,c_1,c_2,d_1,d_2) = (1,-1+\sqrt{2},-1,-1,1,1)$ and $(x_0^{4,5},y_0^{4,5},z_0^{4,5}) = (\pm\frac{\pi}{4},0,\frac{1}{2}) + (\pm 1.3,\pm 1.3,\pm 1.6)\times 10^{-3}$, $(x_0^{6,7},y_0^{6,7},z_0^{6,7}) = (\pm\frac{3\pi}{4},0,\frac{1}{2}) + (\pm 1.3,\pm 1.3,\pm 1.6)\times 10^{-3}, (x_0^{8,9},y_0^{8,9},z_0^{8,9}) = (\pm\frac{5\pi}{4},0,\frac{1}{2}) + (\pm 1.3,\pm 1.3,\pm 1.6)\times 10^{-3}$.

## 5. SINGULARLY DEGENERATE HETEROCLINIC CYCLES WITH NEARBY ISOLATED LORENZ-LIKE ATTRACTORS

Except for conjoined Lorenz-like attractors shown in Figs. 1-3, with suitable choices of parameters and initial conditions, this section detects singularly degenerate heteroclinic cycles and heteroclinic orbits, whose broken generates infinitely many isolated coexisting Lorenz-like attractors as the ones in [17,19]. Unfortunately, we are not able to give a rigorous proof of the existence of singularly degenerate heteroclinic cycles of system (3.1) by now. To achieve this target, we only resort to the numerical simulation. Aiming at coining some new phenomena, the following numerical results are illustrated.

**Case (1):** $d_1 = 0, 0.06$, $(a,b,c_1,c_2,d_2) = (2,0.5,-3,-1,1)$, $E_z^1 = (0,0,-3)$, $E_z^2 = (0,0,0.1)$, $E_z^3 = (0,0,0.4)$, $E_z^4 = (0,0,-2)$, $E_z^5 = (0,0,0)$.

**Numerical Result. 4.1:**

(1) For $d_1 = 0$, the 1D $W^u(E_z^{1,2,3})$ of each normally hyperbolic saddles $E_z^{1,2,3}$ tend toward one of the normally hyperbolic stable foci $(0,0,1.4133)$, $(0,0,2.8513)$ and $(0,0,1.2887)$ in the line $(0,0,z)$ as $t\to\infty$, which creates singularly degenerate heteroclinic cycles, while 1D $W^u(E_z^{4,5})$ of each normally hyperbolic saddles $E_z^{4,5}$ tend toward the normally hyperbolic stable foci $(\pm\pi,0,1.4344)$ and $(\pm\pi,0,2.4165)$ in the line $(\pm\pi,0,z)$ as $t\to\infty$, which forms heteroclinic orbits, as shown in Fig. 7(a).

(2) For $d_1 = 0.06$, the broken of singularly degenerate heteroclinic cycles consisting of $E_z^3$ and $(0,0,1.2887)$ , and heteroclinic orbits to $E_z^4$ and $(\pm\pi,0,1.4344)$ creates chaotic attractors that circle around the line $(0,0,z)$; the broken of singularly degenerate heteroclinic cycles consisting of $E_z^{1,2}$, and heteroclinic orbits to $E_z^5$ and $(\pm\pi,0,2.4165)$ creates chaotic attractors that circle around $(\pm\pi,0,z)$, as depicted in Fig. 7(b).

**Case (2):** $d_1 = 0, 0.06$, $b = 0.5, 0.738, 0.759$, $(a,c_1,c_2,d_2) = (2,-3,-1,1)$, $E_z^6 = (0,0,-1)$, $E_z^{7,8} = (\pm\pi,0,-1)$, $E_z^{9,10} = (\pm\pi,0,0)$, $E_z^{11,12} = (\pm\pi,0,-2)$, $E_z^{13} = (0,0,-2)$.

**Numerical Result. 4.2:**

(3) For $d_1 = 0$ and $b = 0.5$, the 1D $W^u(E_z^{6,7,8})$ of each normally hyperbolic saddles $E_z^{6,7,8}$ tend toward one of the normally hyperbolic stable foci $(0,0,2.2345)$ and $(\pm\pi,0,2.2345)$ as $t \to \infty$, which forms singularly degenerate heteroclinic cycles; while 1D $W^u(E_z^{9,10,11,12,13,5})$ of each normally hyperbolic saddles $E_z^{9,10,11,12,13,5}$ tend toward the normally hyperbolic stable foci $(\pm 2\pi,0,2.4165)$, $(0,0,2.4165)$, $(\pm 2\pi,0,\ 1.4344)$, $(0,0,1.4344)$, $(\pm\pi,0,1.4344)$ and $(\pm\pi,0,2.4165)$, as $t \to \infty$, which creates heteroclinic orbits, as illustrated in Fig. 8(a).

(4) For $d_1 = 0.06$, the broken of singularly degenerate heteroclinic cycles consisting of $E_z^{6,7,8}$ , and heteroclinic orbits to $E_z^{9,10}$ (resp. $E_z^5$) and $(\pm 2\pi,0,2.4165)$ and $(0,0,2.4165)$ (resp. $(\pm\pi,0,2.4165)$) creates chaotic attractors that circle around $(\pm 2\pi,0,z)$ and $(0,0,z)$ (resp. $(\pm\pi,0,z)$); the broken of heteroclinic orbits to $E_z^{11,12,13}$ generates chaotic attractors that circle around $(\pm\pi,0,z)$ and $(0,0,z)$, as depicted in Fig. 8(b).

**Numerical Result. 4.3:**

(5) For $d_1 = 0$ and $b = 0.738$, the 1D $W^u(E_z^{5,9,10})$ of each normally hyperbolic saddles $E_z^{5,9,10}$ tend toward normally hyperbolic stable foci $(0,0,$ $4.0264)$ and $(\pm\pi,0,\ 4.0264)$ as $t \to \infty$, which forms singularly degenerate heteroclinic cycles; while 1D $W^u(E_z^{6,7,8,11,12,13})$ of each normally hyperbolic saddles $E_z^{6,7,8,11,12,13}$ tends toward the normally hyperbolic stable foci $(\pm\pi,0,2.0143)$, $(\pm 2\pi,0,\ 2.0143)$, $(0,0,2.0143)$, $(\pm 2\pi,0,$ $2.0641)$, $(\pm\pi,0,2.0641)$ and $(0,0,2.0641)$ as $t \to \infty$, which generates heteroclinic orbits, as displayed in Fig. 9(a).

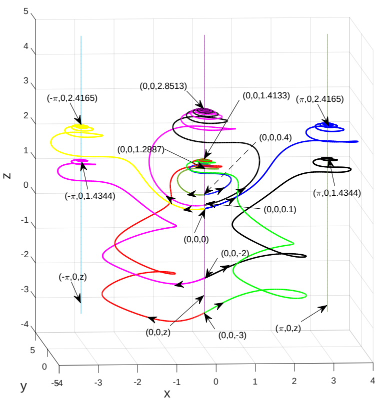
(6) For $d_1 = 0.06$, the broken of singularly degenerate heteroclinic cycles consisting of $E_z^{5,9,10}$ generates chaotic attractors that circle around $(\pm\pi,$ $0,z)$, $(\pm 2\pi,0,z)$ and $(0,0,z)$; the broken of heteroclinic orbits to $E_z^{6,7,8}$ (resp. $E_z^{11,12,13}$) creates chaotic attractors that circle around $(\pm\pi,0,z)$, $(\pm 2\pi,0,z)$ and $(0,0,z)$ (resp. $(\pm\pi,0,z)$ and $(0,0,z)$), as depicted in Fig. 9(b).

**Numerical Result. 4.4:**

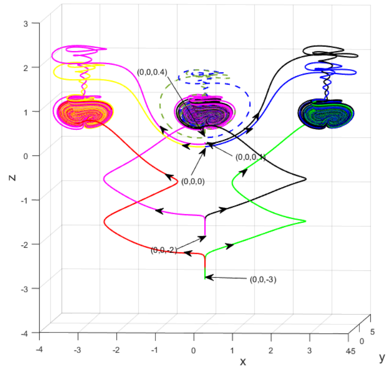
(7) For $d_1 = 0$ and $b = 0.759$, the 1D $W^u(E_z^{5,9,10})$ of each normally hyperbolic saddles $E_z^{5,9,10}$ tend toward normally hyperbolic stable foci $(0,0,$ $2.4165)$ and $(\pm\pi, 0,\ 2.4165)$ as $t \to \infty$, which forms singularly degenerate heteroclinic cycles, while 1D $W^u(E_z^{6,7,8,11,12,13})$ of each normally hyperbolic saddles $E_z^{6,7,8,11,12,13}$ tends toward the normally hyperbolic stable foci $(\pm\pi, 0, 2.2345)$, $(\pm 2\pi, 0, 2.2345)$, $(0,0,2.2345)$, $(\pm 2\pi, 0$ $1.4344)$, $(\pm\pi, 0, 1.4344)$ and $(0,0,1.4344)$ as $t \to \infty$, which generates heteroclinic orbits, as displayed in Fig. 10(a).

(8) For $d_1 = 0.06$, the broken of singularly degenerate heteroclinic cycles consisting of $E_z^{5,9,10}$ generates chaotic attractors that circle around $(\pm\pi,$ $0, z)$ and $(0,0,z)$; the broken of heteroclinic orbits to $E_z^{6,7,8}$ (respectively $E_z^{11,12,13}$) creates chaotic attractors that circle around $(\pm\pi, 0, z)$, $(\pm 2\pi, 0, z)$ and $(0,0,z)$ (resp. $(\pm\pi, 0, z)$ and $(0,0,z)$), as depicted in Fig. 10(b).

*Remark* 2. Numerical Result. 4.1-4.4 demonstrate that the creation of singularly degenerate heteroclinic cycles and heteroclinic orbits with nearby isolated Lorenz-like attractors depends not only the initial conditions but also parameters.

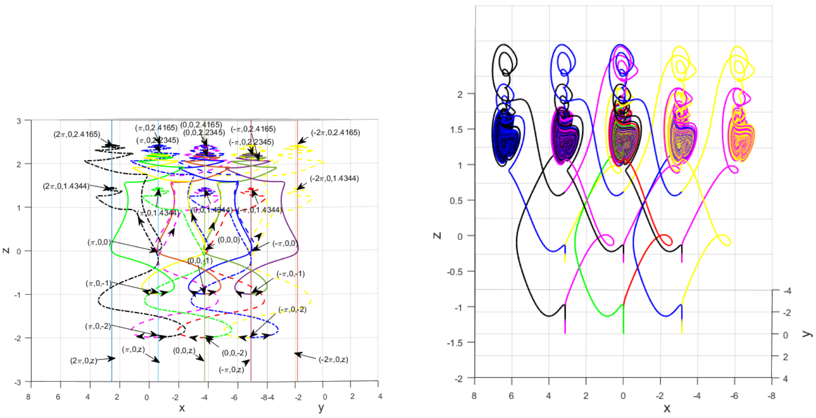


(a) $d_1 = 0$



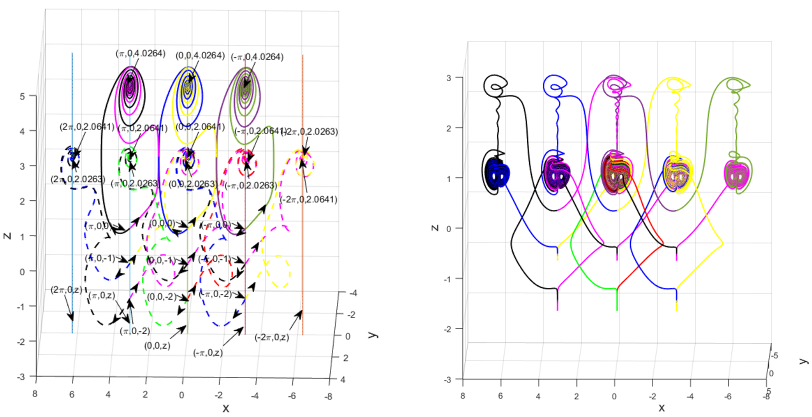
(b) $d_1 = 0.06$

FIG. 7. Coexistence of singularly degenerate heteroclinic cycles and heteroclinic orbits with nearby isolated strange attractors of system (3.1) when $(a,b,c_1,c_2,d_2) = (2, 0.5, -3, -1, 1)$ and initial conditions $E_z^1 = (0,0,-3)$, $E_z^2 = (0,0,0.1)$, $E_z^3 = (0,0,0.4)$, $E_z^4 = (0,0,-2)$, $E_z^5 = (0,0,0)$.



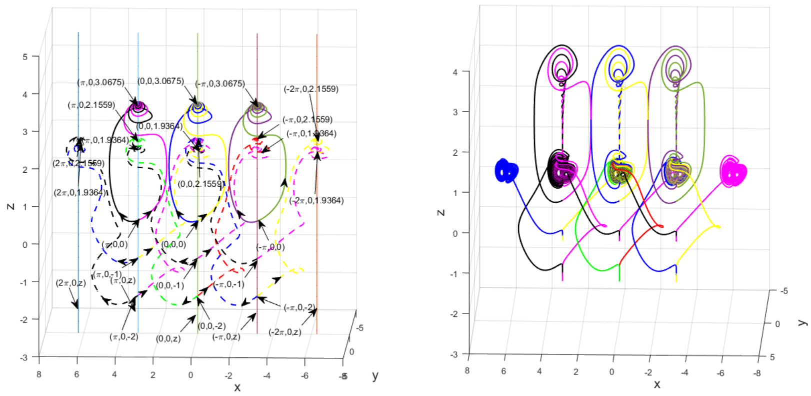
(a) $d_1 = 0$ (b) $d_1 = 0.06$

FIG. 8. Coexistence of singularly degenerate heteroclinic cycles and heteroclinic orbits with nearby isolated strange attractors of system (3.1) when $(a,b,c_1,c_2,d_2) = (2,0.5,-3,-1,1)$, $E_z^6 = (0,0,-1)$, $E_z^{7,8} = (\pm\pi,0,-1)$, $E_z^{9,10} = (\pm\pi,0,0)$, $E_z^{11,12} = (\pm\pi,0,-2)$, $E_z^{13} = (0,0,-2)$ and $E_z^5 = (0,0,0)$.



(a) $d_1 = 0$ (b) $d_1 = 0.06$

FIG. 9. Coexistence of singularly degenerate heteroclinic cycles with and heteroclinic orbits nearby isolated strange attractors of system (3.1) when $(a,b,c_1,c_2,d_2) = (2,0.738,-3,-1,1)$, $E_z^6 = (0,0,-1)$, $E_z^{7,8} = (\pm\pi,0,-1)$, $E_z^{9,10} = (\pm\pi,0,0)$, $E_z^{11,12} = (\pm\pi,0,-2)$, $E_z^{13} = (0,0,-2)$ and $E_z^5 = (0,0,0)$.



(a) $d_1 = 0$ (b) $d_1 = 0.06$

FIG. 10. Coexistence of singularly degenerate heteroclinic cycles and heteroclinic orbits with nearby isolated strange attractors of system (3.1) when $(a,b,c_1,c_2,d_2) = (2,0.759,-3,-1,1)$, $E_z^6 = (0,0,-1)$, $E_z^{7,8} = (\pm\pi,0,-1)$, $E_z^{9,10} = (\pm\pi,0,0)$, $E_z^{11,12} = (\pm\pi,0,-2)$, $E_z^{13} = (0,0,-2)$ and $E_z^5 = (0,0,0)$.

## 6. EXISTENCE OF HERTEROCLINIC ORBITS

In this section, we prove that there exists an infinite set of herteroclinic orbits in system (3.1) when $a > 0$, $b > 0$, $d_1 > 0$, $c_1d_2 > 0$, $c_1d_2 + c_2d_1 < 0$ and $0 < \frac{-ad_1}{c_1d_2+c_2d_1} \leq 1$, as [10, 11, 14–16].

To facilitate derivation, denote by $\phi_t(q_0) = (x(t;x_0), y(t;y_0), z(t;z_0))$ any one solution of system (3.1) with the initial point $q_0 = (x_0,y_0,z_0) \in \mathbb{R}^3$. Since in this case $W^u_{loc}$ is one-dimensional according to Proposition 3, we can denote $\gamma = \{\phi_t(q_0) | t \in \mathbb{R}\}$ be the unstable manifold of system (3.1) at points of $E_0$.

Firstly, set the first Lyapunov function

$$V(\phi_t(q_0)) = \frac{1}{2}\left[-\frac{d_1}{c_1d_2+c_2d_1}y^2 - \frac{c_1}{d_2(c_1d_2+c_2d_1)}(-d_1z + d_2\sin^2(x))^2 + \left(\frac{ad_1}{c_1d_2+c_2d_1} + \sin^2(x)\right)^2\right].$$

Then, the derivative of $V$ along the trajectories of system (3.1) is calculated as follows:

$$\frac{dV(\phi_t(q_0))}{dt}\Big|_{(3.1)} = \frac{c_1d_1}{d_2(c_1d_2+c_2d_1)}(-d_1z + d_2\sin^2(x))^2 + \frac{bd_1}{c_1d_2+c_2d_1}y^2. \tag{6.1}$$

First of all, one introduces the following result.

**Proposition 5.** *When $a > 0$, $b > 0$, $d_1 > 0$, $c_1 d_2 > 0$, $c_1 d_2 + c_2 d_1 < 0$ and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$, we derive the following two assertions.*

*(a) If $\exists\, t_{1,2}$, such that $t_1 < t_2$ and $V(\phi_{t_1}(q_0)) = V(\phi_{t_2}(q_0))$, then $q_0$ is one of the equilibrium points of system (3.1).*

*(b) If $\lim_{t\to-\infty} \phi_t(q_0) \in E_0$ and $q_0 \neq E_0$, then $V(E_0) > V(\phi_t(q_0))$ for all $t \in \mathbb{R}$.*

*Proof.*

(a) Since $\frac{dV(\phi_t(q_0))}{dt}\big|_{(3.1)} \leq 0$ when $a > 0$, $b > 0$, $d_1 > 0$, $c_1 d_2 > 0$, $c_1 d_2 + c_2 d_1 < 0$ and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$, it follows from Eq. (6.1) that $\frac{dV(\phi_t(q_0))}{dt}\big|_{(3.1)} = 0$, $\forall t \in (t_1, t_2)$, which thus suggests that $q_0$ is one of equilibria, i.e.,

$$\dot{x}(t; x_0) \equiv \dot{y}(t; y_0) \equiv \dot{z}(t; z_0) \equiv 0. \tag{6.2}$$

Actually, $\dot{x}(t; x_0) = y = 0$ implies $x(t) = x_0$ and $\dot{y}(t, y_0) = 0$, $y(t) = y_0 = 0$, $\forall t \in \mathbb{R}$.

In a word, $\phi_t(q_0) \in \{y = 0\} \cap \left\{-d_1 z + d_2 \sin^2 x = 0\right\}$ leads to (6.2).

(b) We first prove that $V(E_0) > V(\phi_t(q_0))$ for all $t \in \mathbb{R}$. Suppose that $V(E_0) \leq V(p(t; q_0))$ for some $t \in \mathbb{R}$. Then the above result (a) reads that $q_0$ is one of equilibria of system (3.1) and $q_0 \notin E_0$. This contradicts the fact that $\lim_{t\to-\infty} \phi_t(q_0) \in E_0$. Hence, it follows that $V(E_0) > V(\phi_t(q_0))$ for all $t \in \mathbb{R}$.

□

Using Proposition 5, the existence of heteroclinic orbits to points of $E_0$ and $E_\pm$ is derived in the following statement.

**Proposition 6.** *Consider $a > 0$, $b > 0$, $d_1 > 0$, $c_1 d_2 > 0$, $c_1 d_2 + c_2 d_1 < 0$ and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$. Then the following statements are true.*

*(i) Neither homoclinic orbits nor heteroclinic orbits to points of $E_+$ or $E_-$ exist in system (3.1).*

*(ii) System (3.1) has infinitely many heteroclinic orbits to points of $E_0$ and $E_\pm$.*

*Proof.*

(i) Let us first prove that there is no heteroclinic/homoclinic orbits to points of $E_+$ or $E_-$ in system (3.1) when $a > 0$, $b > 0$, $d_1 > 0$, $c_1 d_2 > 0$, $c_1 d_2 + c_2 d_1 < 0$ and $0 < \frac{-ad_1}{c_1 d_2 + c_2 d_1} \leq 1$.

Suppose that $p(t) = (x, y, z)$ (it belongs to the set of $\phi_t(q_0)$) is a homoclinic orbit of system (3.1) or a heteroclinic orbit to $E'_+$ or $E'_-$, where $\forall E'_+ \in E_+$ and $\forall E'_- \in E_-$. Namely, $p(t)$ is a solution of system (3.1) such that $\lim\limits_{t\to-\infty} p(t) = e_-$, $\lim\limits_{t\to+\infty} p(t) = e_+$, where points $e_-$ and $e_+$ satisfy either $e_- = e_+ \in \left\{E'_-, E'_0, E'_+\right\}$ or $\{e_-, e_+\} = \{E'_-, E'_+\}$ with $\forall E'_0 \in E_0$. From (6.1), one has

$$V(e_-) \geq V(p(t)) \geq V(e_+).$$

In either case, one only has the relation $V(e_-) = V(e_+)$, which suggests $V(p(t)) \equiv V(e_-)$. According to the assertion (a) of Proposition 5, $p(t)$ is just one of the equilibrium points of system (3.1). Therefore, neither homoclinic orbits nor heteroclinic orbits to points of $E_+$ and $E_-$ exist in system (3.1).

(ii) Next, let us show that $\gamma$ is any one heteroclinic orbit to $E_0^{'}$ and $E_+^{'}$, i.e., $\lim_{t\to+\infty} p(t) = E_+^{'}$. From the definition of $\gamma$ and the conclusion (ii) of Proposition 5 one can get $V(E_0^{'}) > V(p(t;q_0))$. This demonstrates that $p(t)$ does not approach $E_0^{'}$ as $t \to +\infty$. Therefore, $\lim_{t\to+\infty} p(t) = E_+^{'}$.

Lastly, we prove that, if system (3.1) has a heteroclinic orbit to $E_0^{'}$ and $E_+^{'}$, then this orbit belongs to $\gamma$.

Denote by $p^*(t) = (x^*(t), y^*(t), z^*(t))$ be a solution of system (3.1) such that

$$\lim_{t\to-\infty} p^*(t) = e_1^-, \quad \lim_{t\to+\infty} p^*(t) = e_1^+,$$

where $e_1^-$ and $e_1^+$ satisfy $\{e_1^-, e_1^+\} = \{E_0^{'}, E_+^{'}\}$. Like for (6.2), one obtains from (6.1) that for all $t \in \mathbb{R}$, $V(e_1^-) \geq V(p^*(t)) \geq V(e_1^+)$. Since $V(E_0^{'}) > V(E_+^{'})$, we have $e_1^- = E_0^{'}$ and $e_1^+ = E_+^{'}$, i.e.,

$$\lim_{t\to-\infty} p^*(t) = E_0^{'}, \quad \lim_{t\to+\infty} p^*(t) = E_+^{'},$$

which yields $p^*(t) \in \gamma$ based on the assertion (b) of Proposition 5. As the orbits are symmetrical w.r.t. the $z$–axis, there still exists a unique heteroclinic orbit $q^*(t) \in \gamma$ symmetrical to $p^*(t)$, which satisfies

$$\lim_{t\to-\infty} q^*(t) = E_0^{'}, \quad \lim_{t\to+\infty} q^*(t) = E_-^{'}.$$

As $E_0$ and $E_\pm$ contain infinitely many isolated stationary points $E_0^{'}$ and $E_\pm^{'}$, there exist infinitely many heteroclinic orbits to points of $E_0$ and $E_\pm$. This theoretical result is also verified via numerical simulation, as shown in Fig. 11. The proof is over. □

## ACKNOWLEDGEMENTS

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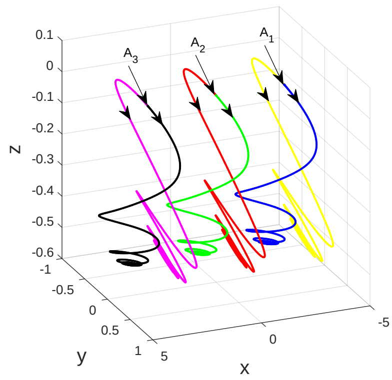


FIG. 11. Infinitely many pairs of heteroclinic orbits to points of $E_0$ and $E_\pm$ of system (3.1) $(a,b,c_1,c_2,d_1,d_2) = (3,0.1,-3,-9,1,-1)$ and $(A_1)$ $(x_0^{1,2},y_0^{1,2},z_0^{1,2}) = (-\pi,0,0)+(\pm1.3,\pm13,\pm16)\times10^{-8}$, $(A_2)$ $(x_0^{3,4},y_0^{3,4},z_0^{3,4}) = (\pm1.3,\pm13,\pm16)\times10^{-8}$, $(A_3)$ $(x_0^{5,6},y_0^{5,6},z_0^{5,6}) = (\pi,0,0)+(\pm1.3,\pm13,\pm16)\times10^{-8}$.

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