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AN INVERSE TIME-DEPENDENT SOURCE PROBLEM FOR A TIME-FRACTIONAL DIFFUSION EQUATION WITH NONLOCAL BOUNDARY CONDITIONS

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Abstract. In this paper, we investigate an inverse time-dependent source problem for a time-fractional diffusion equation with nonlocal boundary and integral over-determination conditions. The fractional derivative is described in the generalized Caputo sense. The nonlocal boundary conditions are regular but not strongly regular. The special thing about this problem is: the system of eigenfunctions is not complete, but the system of eigen-and associated functions forming a basis in $L^2(0,1)$. Under some natural regularity and consistency conditions on the input data the existence, uniqueness and continuously dependence upon the data of the solution are shown by using the generalized Fourier method, the estimates of Mittag-Leffler function and Banach's contraction mapping principle.

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1. INTRODUCTION

Fractional calculus (FC) is the study of integrals and derivatives of arbitrary order, which has been considered by many researchers in recent years. The arbitrary order integrals and derivatives are used in the modelling of many physical, chemical, and biological phenomena, [2, 5, 8, 20, 24, 28]. The FC has applications in the reaction-diffusion equations [18, 27], and is used to explain the well-known phenomena of anomalous diffusion observed in experiments, [4, 18]. The time fractional diffusion equations are obtained by replacing the standard time derivative with fractional derivative in time variable to explain the sub-diffusion or super-diffusion, [19, 21].

In this paper, we are interested with the following time-fractional diffusion equation

$$ {}^{c}\mathcal{D}_t^{\alpha,\rho} u(x,t) = u_{xx} + F(x,t), \quad (x,t) \in \Omega_T, \tag{1.1}$$

with the initial condition

$$u(x,0)=\varphi(x), \quad 0<x<1, \tag{1.2}$$

and family of nonlocal boundary conditions

$$u(0,t)=u(1,t), \quad \beta u_x(0,t)=u_x(1,t), \quad 0<t\leq T, \quad \beta\in\mathbb{R}\backslash\{-1,1\}, \tag{1.3}$$

where $\Omega_T:=\{(x,t):0<x<1,\ 0<t\leq T\}$ for some fixed $T>0$, ${}^c\mathcal{D}_t^{\alpha,\rho}$ stands for left-sided generalized Caputo fractional derivative of order $0<\alpha\leq 1$, $\rho>0$ is a real constant, $F(x,t)$ is the source term and $\varphi(x)$ is the initial data. The fractional derivative in (1.1) is a generalization of left Caputo and Caputo-Hadamard fractional derivatives, which can be obtained by taking $\rho=1$ and $\rho\to 0^+$, respectively. For $\beta=0$, the boundary conditions (1.3) are well-known and called in literature as Samarskii-Ionkin conditions, [10]. More general boundary conditions of the type (1.3) have been considered in [7, 12, 22, 25, 29].

The determination of a function $u(x,t)$ which satisfies the initial boundary value problem (IBVP) (1.1)-(1.3) such that $u(\cdot,t)\in\mathcal{C}^2(0,1)$ and ${}^c\mathcal{D}_t^{\alpha,\rho}u(x,\cdot)\in\mathcal{C}(0,T)$, whenever the source term $F(x,t)$ and the initial data $\varphi(x)$ are given and continuous, is called strong or classical solution of the IBVP (1.1)-(1.3). This problem is usually known as the direct problem.

Letting the source term have the form $F(x,t)=r(t)f(x,t)$, the inverse source problem consists of determining $u(x,t)$ and $r(t)$, from the initial data $\varphi(x)$, the source term $f(x,t)$ and nonlocal boundary conditions (1.3). This problem is not uniquely solvable. To have the inverse source problem uniquely solvable, we impose the integral over-determination condition

$$\int_0^1 u(x,t)\,dx=g(t), \quad t\in[0,T], \tag{1.4}$$

where $g\in AC[0,T]$ (the space of absolutely continuous functions). The solvability of inverse source problems with such condition has been considered earlier [3, 11, 26].

In [26] the inverse source problem (1.1)-(1.4) is studied for $\alpha=\rho=1$. The case $0<\alpha<1$ and $\rho\neq 1$ is considered for the first time in this paper. We have analysed the inverse source problem (1.1)-(1.4). Our strategy is mainly based on Fourier's method for construction of the series solution using a bi-orthogonal system of functions obtained from the eigen-and associated functions of a spectral problem and its conjugate problem, see [16]. We provide existence, uniqueness, and stability results for solution of inverse source problem.

The rest of the paper is organized as follows: in Section 2, we provide some preliminaries and basic result needed for the forthcoming sections. In Section 3, we present associated spectral problem of this problem and its properties, whilst the bi-orthogonal system is constructed. Our main results concerning the existence, uniqueness and continuous dependence of the solution of the inverse problem is present in Section 4.

2. Preliminaries and some basic results

In this section, we recall some definitions, notations from fractional calculus, and some basic results for the convenience of the readers.

Definition 2.1 ([15, page 861])**.** Let $[a,b]$ be a finite interval and $f\colon [a,b] \to \mathbb{R}$ be an integrable function. The generalized left fractional integral (in the sense of Katugampola) is defined by

$$\left(I_a^{\alpha,\rho}\right) f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} f(s)\, \frac{ds}{s^{1-\rho}}, \quad 0<\alpha<1, \quad \rho>0,$$

where $\Gamma(\cdot)$ is the Euler Gamma function defined by $\Gamma(\alpha) := \int_0^{+\infty} t^{\alpha-1} e^{-t}\, dt$.

Definition 2.2 ([17, 1.1.5])**.** Let $[a,b]$ be a finite interval. Then, $AC[a,b]$ is the space of absolute continuous functions on $[a,b]$, defined by

$$AC[a,b] = \left\{ f\colon\ [a,b] \to \mathbb{R} \text{ such that } f(x) = c + \int_a^x \varphi(t)\, dt,\, \varphi \in L^1(a,b) \right\}.$$

Definition 2.3 ([14])**.** Let $\rho > 0$ and $f \in AC[a,b]$. The left generalized Caputo fractional derivative of f of order $0<\alpha<1$ is defined by

$${}_a^C\mathcal{D}^{\alpha,\rho} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{-\alpha} f'(s)\, ds.$$

If $\alpha = \rho = 1$, then ${}_a^c\mathcal{D}^{\alpha,\rho} f(t) = f'(t)$.

Theorem 2.1 ([14])**.** *Let $f \in AC[a,b]$, $0<\alpha<1$ and $\rho>0$. Then, we have:*

$$I_a^{\alpha,\rho}\left({}_a^C\mathcal{D}^{\alpha,\rho} f(x)\right) = f(x) - f(a).$$

Definition 2.4 ([1])**.** Let $f\colon\ [0,+\infty[\ \to \mathbb{R}$ be a real valued function. The ρ-Laplace transform of f is defined by

$$\mathcal{L}_\rho\{f(t)\}(s) = \int_0^{+\infty} e^{-s\frac{t^\rho}{\rho}} f(t)\, \frac{dt}{t^{1-\rho}}, \quad \rho>0,$$

for all values of s, the integral is valid.

Theorem 2.2 ([1])**.** *If the ρ-Laplace transform of $f\colon\ [0,+\infty[\ \to \mathbb{R}$ exists for $s>c_1$ and the ρ-Laplace transform of $g\colon\ [0,+\infty[\ \to \mathbb{R}$ for $s>c_2$. Then, for any constants a and b, the ρ-Laplace transform of $af+bg$ exists and*

$$\mathcal{L}_\rho\{af(t)+bg(t)\}(s) = a\mathcal{L}_\rho\{f(t)\}(s) + b\mathcal{L}_\rho\{g(t)\}(s), \quad \textit{for } s > \max\{c_1,c_2\}.$$

Definition 2.5 ([13])**.** Let f and g be two functions which are piecewise continuous at each interval $[0,T]$. We define the ρ-convolution of f and g by

$$(f*g)(t) = \int_0^t f\left[\left(t^\rho - s^\rho\right)^{1/\rho}\right] g(s)\, \frac{ds}{s^{1-\rho}}.$$

Theorem 2.3 ([13])**.** *Let f and g be two functions which are piecewise continuous at each interval $[0,T]$. Then,*

$$\mathcal{L}_\rho\left\{(f*g)(t)\right\}=\mathcal{L}_\rho\left\{f(t)\right\}\mathcal{L}_\rho\left\{g(t)\right\}.$$

Theorem 2.4 ([13])**.** *Let $\alpha>0$ and $f\in AC[0,T]$. Then,*

$$\mathcal{L}_\rho\left\{\left({}_0^C\mathcal{D}^{\alpha,\rho}f\right)(t)\right\}(s)=s^\alpha\mathcal{L}_\rho\left\{f(t)\right\}-s^{\alpha-1}f(0).$$

Definition 2.6 ([6])**.** The Mittag-Leffler function of two parameters is defined as

$$E_{\xi,\eta}(x):=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\xi k+\eta)},\quad z\in\mathbb{C},\quad \operatorname{Re}(\xi)>0,\quad \operatorname{Re}(\eta)>0.$$

For $\eta=1$, the Mittag-Leffler function is reduced to classical one parameter Mittag-Leffler function, that is,

$$E_{\xi,1}(x):=E_\xi(x)=\sum_{k=0}^{+\infty}\frac{z^k}{\Gamma(\xi k+1)}.$$

Let $e_\xi(t,\mu):=E_\xi\left(-\mu t^\xi\right)$ and $e_{\xi,\eta}(t,\mu):=t^{\eta-1}E_{\xi,\eta}\left(-\mu t^\xi\right)$, where μ is a positive real number. The Mittag-Leffler functions $e_\xi(t,\mu)$, $e_{\xi,\eta}(t,\mu)$ for $0<\xi\leq 1$, $0<\xi\leq\eta\leq 1$, respectively, are completely monotone functions, i.e.

$$(-1)^n\frac{\partial^n}{\partial t^n}\left[e_\xi(t,\mu)\right]\geq 0\ \text{ and }\ (-1)^n\frac{\partial^n}{\partial t^n}\left[e_{\xi,\eta}(t,\mu)\right]\geq 0,\quad n\in\mathbb{N}.$$

Using Theorem 1.6 in [23], we can have the following estimate

$$\left|\mu e_{\xi,\xi}(t,\mu)\right|\leq\frac{N\mu t^\xi}{t\left(1+\mu t^\xi\right)}\leq\frac{N}{t}\leq C,\quad t\in\,]\varepsilon,T\,],\tag{2.1}$$

where $\varepsilon>0$, N and C are some constants.

Lemma 2.1 ([13])**.** *Let $\xi>0$ and $\left|\frac{\lambda}{s^\xi}\right|<1$. Then, we have:*

$$\mathcal{L}_\rho\left\{e_\xi\left(\frac{t^\rho}{\rho},\lambda\right)\right\}=\frac{s^{\alpha-1}}{s^\alpha+\lambda}\text{ and }\mathcal{L}_\rho\left\{e_{\xi,\xi}\left(\frac{t^\rho}{\rho},\lambda\right)\right\}=\frac{1}{s^\alpha+\lambda}.$$

Theorem 2.5 ([13])**.** *The Cauchy problem*

$$\begin{cases}{}_0^C\mathcal{D}^{\alpha,\rho}y(t)+\lambda y(t)=f(t), & t>0,\quad 0<\alpha<1,\quad \rho>0,\quad \lambda\in\mathbb{R},\\ y(0)=y_0, & y_0\in\mathbb{R},\end{cases}$$

has the solution

$$y(t)=y_0e_\alpha\left(\frac{t^\rho}{\rho},\lambda\right)+\int_0^t e_{\alpha,\alpha}\left(\frac{t^\rho-s^\rho}{\rho},\lambda\right)f(s)\frac{ds}{s^{1-\rho}}.$$

3. The auxiliary spectral problem and bi-orthogonal systems

Applying the Fourier method for solving the direct problem (1.1)-(1.3) leads to the spectral problem

$$-X'' = \lambda X, \quad 0 < x < 1, \tag{3.1}$$

$$X(0) = X(1), \quad \beta X'(0) = X'(1), \quad \beta \in \mathbb{R}\backslash\{-1,1\}. \tag{3.2}$$

According to [25, Theorem 3.109], the boundary conditions (3.2) are regular but not strongly regular. Obviously, the boundary-value problem (3.1)-(3.2) is not self-adjoint. But the problem

$$-Y'' = \lambda Y, \quad 0 < x < 1, \tag{3.3}$$

$$Y(0) = \beta Y(1), \quad Y'(0) = Y'(1), \quad \beta \in \mathbb{R}\backslash\{-1,1\}, \tag{3.4}$$

will be a conjugated problem.

The two spectral problems (3.1)-(3.2) and (3.3)-(3.4) have the same double eigenvalues $\lambda_k = (2\pi k)^2$ (except for the first $\lambda_0 = 0$). Following the idea presented in [9], the system of eigen-and associated functions of problem (3.1)-(3.2) is given by:

$$X_0(x) = 2, \quad X_{2k-1}(x) = 4\cos(2\pi kx), \quad X_{2k}(x) = 4(1-b-ax)\sin(2\pi kx) \tag{3.5}$$

and the system of eigen-and associated functions of conjugate problem (3.3)-(3.4) is the following:

$$Y_0(x) = ax+b, \quad Y_{2k-1}(x) = (ax+b)\cos(2\pi kx), \quad Y_{2k}(x) = \sin(2\pi kx), \tag{3.6}$$

where $a = (1-\beta)/(1+\beta)$ and $b = \beta/(1+\beta)$.

Lemma 3.1. *The systems of functions* (3.5) *and* (3.6) *are bi-orthonormal in* $L^2(0,1)$.

Proof. It is easy to show that the systems (3.5) and (3.6) form a bi-orthogonal system on $[0,1]$, i.e. $\langle X_i, Y_i \rangle = 1$ and $\langle X_i, Y_j \rangle = 0, i \neq j$. □

Lemma 3.2. *The systems of functions* (3.5) *and* (3.6) *are complete in* $L^2(0,1)$.

Proof. Let $f \in L^2(0,1)$ be orthogonal with the system of functions (3.5). $f(x)$ can be presented by the series

$$f(x) = \sum_{n=1}^{+\infty} B_n \sin(2\pi nx), \tag{3.7}$$

which converges in $L^2(0,1)$. Since $f(x)$ is orthogonal with (3.5), we have

$$\begin{aligned} 0 &= \int_0^1 4(1-b-ax) f(x) \sin(2\pi kx)\, dx \\ &= \sum_{n=1}^{+\infty} B_n \int_0^1 4(1-b-ax)\sin(2\pi kx)\sin(2\pi nx)\, dx = B_k, \quad k \in \mathbb{N}^*. \end{aligned}$$

From (3.7), $f(x)=0$. Then, (3.5) is complete in $L^2(0,1)$. □

Theorem 3.1. *The system of functions (3.5) forms a Riesz basis in* $L^2(0,1)$.

Proof. From [25, page 211], the system (3.5) is a Riesz basis in $L^2(0,1)$ if there exist two constants $m,M>0$ such that for any $f\in L^2(0,1)$, the following inequality holds:

$$m\|f\|^2_{L^2(0,1)} \le \sum_{i=0}^{+\infty} f_i^2 \le M\|f\|^2_{L^2(0,1)},$$

where

$$f_i=\langle f,Y_i\rangle=\int_0^1 f(x)Y_i(x)\,dx \text{ and } \bar{f}_i=\langle f,X_i\rangle=\int_0^1 f(x)X_i(x)\,dx.$$

For $i=0$, and using the Cauchy-Schwarz inequality we have

$$\begin{aligned} f_0^2=\langle f,Y_0\rangle^2=\left[\int_0^1 Y_0(x)f(x)\,dx\right]^2 &\le \int_0^1 Y_0^2(x)\,dx\int_0^1 f^2(x)\,dx \\ &\le \frac{1+\beta+\beta^2}{3(1+\beta)^2}\|f\|^2_{L^2(0,1)}. \end{aligned} \tag{3.8}$$

For $i=2k-1$, and using the Bessel inequality we obtain:

$$\begin{aligned} \sum_{k=1}^{+\infty} f_{2k-1}^2=\sum_{k=1}^{+\infty}\langle f,Y_{2k-1}\rangle^2 &\le \|Y_{2k-1}\|^2_{L^2(0,1)}\|f\|^2_{L^2(0,1)} \\ &\le \frac{7-11\beta+7\beta^2}{6(1+\beta)^2}\|f\|^2_{L^2(0,1)}. \end{aligned} \tag{3.9}$$

For $i=2k$, and using the Bessel inequality we obtain:

$$\sum_{k=1}^{+\infty} f_{2k}^2=\sum_{k=1}^{+\infty}\langle f,Y_{2k}\rangle^2 \le \|Y_{2k}\|^2_{L^2(0,1)}\|f\|^2_{L^2(0,1)} \le \frac{1}{2}\|f\|^2_{L^2(0,1)}. \tag{3.10}$$

From (3.8)-(3.10), we have

$$\sum_{i=0}^{+\infty} f_i^2=f_0^2+\sum_{k=1}^{+\infty} f_{2k-1}^2+\sum_{k=1}^{+\infty} f_{2k}^2 \le M\|f\|^2_{L^2(0,1)}, \tag{3.11}$$

where $M=\frac{4-\beta+4\beta^2}{2(1+\beta)^2}$.

On the other hand we have:

$$\bar{f}_0^2=\langle f,X_0\rangle^2=\left[\int_0^1 X_0(x)f(x)\,dx\right]^2 \le 4\|f\|^2_{L^2(0,1)}. \tag{3.12}$$

Using the Bessel inequality, we obtain:

$$\sum_{k=1}^{+\infty}\bar{f}_{2k-1}^2=\sum_{k=1}^{+\infty}\langle f,X_{2k-1}\rangle^2 \le 8\|f\|^2_{L^2(0,1)}, \tag{3.13}$$

$$\sum_{k=1}^{+\infty} \bar{f}_{2k}^2 = \sum_{k=1}^{+\infty} \langle f, X_{2k} \rangle^2 \leq \frac{8\left(1+\beta+\beta^2\right)}{3\left(1+\beta\right)^2} \|f\|_{L^2(0,1)}^2 . \tag{3.14}$$

Then, from (3.12)-(3.14) we have:

$$\sum_{i=0}^{+\infty} \bar{f}_i^2 \leq \frac{44+80\beta+44\beta^2}{3\left(1+\beta\right)^2} \|f\|_{L^2(0,1)}^2 . \tag{3.15}$$

Using the Cauchy-Schwarz inequality and (3.15), we get

$$\begin{aligned} \|f\|_{L^2(0,1)}^2 = \langle f, f \rangle &= \sum_{i=0}^{+\infty} \bar{f}_i f_i \\ &\leq \left[\sum_{i=0}^{+\infty} \bar{f}_i^2\right]^{1/2} \left[\sum_{i=0}^{+\infty} f_i^2\right]^{1/2} \\ &\leq \left[\frac{44+80\beta+44\beta^2}{3\left(1+\beta\right)^2}\right]^{1/2} \|f\|_{L^2(0,1)} \left[\sum_{i=0}^{+\infty} f_i^2\right]^{1/2} . \end{aligned}$$

Consequently, we have:

$$m\|f\|_{L^2(0,1)}^2 \leq \sum_{i=0}^{+\infty} f_i^2, \quad m = \frac{3\left(1+\beta\right)^2}{44+80\beta+44\beta^2}. \tag{3.16}$$

From (3.11) and (3.16), the system (3.5) is a Riesz basis in $L^2(0,1)$. □

Corollary 3.1. *From Lemme 3.1 and Theorem 3.1, the systems (3.5) and (3.6) are equivalent bases in $L^2(0,1)$.*

4. Main results

4.1. *Existence and uniqueness of the solution*

In this subsection, we give the main result on existence and uniqueness of the solution of the inverse problem (1.1)-(1.4) is presented as follows.

Theorem 4.1. *Let the following assumptions be satisfied*

(A1) $\varphi \in C^4(0,1)$, $\varphi(1) = \varphi(0)$, $\varphi'(1) = \beta\varphi'(0)$, $\varphi''(1) = \varphi''(0)$, $\varphi'''(1) = \beta \times\varphi'''(0)$;

(A2) $f(x,\cdot) \in C[0,T]$ *and for* $t \in [0,T]$, $f(\cdot,t) \in C^4[0,1]$;

$$f(0,t) = f(1,t), \qquad f_x(1,t) = \beta f_x(0,t), \qquad f_{xx}(0,t) = f_{xx}(1,t),$$

$$f_{xxx}(1,t) = \beta f_{xxx}(0,t), \qquad \int_0^1 f(x,t)\,dx \neq 0$$

and there exists a constant $M > 0$ *such that*

$$0 < \left|\int_0^1 f(x,t)\,dx\right|^{-1} \leq M;$$

(A3) *$g \in C^1(0,T)$, and g satisfies the consistency condition $\int_0^1 \varphi(x)\,dx = g(0)$.*
If the following condition

$$T < \left(\frac{\alpha |1+\beta| \rho^{\alpha}}{MC' |1-\beta|} \right)^{1/\rho\alpha}, \tag{4.1}$$

where C' is defined in (4.16), then the inverse problem (1.1)-(1.4) has a unique solution.

Proof. According to assumptions (A1)-(A3), there are positive constants, L_1, L_2, M_i, $i = 0, \ldots, 2$, such that

$$L_1 := \max_{0\le t\le T} e_{\alpha}\left(\frac{t^{\rho}}{\rho}, \lambda_k \right), \quad L_2 := \max_{0\le s\le t\le T} E_{\alpha,\alpha}\left[-\lambda_k \left(\frac{t^{\rho}-s^{\rho}}{\rho} \right)^{\alpha} \right],$$
$$M_0 := \|r\|_{C(0,T)}. \quad M_1 := \max\left(\|f_0\|_{C(0,T)}, \left\| f_{2k-1}^{(4)} \right\|_{C(0,T)}, \left\| f_{2k}^{(4)} \right\|_{C(0,T)} \right),$$
$$M_2 := \max\left(|\varphi_0|, \left|\varphi_{2k-1}^{(4)}\right|, \left|\varphi_{2k}^{(4)}\right| \right).$$

The proof of this theorem takes place in three steps:

Step 1: Construction of solution: By applying the Fourier's method, the solution $u(x,t)$ of the direct problem (1.1)-(1.3), can be developed in uniformly convergent series form using the eigenfunctions (3.5) in $L^2(0,1)$ as follows

$$u(x,t) = 2u_0(t) + \sum_{k=1}^{+\infty} u_{2k-1}(t) X_{2k-1}(x) + \sum_{k=1}^{+\infty} u_{2k}(t) X_{2k}(x), \tag{4.2}$$

We define the coefficients $u_0(t)$, $u_{2k-1}(t)$ and $u_{2k}(t)$ for $k \in \mathbb{N}^*$ by multiplying (4.2) by the eigenfunctions of (3.6) and integrating over $[0,1]$ and using Lemma 3.1, we get

$$\begin{aligned} u_0(t) &= \langle u(x,t), Y_0(x) \rangle, \quad u_{2k-1}(t) = \langle u(x,t), Y_{2k-1}(x) \rangle, \\ u_{2k}(t) &= \langle u(x,t), Y_{2k}(x) \rangle, \end{aligned} \tag{4.3}$$

where $\langle \cdot, \cdot \rangle$ represents the inner product in $L^2(0,1)$.

The expansion coefficients of the functions $f(x,t)$ and $\varphi(x)$ into eigenfunctions (3.6) are given by

$$\begin{aligned} f_0(t) &= \langle f(x,t), Y_0(x) \rangle, \quad f_{2k-1}(t) = \langle f(x,t), Y_{2k-1}(x) \rangle, \\ f_{2k}(t) &= \langle f(x,t), Y_{2k}(x) \rangle, \end{aligned} \tag{4.4}$$

and

$$\begin{aligned} \varphi_0 &= \langle \varphi(x), Y_0(x) \rangle, \quad \varphi_{2k-1} = \langle \varphi(x), Y_{2k-1}(x) \rangle, \\ \varphi_{2k} &= \langle \varphi(x), Y_{2k}(x) \rangle. \end{aligned} \tag{4.5}$$

From (1.1), (4.3)-(4.5), Lemma 3.1, integration by parts twice and (1.3), we obtain

$$\begin{cases} {}^{c}\mathcal{D}_{t}^{\alpha,\rho}u_{0}(t)=r(t)f_{0}(t), \\ u_{0}(0)=\varphi_{0}, \end{cases} \tag{4.6}$$

$$\begin{cases} {}^{c}\mathcal{D}_{t}^{\alpha,\rho}u_{2k}(t)+\lambda_{k}u_{2k}(t)=r(t)f_{2k}(t), \\ u_{2k}(0)=\varphi_{2k}, \end{cases} \tag{4.7}$$

$$\begin{cases} {}^{c}\mathcal{D}_{t}^{\alpha,\rho}u_{2k-1}(t)+\lambda_{k}u_{2k-1}(t)=-4\pi aku_{2k}(t)+r(t)f_{2k-1}(t), \\ u_{2k-1}(0)=\varphi_{2k-1}. \end{cases} \tag{4.8}$$

Applying $I_{0}^{\alpha,\rho}$ on (4.6) and using Theorem 2.1, we obtain

$$u_{0}(t)=\varphi_{0}+\frac{1}{\Gamma(\alpha)}\int_{0}^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}r(s)f_{0}(s)\frac{ds}{s^{1-\rho}}. \tag{4.9}$$

Applying Theorem 2.5 on (4.7) and (4.8) , we obtain:

$$u_{2k}(t)=\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)+\int_{0}^{t}e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_{k}\right)r(s)f_{2k}(s)\frac{ds}{s^{1-\rho}}, \tag{4.10}$$

and

$$\begin{aligned} u_{2k-1}(t)=\varphi_{2k-1}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)&+\int_{0}^{t}e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_{k}\right)r(s)f_{2k-1}(s)\frac{ds}{s^{1-\rho}} \\ &-4\pi ak\int_{0}^{t}e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_{k}\right)u_{2k}(s)\frac{ds}{s^{1-\rho}}. \end{aligned} \tag{4.11}$$

After substituting expressions $u_{0}(t)$, $u_{2k}(t)$, and $u_{2k-1}(t)$, respectively described by (4.9), (4.10), and (4.11), into (4.2), we have:

$$\begin{aligned} u(x,t)=2\varphi_{0}&+\frac{2}{\Gamma(\alpha)}\int_{0}^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}r(s)f_{0}(s)\frac{ds}{s^{1-\rho}} \\ &+\sum_{k=1}^{+\infty}\left\{\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)+\int_{0}^{t}e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_{k}\right)r(s)f_{2k}(s)\frac{ds}{s^{1-\rho}}\right\}X_{2k}(x) \\ &+\sum_{k=1}^{+\infty}\left\{\varphi_{2k-1}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_{k}\right)+\int_{0}^{t}e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_{k}\right)r(s)f_{2k-1}(s)\frac{ds}{s^{1-\rho}}\right. \\ &\left.-4\pi ak\int_{0}^{t}e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_{k}\right)u_{2k}(s)\frac{ds}{s^{1-\rho}}\right\}X_{2k-1}(x), \end{aligned}$$

Taking the generalized Caputo fractional derivative ${}^{c}\mathcal{D}_{t}^{\alpha,\rho}$ of the over-determination condition (1.4), and integrating the equation (1.1) on $[0,1]$ and using (1.3), we obtain

$$r(t)=\frac{{}^{c}\mathcal{D}_{t}^{\alpha,\rho}g(t)+(1-\beta)u_{x}(0,t)}{\int_0^1 f(x,t)\,dx}\text{ where }\int_0^1 f(x,t)\,dx=2f_0(t)+\frac{2a}{\pi}\sum_{k=1}^{+\infty}\frac{f_{2k}(t)}{k}$$

and

$$u_x(0,t)=\sum_{k=1}^{+\infty}8\pi k(1-b)\left(\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_k\right)+\int_0^t e_{\alpha,\alpha}\left(\frac{t^{\rho}-s^{\rho}}{\rho},\lambda_k\right)r(s)f_{2k}(s)\frac{ds}{s^{1-\rho}}\right).$$

Hence, we get following implicit representation of $r(t)$

$$r(t)=\eta(t)+\left[2f_0(t)+\frac{2a}{\pi}\sum_{k=1}^{+\infty}\frac{f_{2k}(t)}{k}\right]^{-1}\int_0^t K(t,s)\,r(s)\frac{ds}{s^{1-\rho}},\tag{4.12}$$

where

$$\eta(t)=\frac{{}^{c}\mathcal{D}_{t}^{\alpha,\rho}g(t)+a\sum_{k=1}^{+\infty}8\pi k\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_k\right)}{\int_0^1 f(x,t)\,dx},\tag{4.13}$$

and

$$K(t,s)=a\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}\sum_{k=1}^{+\infty}8\pi k f_{2k}(s)E_{\alpha,\alpha}\left[-\lambda_k\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right].$$

Step 2: Existence of the solution: We consider the following map:

$$\mathcal{P}(r(t)):=\eta(t)+\left[2f_0(t)+\frac{2a}{\pi}\sum_{k=1}^{+\infty}\frac{f_{2k}(t)}{k}\right]^{-1}\int_0^t K(t,s)\,r(s)\frac{ds}{s^{1-\rho}}.$$

on the space $\mathcal{C}[0,T]$ with $\|\phi\|:=\max_{0\le t\le T}|\phi(t)|$. To show $\mathcal{P}$ is well defined. Since, under the assumptions (A1), (A2) and integration by parts four times, for $t,s\in[0,T]$, we obtain

$$\sum_{k=1}^{+\infty}8\pi k\varphi_{2k}e_{\alpha}\left(\frac{t^{\rho}}{\rho},\lambda_k\right)\le\sum_{k=1}^{+\infty}\frac{L_1\left|\varphi_{2k}^{(4)}\right|}{2\pi^3k^3},\tag{4.14}$$

$$\sum_{k=1}^{+\infty}8\pi k f_{2k}(s)E_{\alpha,\alpha}\left[-\lambda_k\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha}\right]\le\sum_{k=1}^{+\infty}\frac{L_2\left|f_{2k}^{(4)}(s)\right|}{2\pi^3k^3},\tag{4.15}$$

where $\varphi_{2k}^{(4)}=\int_0^1\varphi^{(4)}(x)\sin(2\pi kx)\,dx$, $f_{2k}^{(4)}(t)=\int_0^1\frac{\partial^4 f(x,t)}{\partial x^4}\sin(2\pi kx)\,dx$.

Using the Cauchy-Schwarz and Bessel inequalities, we obtain

$$\sum_{k=1}^{+\infty}\frac{L_2\left|f_{2k}^{(4)}(s)\right|}{2\pi^3k^3}\leq\left[\sum_{k=1}^{+\infty}\frac{L_2^2}{4\pi^6k^6}\right]^{1/2}\left[\sum_{k=1}^{+\infty}\left(f_{2k}^{(4)}(s)\right)^2\right]^{1/2}\leq c\left\|\frac{\partial^4 f(x,t)}{\partial x^4}\right\|_{L^2(0,1)},$$

where c is a constant independent of t and k. Thus, we have

$$\begin{aligned}\sum_{k=1}^{+\infty}8\pi k f_{2k}(s)E_{\alpha,\alpha}\left[-\lambda_k\left(\frac{t^\rho-s^\rho}{\rho}\right)^\alpha\right]\leq C',\\ C'=c\max_{0\leq t\leq T}\left\|\frac{\partial^4 f(x,t)}{\partial x^4}\right\|_{L^2(0,1)}.\end{aligned}\tag{4.16}$$

By (4.14) and (4.15), the series functions

$$\sum_{k=1}^{+\infty}8\pi k f_{2k}(s)E_{\alpha,\alpha}\left[-\lambda_k\left(\frac{t^\rho-s^\rho}{\rho}\right)^\alpha\right]\text{ and }\sum_{k=1}^{+\infty}8\pi k\varphi_{2k}e_\alpha\left(\frac{t^\rho}{\rho},\lambda_k\right)$$

are uniformly convergent. Then, $\eta(t)$ and $K(t,s)$ are continuous functions on $[0,T]$ and $[0,T]\times[0,T]$, respectively. Hence, the operator $\mathcal{P}$ is well defined.

Let $r_1,r_2\in\mathcal{C}(0,T)$. From (4.16) and the change of variable $\tau=\frac{t^\rho-s^\rho}{\rho}$, we get

$$\|\mathcal{P}(r_1)-\mathcal{P}(r_2)\|\leq\frac{MC'|1-\beta|T^{\rho\alpha}}{\alpha|1+\beta|\rho^\alpha}\|r_1-r_2\|.\tag{4.17}$$

With the condition (4.1), $\frac{MC'|1-\beta|T^{\rho\alpha}}{\alpha|1+\beta|\rho^\alpha}<1$, then the mapping $\mathcal{P}$ is a contraction. Consequently, by Banach fixed point theorem, the mapping $\mathcal{P}$ has a unique fixed point $r\in\mathcal{C}[0,T]$.

To establish the regularity of the obtained solution, it remains to show

$$u(x,t),u_x(x,t),u_{xx}(x,t),{}^c\mathcal{D}_t^{\alpha,\rho}u(x,t)\in\mathcal{C}(\Omega_T).$$

Under assumptions (A1)-(A2) and integration by parts four times, we have

$$\begin{aligned}f_{2k}(t)&=\frac{f_{2k}^{(4)}(t)}{16\pi^4k^4},\quad f_{2k-1}(t)=\frac{1}{16\pi^4k^4}\left(f_{2k-1}^{(4)}(t)-\frac{2a}{\pi k}f_{2k}^{(4)}(t)\right),\\ \varphi_{2k}&=\frac{\varphi_{2k}^{(4)}}{16\pi^4k^4},\quad \varphi_{2k-1}=\frac{1}{16\pi^4k^4}\left(\varphi_{2k-1}^{(4)}-\frac{2a}{\pi k}\varphi_{2k}^{(4)}\right).\end{aligned}\tag{4.18}$$

From (4.9)-(4.11), (4.18) and (2.1), we get

$$\begin{aligned}
|u_0(t)| &\leq M_2 + \frac{M_0 M_1 T^{\rho\alpha}}{\rho^\alpha \Gamma(\alpha+1)} := M_3, \quad t \in [0,T], \\
|u_{2k}(t)| &\leq \frac{L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha}{16\pi^4 k^4}, \quad t \in [0,T], \\
|u_{2k-1}(t)| &\leq \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)(1+|a|+|a|CT^\rho/\rho)}{16\pi^4 k^4}, \\
&\qquad t \in [\varepsilon, T], \quad \varepsilon > 0.
\end{aligned} \tag{4.19}$$

By using (4.2) and (4.19), following relations hold for $x \in [0,1]$ and $t \in [\varepsilon, T]$ with $\varepsilon > 0$ such that

$$\begin{aligned}
|u(x,t)| &\leq 2M_1 + \sum_{k=1}^{+\infty} \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)(1+|a|+|a|CT^\rho/\rho)}{4\pi^4 k^4} \\
&\quad + \sum_{k=1}^{+\infty} \frac{(1+|b|+|a|)(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)}{4\pi^4 k^4}, \\
|u_x(x,t)| &\leq \sum_{k=1}^{+\infty} \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)(1+|a|+|a|CT^\rho/\rho)}{2\pi^3 k^3} \\
&\quad + \sum_{k=1}^{+\infty} \frac{(|a|+2\pi k(1+|b|+|a|))(1+|b|+|a|)}{4\pi^4 k^4} \\
&\qquad \times (L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha) \\
|u_{xx}(x,t)| &\leq \sum_{k=1}^{+\infty} \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)(1+|a|+|a|CT^\rho/\rho)}{\pi k^2} \\
&\quad + \sum_{k=1}^{+\infty} \frac{(a+\pi k(1+|b|+|a|))(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)}{\pi^3 k^3}.
\end{aligned} \tag{4.20}$$

From (4.6)-(4.8), (4.19) and for $t \in [\varepsilon, T]$, we have

$$\begin{aligned}
\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u_0(t)\right| &\leq M_0 M_2, \\
\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u_{2k}(t)\right| &\leq \frac{M_0 M_2}{16\pi^4 k^4} + \frac{L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha}{4\pi^2 k^2}, \\
\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u_{2k-1}(t)\right| &\leq \frac{(1+|a|) M_0 M_2}{16\pi^4 k^4} + \frac{(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)}{4\pi^2 k^2} \\
&\qquad \times (1+|a|+|a|CT^\rho/\rho) + \frac{|a|(L_1 M_2 + L_2 M_0 M_1 T^{\rho\alpha}/\alpha\rho^\alpha)}{4\pi^3 k^3}.
\end{aligned}$$

Consequently,

$$\left|{}_0^c\mathcal{D}_t^{\alpha,\rho} u(x,t)\right| \leq 2M_0 M_2 + \sum_{k=1}^{+\infty} \frac{(2+|b|+2|a|) M_0 M_2}{4\pi^4 k^4} \tag{4.21}$$

$$+\sum_{k=1}^{+\infty}\frac{|a|\left(L_1M_2+L_2M_0M_1T^{\rho\alpha}/\alpha\rho^{\alpha}\right)}{\pi^3k^3}$$
$$+\sum_{k=1}^{+\infty}\frac{\left(L_1M_2+L_2M_0M_1T^{\rho\alpha}/\alpha\rho^{\alpha}\right)\left(2+|b|+2|a|+|a|CT^{\rho}/\rho\right)}{\pi^2k^2}.$$

From (4.20), (4.21) and by Weierstrass M-test, the series corresponding to $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, ${}_0^c\mathcal{D}_t^{\alpha,\rho}u(x,t)$ are uniformly convergent on $[0,1]\times[\varepsilon,T]$ for $\varepsilon>0$. Hence, $u(x,t)$, $u_x(x,t)$, $u_{xx}(x,t)$, ${}_0^c\mathcal{D}_t^{\alpha,\rho}u(x,t)$ are continuous functions on Ω_T.

Step 3: Uniqueness of the solution: Let $\{u(x,t),r_1(t)\}$ and $\{v(x,t),r_2(t)\}$ be two solution sets of the inverse problem (1.1)-(1.4). By using (4.2), we obtain

$$\begin{aligned}u(x,t)-v(x,t)=2\left(u_0(t)-v_0(t)\right)&\\+\sum_{k=1}^{+\infty}\left(u_{2k-1}(t)-v_{2k-1}(t)\right)X_{2k-1}(x)&\\+\sum_{k=1}^{+\infty}\left(u_{2k}(t)-v_{2k}(t)\right)X_{2k}(x),&\end{aligned}\tag{4.22}$$

Due to the estimate (4.17) and condition (4.1), we have $r_1=r_2$, and by substituting $r_1=r_2$ in (4.22) and (4.9)-(4.11), we obtain $u=v$.

□

4.2. *Continuous dependence of the solution on the data*

Let $\mathcal{H}$ be the set of triples $\{\varphi,f,g\}$ where the functions φ, f and g satisfy the assumptions of Theorem 4.1 and

$$\|\varphi\|_{C^4(0,1)}\le M_4,\quad \|f\|_{C^4(\Omega_T)}\le M_5,\quad \|g\|_{C^1(0,1)}\le M_6.$$

For $\phi\in\mathcal{H}$, we define the norm $\|\phi\|_{\mathcal{H}}:=\|\varphi\|_{C^4(0,1)}+\|f\|_{C^4(\Omega_T)}+\|g\|_{C^1(0,1)}$.

By using the Cauchy-Schwarz and Bessel inequalities, the series functions

$$\sum_{k=1}^{+\infty}\frac{\left|f_{2k}^{(4)}(s)\right|}{2\pi^3k^3}\le M_7,$$

is uniformly convergent, where $f_{2k}^{(4)}(s)$ are the coefficients of the sine Fourier expansion of the function $\frac{\partial^4 f(x,s)}{\partial x^4}$.

Theorem 4.2. *The solution $\{u(x,t),r(t)\}$ of the inverse problem (1.1)-(1.4) under the assumptions of Theorem 4.1, depends continuously upon the data for $T<\left(\frac{\alpha|1+\beta|\rho^{\alpha}}{MC'|1-\beta|}\right)^{1/\rho\alpha}$.*

Proof. Let $\{u(x,t), r(t)\}$ and $\{\tilde{u}(x,t), \tilde{r}(t)\}$ be two solution sets of the inverse problem (1.1)-(1.4), corresponding to the data $\phi = \{\varphi, f, g\}$ and $\phi = \{\tilde{\varphi}, \tilde{f}, \tilde{g}\}$, respectively.

For $g, \tilde{g} \in C^1(0,T)$, we have $\left\| {}_0^c\mathcal{D}_t^{\alpha,\rho} g - {}_0^c\mathcal{D}_t^{\alpha,\rho} \tilde{g} \right\|_{C(0,T)} \le M_8 \|g-\tilde{g}\|_{C^1(0,T)}$, where $M_8 = \frac{T^{1-\rho\alpha}}{\rho^{1-\alpha}\Gamma(2-\alpha)}$. From (4.13), we have

$$\begin{aligned}\eta(t)-\tilde{\eta}(t) = {} & \left(\int_0^1 f(x,t)\,dx \int_0^1 \tilde{f}(x,t)\,dx\right)^{-1} \\ & \times \Bigg[\int_0^1 \tilde{f}(x,t)\,dx \left({}_0^c\mathcal{D}_t^{\alpha,\rho} g(t) - {}_0^c\mathcal{D}_t^{\alpha,\rho}\tilde{g}(t)\right) \\ & + a\sum_{k=1}^{+\infty} 8\pi k (\varphi_{2k}-\tilde{\varphi}_{2k}) E_\alpha\left(-\lambda_k\left(\frac{t^\rho}{\rho}\right)^\alpha\right) \\ & + {}_0^c\mathcal{D}_t^{\alpha,\rho}\tilde{g}(t)\left(\int_0^1 \tilde{f}(x,t)\,dx - \int_0^1 f(x,t)\,dx\right) \\ & + a\sum_{k=1}^{+\infty} 8\pi k \tilde{\varphi}_{2k} E_\alpha\left(-\lambda_k\left(\frac{t^\rho}{\rho}\right)^\alpha\right) \\ & \times \left(\int_0^1 \tilde{f}(x,t)\,dx - \int_0^1 f(x,t)\,dx\right)\Bigg].\end{aligned}$$

From (4.18), we have $\varphi_{2k}-\tilde{\varphi}_{2k} = \int_0^1 (\varphi(x)-\tilde{\varphi}(x)) X_{2k}(x)\,dx = \frac{\varphi_{2k}^{(4)}-\tilde{\varphi}_{2k}^{(4)}}{16\pi^4 k^4}$.

We have the estimate

$$\|\eta-\tilde{\eta}\| \le N_1 \|\varphi-\tilde{\varphi}\|_{C^4(0,1)} + N_2 \left\|f-\tilde{f}\right\|_{C(\Omega_T)} + N_3 \|g-\tilde{g}\|_{C^1(0,1)},$$

where $N_1 = M^2|a| L_1 C^*$, $\quad N_2 = M^2(|a| L_1 M_7 + M_6 M_8)$, $\quad N_3 = M^2 M_5 M_8$.

From (4.12), we have the estimate

$$\begin{aligned}\|r-\tilde{r}\| \le {} & \|\eta-\tilde{\eta}\| + \frac{MM_0|a|T^{\rho\alpha}}{\alpha\rho^\alpha}\left\|f^{(4)}-\tilde{f}^{(4)}\right\|_{C(\Omega_T)} + \frac{M|a|C'T^{\rho\alpha}}{\alpha\rho^\alpha}\|r-\tilde{r}\| \\ & + \frac{M^2 M_0 C' T^{\rho\alpha}}{\alpha\rho^\alpha}\left\|f-\tilde{f}\right\|_{C(\Omega_T)}.\end{aligned}$$

Due to the estimate of $\|\eta-\tilde{\eta}\|$, we have

$$\begin{aligned}\left(1-\frac{M|a|C'T^{\rho\alpha}}{\alpha\rho^\alpha}\right)\|r-\tilde{r}\| \le {} & N_1\|\varphi-\tilde{\varphi}\|_{C^4(0,1)} \\ & + \left(N_2 + \frac{MM_0|a|T^{\rho\alpha}}{\alpha\rho^\alpha} + \frac{M^2M_0C'T^{\rho\alpha}}{\alpha\rho^\alpha}\right)\left\|f-\tilde{f}\right\|_{C^4(\Omega_T)} \\ & + N_3\|g-\tilde{g}\|_{C^1(0,1)}.\end{aligned}$$

Hence

$$\left(1-\frac{M\,|a|\,C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}\right)\|r-\tilde{r}\| \le N_4\left\|\phi-\tilde{\phi}\right\|_{\mathcal{H}},$$

where $N_4 := \max\left\{N_1, N_2+\frac{MM_0|a|T^{\rho\alpha}}{\alpha\rho^{\alpha}}+\frac{M^2M_0C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}, N_3\right\}$.

For $T<\left(\frac{\alpha\rho^{\alpha}}{M|a|C'}\right)^{1/\rho\alpha}$, we have $\|r-\tilde{r}\| \le \frac{N_4}{1-\frac{M|a|C'T^{\rho\alpha}}{\alpha\rho^{\alpha}}}\left\|\phi-\tilde{\phi}\right\|_{\mathcal{H}}$.

From (4.2), a similar estimate can be also obtained for the difference $u-\tilde{u}$:

$$\|u-\tilde{u}\|_{C(\bar{\Omega}_T)} \le N_5\left\|\phi-\tilde{\phi}\right\|_{\mathcal{H}}.$$

□

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