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ON SOME FIXED POINT THEOREMS FOR ĆIRIĆ OPERATORS

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Abstract. In this paper, we will present existence and localization fixed point theorems and stability results for the fixed point problem involving some very general classes of operators (single-valued and multi-valued), namely for Ćirić type operators. Then, an application to homotopy principles is given. Our results complement and extend the works in the literature.

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1. INTRODUCTION AND PRELIMINARIES

Let (X,d) be a metric space and $\mathrm{f}\colon X \to X$ be an operator. For each $x \in X$, we denote $O(x,\infty) = \{x, \mathrm{f}(x), \ldots, \mathrm{f}^n(x), \ldots\}$.

Let x_0 be a given point in X and $r > 0$. The set $B(x_0;r) := \{x \in X\colon \mathrm{d}(x_0,x) < r\}$ is the open ball of center x_0 and radius r, while $\tilde{B}(x_0;r) := \{x \in X\colon \mathrm{d}(x_0,x) \leq r\}$ is the closed ball of center x_0 and radius r.

We will now recall some definitions and well-known results, which will be useful throughout the paper.

Definition 1 ([4, Cirić Definition page 268])**.** Let (X,d) be a metric space and $\mathrm{f}\colon X \to X$ be an operator. Then X is said to be f-orbitally complete if every Cauchy sequence contained in $O(x,\infty)$, for some $x \in X$, converges in X.

In the above context, a sequence of Picard iterates starting from $x_0 \in X$ is a sequence $x_n := \mathrm{f}^n(x_0)$, for $n \in \mathbb{N}^*$.

Definition 2 ([4, Cirić Definition 1])**.** Let (X,d) be a metric space. Then, $\mathrm{f}\colon Y \subseteq X \to X$ is a single-valued Ćirić type operator with constant q if there exists a number $q \in (0,1)$, such that for all $x, y \in Y$ we have

$$\mathrm{d}(\mathrm{f}(x),\mathrm{f}(y)) \leq q \cdot \max\{\mathrm{d}(x,y), \mathrm{d}(x,\mathrm{f}(x)), \mathrm{d}(y,\mathrm{f}(y)), \mathrm{d}(x,\mathrm{f}(y)), \mathrm{d}(y,\mathrm{f}(x))\}.$$

Let (X,d) be a metric space. By $P(X)$ we denote the family of all nonempty subsets of X, and the family of all nonempty and closed subsets of X is denoted with $P_{cl}(X)$. Throughout the paper, we consider the following distances (see, e.g., [9, 10]):

(1) The gap functional (generated by d) between a point $a \in X$ and a set $Y \in P(X)$ is

$$\mathrm{D}(a,Y) := \inf\{\mathrm{d}(a,y) \mid y \in Y\}.$$

(2) The Pompeiu-Hausdorff functional (generated by d) between two sets $A, B \in P(X)$ is

$$\mathrm{H}(A,B) := \max\left\{\sup_{a\in A}(\inf_{b\in B}\mathrm{d}(a,b)), \sup_{b\in B}(\inf_{a\in A}\mathrm{d}(a,b))\right\}.$$

If $\mathrm{F}\colon X \to P(X)$ is a multi-valued operator, then its fixed point set is denoted by $\mathrm{Fix}(F) := \{x \in X \mid x \in \mathrm{F}(x)\}$, while the graph of F is the set $\mathrm{Graph}(F) := \{(x,y) \in X \times X \mid y \in \mathrm{F}(x)\}$. The set of all strict fixed points of F is denoted by $\mathrm{SFix}(F)$, i.e., there exists $x^* \in X$ such that $\mathrm{F}(x^*) = \{x^*\}$.

In this paper, we will present several existence and localization fixed point theorems and stability results for the fixed point problem involving some very general classes of operators (single-valued and multi-valued), namely for Ćirić type operators. Then, some applications to homotopy principles are given. Our results complement and extend some works in the literature, see e.g. [1–4, 6, 8, 11].

2. A STUDY OF THE FIXED POINT EQUATION WITH GENERALIZED ĆIRIĆ OPERATORS

In this section, the single-valued case is taken into consideration. We first recall Ćirić's Theorem which appeared in the well-known paper from 1974, see [4].

Theorem 1 ([4, Cirić Theorem 1])**.** *Let (X,d) be a metric space and $\mathrm{f}\colon X \to X$ be a Ćirić type operator with constant $q \in (0,1)$. Suppose that X is* f*-orbitally complete. Then:*

(i) f *has a unique fixed point x^* in X and $\lim_{n\to\infty} f^n(x) = x^*$, i.e.,* f *is a Picard operator;*

(ii) $\mathrm{d}(f^n(x),x^*) \leq \dfrac{q^n}{1-q}\mathrm{d}(x,\mathrm{f}(x))$*, for every $x \in X$ and every $n \in \mathbb{N}^*$.*

Our first main result, which generalizes the above theorem, is an existence, uniqueness and localization for the unique fixed point of a single-valued Ćirić type operator.

Theorem 2. *Let (X,d) be a complete metric space, $x_0 \in X$ and $r > 0$. We consider $\mathrm{f}\colon B(x_0;r) \to X$ a single-valued Ćirić type operator with constant $q \in \left(0,\frac{1}{2}\right)$. We also suppose that*

$$\mathrm{d}(x_0,\mathrm{f}(x_0)) < \frac{1-2q}{1-q}r.$$

Then f *has a unique fixed point* $x^* \in B(x_0;r)$*,* $f^n(x_0) \in B(x_0;r)$*, for all* $n \in \mathbb{N}$ *and the sequence* $(f^n(x_0))_{n\in\mathbb{N}}$ *of Picard iterates starting from* x_0 *converges to* x^* *as* $n \longrightarrow \infty$*.*

Proof. Let $0 < s < r$ such that

$$\mathrm{d}(x_0,\mathrm{f}(x_0)) \leq \frac{1-2q}{1-q}s < \frac{1-2q}{1-q}r.$$

The sequence $(x_n)_{n\in\mathbb{N}}$, with $x_n := \mathrm{f}^n(x_0)$, has the recurrent form $x_{n+1} = \mathrm{f}(x_n)$, for all $n \in \mathbb{N}$. Then,

$$\begin{aligned}
\mathrm{d}(x_1,x_2) &= \mathrm{d}(\mathrm{f}(x_0),\mathrm{f}(x_1)) \\
&\leq q\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_2)\mathrm{d}(x_1,x_1)\} \\
&= q\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_2)\} \\
&\leq q\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)\} \\
&= q\left(\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)\right),
\end{aligned}$$

implying

$$\mathrm{d}(x_1,x_2) \leq \frac{q}{1-q}\mathrm{d}(x_0,x_1).$$

We denote $h := \dfrac{q}{1-q}$, thus $\dfrac{1-2q}{1-q} = 1-h$ with $h \in (0,1)$. Using the mathematical induction, we can prove the inequality

$$\mathrm{d}(x_{n-1},x_n) \leq h^{n-1}\mathrm{d}(x_0,x_1)$$

holds for all $n \in \mathbb{N}^*$. We also know that $\mathrm{d}(x_0,x_1) \leq (1-h)s$.

By taking a point $n \in \mathbb{N}^*$ arbitrarily, we obtain

$$\begin{aligned}
\mathrm{d}(x_0,x_n) &\leq \mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)+\cdots+\mathrm{d}(x_{n-1},x_n) \\
&\leq \mathrm{d}(x_0,x_1)+h\mathrm{d}(x_0,x_1)+\cdots+h^{n-1}\mathrm{d}(x_0,x_1) \\
&= \mathrm{d}(x_0,x_1)(1+h+\cdots+h^{n-1}) \\
&= \frac{1-h^n}{1-h}\mathrm{d}(x_0,x_1) \leq \frac{1}{1-h}\mathrm{d}(x_0,x_1) \leq s,
\end{aligned}$$

proving that all elements of the sequence are in the closed ball $\tilde{B}(x_0;s)$.

We will continue by proving that the sequence considered is Cauchy in X. Let $m \in \mathbb{N}$ and $n \in \mathbb{N}^*$. We compute

$$\begin{aligned}
\mathrm{d}(x_m,x_{m+n}) &\leq \mathrm{d}(x_m,x_{m+1})+\cdots+\mathrm{d}(x_{m+n-1},x_{m+n}) \\
&\leq h^m\mathrm{d}(x_0,x_1)+\cdots+h^{m+n-1}\mathrm{d}(x_0,x_1) \\
&= h^m\mathrm{d}(x_0,x_1)\left(1+h+\cdots+h^{n-1}\right) \\
&= h^m\frac{1-h^n}{1-h}\mathrm{d}(x_0,x_1)
\end{aligned}$$

$$\leq \frac{h^m}{1-h}\mathrm{d}(x_0,x_1).$$

This relation leads us to the conclusion that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Moreover, due to the completeness of (X,d), we obtain it is also convergent to a point $x^* \in \tilde{B}(x_0;s)$. We will prove that x^* is a fixed point. We compute the following inequality

$$\begin{aligned}
\mathrm{d}(x^*,\mathrm{f}(x^*)) &\leq \mathrm{d}(x^*,x_{n+1}) + \mathrm{d}(x_{n+1},\mathrm{f}(x^*)) \\
&\leq \mathrm{d}(x^*,x_{n+1}) + q\max\{\mathrm{d}(x_n,x^*),\mathrm{d}(x_n,x_{n+1}),\mathrm{d}(x^*,\mathrm{f}(x^*)), \\
&\qquad \mathrm{d}(x^*,x_{n+1}),\mathrm{d}(x_n,\mathrm{f}(x^*))\} \\
&\leq \mathrm{d}(x^*,x_{n+1}) + q\max\{\mathrm{d}(x_n,x_{n+1}),\mathrm{d}(x^*,x_{n+1}),\mathrm{d}(x_n,x^*)+\mathrm{d}(x^*,\mathrm{f}(x^*))\} \\
&\leq \mathrm{d}(x^*,x_{n+1}) + q\left[\mathrm{d}(x_n,x_{n+1}) + \mathrm{d}(x_{n+1},x^*) + \mathrm{d}(x^*,\mathrm{f}(x^*))\right],
\end{aligned}$$

which implies

$$\mathrm{d}(x^*,\mathrm{f}(x^*)) \leq \frac{1+q}{1-q}\mathrm{d}(x_{n+1},x^*) + \frac{q}{1-q}\left[\mathrm{d}(x_n,x_{n+1})\right], \text{ for all } n\in\mathbb{N}.$$

We only need to let $n \longrightarrow \infty$ in the above inequality, and we will obtain $\mathrm{d}(x^*,\mathrm{f}(x^*)) = 0$, proving that x^* is a fixed point for f.

For the uniqueness of the fixed point, we suppose by contradiction that there exists another fixed point y^*, with $x^* \neq y^*$. Then,

$$\begin{aligned}
\mathrm{d}(x^*,y^*) &= \mathrm{d}(\mathrm{f}(x^*),\mathrm{f}(y^*)) \\
&\leq q\max\{\mathrm{d}(x^*,y^*),\mathrm{d}(x^*,x^*),\mathrm{d}(y^*,y^*),\mathrm{d}(x^*,y^*),\mathrm{d}(y^*,x^*)\} = q\mathrm{d}(x^*,y^*),
\end{aligned}$$

which is a contradiction due to the fact that $q < \frac{1}{2}$. □

We now denote by $\mathcal{S}(Y,X)$ the family of all operators from Y to X, where X is a metric space and Y a closed subset of X, and by

$$\mathcal{S}_{\partial Y}(Y,X) := \left\{\mathrm{f}\in\mathcal{S}(Y,X) \text{ such that } \mathrm{f}_{|_{\partial Y}} : \partial Y \to X \text{ is fixed point free}\right\}.$$

We will introduce the concept of a family of single-valued Ćirić type operators with constant q.

Definition 3. Let (X,d) be a metric space and (J,ρ) be a metric space. We say that $\{\mathrm{f}_\lambda : \lambda\in J\} \subset \mathcal{S}(Y,X)$ is a family of single-valued Ćirić type operators with constant $q\in(0,1)$ if the following conditions are satisfied: there exist $p\in(0,1]$ and $M>0$ such that

(i) for all $x_1,x_2\in Y$ and $\lambda\in J$, we have

$$\begin{aligned}
\mathrm{d}(\mathrm{f}_\lambda(x_1),\mathrm{f}_\lambda(x_2)) \leq q\max\{&\mathrm{d}(x_1,x_2),\mathrm{d}(x_1,\mathrm{f}_\lambda(x_1)),\mathrm{d}(x_2,\mathrm{f}_\lambda(x_2)), \\
&\mathrm{d}(x_1,\mathrm{f}_\lambda(x_2)),\mathrm{d}(x_2,\mathrm{f}_\lambda(x_1))\};
\end{aligned}$$

(ii) for all $x\in Y$ and $\lambda,\mu\in J$, we have

$$\mathrm{d}(\mathrm{f}_\lambda(x),\mathrm{f}_\mu(x)) \leq M\left[\rho(\lambda,\mu)\right]^p.$$

The following homotopy result can now be proved.

Theorem 3. *Let* (X,d) *be a complete metric space and* Y *be a closed subset such that* $\operatorname{int} Y \neq \varnothing$*. Let* (J,ρ) *be a connected metric space and* $\{\mathrm{f}_\lambda\colon \lambda \in J\}$ *be a family of single-valued Ćirić type operators with constant* $q \in \left(0,\frac{1}{2}\right)$ *from* $\mathcal{S}_{\partial Y}(Y,X)$*. Then the following conclusions occur:*

(i) If there exists a point $\lambda_0^* \in J$*, such that the equation* $\mathrm{f}_{\lambda_0^*}(x) = x$ *has a solution, then the equation* $\mathrm{f}_\lambda(x) = x$ *has a unique solution for any* $\lambda \in J$*;*

(ii) If $\mathrm{f}_\lambda(x_\lambda) = x_\lambda$*, for any* $\lambda \in J$*, then the operator*

$$\mathrm{j}\colon J \to \operatorname{int} Y, \mathrm{j}(\lambda) = x_\lambda$$

is continuous.

Proof. We will begin the proof by considering two fixed points, x_λ a fixed point of f_λ and x_μ a fixed point of f_μ. Then,

$$\begin{aligned}\mathrm{d}(x_\lambda,x_\mu) &= \mathrm{d}(\mathrm{f}_\lambda(x_\lambda),\mathrm{f}_\mu(x_\mu))\\ &\leq \mathrm{d}(\mathrm{f}_\lambda(x_\lambda),\mathrm{f}_\lambda(x_\mu)) + \mathrm{d}(\mathrm{f}_\lambda(x_\mu),\mathrm{f}_\mu(x_\mu)).\end{aligned}$$

Taking $\mathrm{d}(\mathrm{f}_\lambda(x_\lambda),\mathrm{f}_\lambda(x_\mu))$ separately, we compute

$$\begin{aligned}\mathrm{d}(\mathrm{f}_\lambda(x_\lambda),\mathrm{f}_\lambda(x_\mu)) &\leq q\max\{\mathrm{d}(x_\lambda,x_\mu),\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\lambda)),\mathrm{d}(x_\mu,\mathrm{f}_\lambda(x_\mu)),\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\mu)),\\ &\qquad \mathrm{d}(x_\mu,\mathrm{f}_\lambda(x_\lambda))\}\\ &= q\max\left\{\mathrm{d}(x_\lambda,x_\mu),\mathrm{d}(x_\mu,\mathrm{f}_\lambda(x_\mu)),\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\mu))\right\}\\ &\leq q\max\left\{\mathrm{d}(x_\lambda,x_\mu),\mathrm{d}(x_\lambda,x_\mu)+\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\mu)),\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\mu))\right\}\\ &= q\left[\mathrm{d}(x_\lambda,x_\mu)+\mathrm{d}(x_\lambda,\mathrm{f}_\lambda(x_\mu))\right],\end{aligned}$$

which implies

$$\mathrm{d}(\mathrm{f}_\lambda(x_\lambda),\mathrm{f}_\lambda(x_\mu)) \leq \frac{q}{1-q}\mathrm{d}(x_\lambda,x_\mu).$$

Using the latter inequality together with the first one, we obtain

$$\begin{aligned}\mathrm{d}(x_\lambda,x_\mu) &\leq \frac{q}{1-q}\mathrm{d}(x_\lambda,x_\mu) + \mathrm{d}(\mathrm{f}_\lambda(x_\mu),\mathrm{f}_\mu(x_\mu))\\ &\leq \frac{q}{1-q}\mathrm{d}(x_\lambda,x_\mu) + M\left[\rho(\lambda,\mu)\right]^p\end{aligned}$$

entailing

$$\mathrm{d}(x_\lambda,x_\mu) \leq \frac{1-q}{1-2q}M\left[\rho(\lambda,\mu)\right]^p.$$

Let us consider the set

$$Q = \left\{\lambda \in J \mid \exists x_\lambda \in \operatorname{int} Y \text{ such that } x_\lambda = \mathrm{f}_\lambda(x_\lambda)\right\}.$$

In addition to J being a connected space, by proving that Q is both closed and open, will lead us to $Q = J$, proving (i). For the closedness of Q, let $(\lambda_n)_{n\in\mathbb{N}} \subset Q$ such that

$\lambda_n \to \lambda^*$, and we show that $\lambda^* \in Q$. We consider $x_{\lambda_m} = f_{\lambda_m}(x_{\lambda_m})$ and $x_{\lambda_n} = f_{\lambda_n}(x_{\lambda_n})$ and we know that

$$d(x_{\lambda_m}, x_{\lambda_n}) \leq \frac{1-q}{1-2q} M \left[\rho(\lambda_m, \lambda_n)\right]^p. \tag{2.1}$$

We already know that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ is Cauchy in J, which implies that for an arbitrary $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ with $m,n \in \mathbb{N}$, $m,n > n_\varepsilon$ such that

$$\rho(\lambda_m, \lambda_n) < \varepsilon. \tag{2.2}$$

We will denote $\dfrac{\varepsilon^p M(1-q)}{1-2q} =: \varepsilon' > 0$. Using this notation, together with relations (2.1) and (2.2), we get

$$d(x_{\lambda_m}, x_{\lambda_n}) < \varepsilon',$$

which proves the sequence (x_{λ_n}) is Cauchy in X. Also, since (X, d) is a complete space, we obtain (x_{λ_n}) is a convergent sequence in Y. Let us now denote the limit of this sequence by x_{λ^*}, and we compute

$$\begin{aligned} d(x_{\lambda^*}, f(x_{\lambda^*})) &\leq d(x_{\lambda^*}, x_{\lambda_{n+1}}) + d(x_{\lambda_{n+1}}, f(x_{\lambda^*})) \\ &\leq d(x_{\lambda^*}, x_{\lambda_{n+1}}) + q \max\big\{d(x_{\lambda_n}, x_{\lambda^*}), d(x_{\lambda_n}, x_{\lambda_{n+1}}), d(x_{\lambda^*}, f(x_{\lambda^*})), \\ &\qquad\qquad d(x_{\lambda_n}, f(x_{\lambda^*})), d(x_{\lambda^*}, x_{\lambda_{n+1}})\big\} \\ &\leq d(x_{\lambda^*}, x_{\lambda_{n+1}}) + q\left[d(x_{\lambda_n}, x_{\lambda^*}) + d(x_{\lambda_n}, x_{\lambda_{n+1}}) + d(x_{\lambda^*}, f(x_{\lambda^*}))\right. \\ &\qquad\qquad \left. + \, d(x_{\lambda_n}, f(x_{\lambda^*})) + d(x_{\lambda^*}, x_{\lambda_{n+1}})\right]. \end{aligned}$$

The inequality obtained above implies that for all $n \in \mathbb{N}$,

$$d(x_{\lambda^*}, f(x_{\lambda^*})) \leq \frac{1}{1-2q}\left[(1+2q)d(x_{\lambda_n}, x_{\lambda^*}) + q\left(d(x_{\lambda_n}, x_{\lambda_{n+1}}) + d(x_\lambda^*, x_{\lambda_{n+1}})\right)\right],$$

and by letting $n \longrightarrow \infty$, we get that x_{λ^*} is a fixed point for f. Since f is fixed point free on its boundary, then λ^* belongs to Q, proving it is closed.

In order to show that Q is open, we consider $\lambda_0 \in Q$. Then, there exists a point $x_{\lambda_0} \in \operatorname{int} Y$ such that $x_{\lambda_0} = f_{\lambda_0}(x_{\lambda_0})$. Now, we will prove the existence of an $\varepsilon > 0$ and an open ball $B(\lambda_0; \varepsilon) \subset Q$. Due to $\operatorname{int} Y$ being an open set and $x_{\lambda_0} \in \operatorname{int} Y$, there exists an open ball $B(x_{\lambda_0}; r) \subseteq \operatorname{int} Y$. We consider arbitrary $\varepsilon > 0$ such that $\varepsilon^p < \dfrac{1-2q}{M(1-q)} r$ and an arbitrary $\lambda \in B(\lambda_0; \varepsilon)$, and we prove that $\lambda \in Q$. Let us begin by estimating the following distance

$$\begin{aligned} d(f_\lambda(x_{\lambda_0}), x_{\lambda_0}) &= d(f_\lambda(x_{\lambda_0}), f_{\lambda_0}(x_{\lambda_0})) \\ &\leq M(\rho(\lambda, \lambda_0))^p \leq M\varepsilon^p \\ &\leq \frac{1-2q}{1-q} r. \end{aligned}$$

From the inequality above, we get that the operator

$$\mathrm{f}_\lambda\colon B(x_{\lambda_0};r)\to X$$

is a Ćirić type operator, and using the local fixed point theorem for Ćirić type operators, we obtain that $\mathrm{Fix}(\mathrm{f}_\lambda)\neq\varnothing$, implying $\lambda\in Q$.

Based on what we proved so far, the operator j is single-valued. We consider $\lambda,\ \mu\in J$ and we have

$$\mathrm{d}(\mathrm{j}(\lambda),\mathrm{j}(\mu))\leq\frac{1-q}{1-2q}M\left[\rho(\lambda,\mu)\right]^p.$$

Letting

$$\rho(\lambda,\mu)<\delta:=\left[\frac{\varepsilon(1-2q)}{M(1-q)}\right]^{\frac{1}{p}},$$

we immediately obtain that $\mathrm{d}(\mathrm{j}(\lambda),\mathrm{j}(\mu))<\varepsilon$, proving that j is a continuous operator. □

3. A STUDY OF THE FIXED POINT EQUATION WITH MULTI-VALUED GENERALIZED ĆIRIĆ OPERATORS

We first consider some notions related to our main results.

Definition 4. An operator $\mathrm{F}\colon X\to P_{cl}(X)$ is said to be a multi-valued generalized contraction if for every $x,y\in X$ there exist non-negative numbers p,q,r, which may depend on both x and y, such that $sup\{p+2q+2r\mid x,y\in X\}<1$ and

$$\mathrm{H}(\mathrm{F}(x),\mathrm{F}(y))\leq p\cdot\mathrm{d}(x,y)+q\cdot[\mathrm{D}(x,\mathrm{F}(x))+\mathrm{D}(y,\mathrm{F}(y))]+r\cdot[\mathrm{D}(x,\mathrm{F}(y))+\mathrm{D}(y,\mathrm{F}(x))].$$

Definition 5 ([2, A. Amini-Harandi Definition 2.1])**.** Let (X,d) be a metric space. The set-valued map $\mathrm{F}\colon Y\subseteq X\to P_{b,cl}(X)$ is said to be a multi-valued Ćirić type operator with constant k (named a k-set-valued quasi-contraction in [2]) if

$$\mathrm{H}(\mathrm{F}(x),\mathrm{F}(y))\leq k\max\{\mathrm{d}(x,y),\mathrm{D}(x,\mathrm{F}(x)),\mathrm{D}(y,\mathrm{F}(y)),\mathrm{D}(x,\mathrm{F}(y)),\mathrm{D}(y,\mathrm{F}(x))\},$$

for any $x,y\in X$, where $0\leq k<1$.

We have the following example of a multi-valued Ćirić type operator, which is not a multi-valued generalized contraction.

Example 1. Let

$$X_1=\left\{\frac{m}{n}: m=0,1,2,4,6,\dots;\ n=1,3,7,\dots,2k+1,\dots\right\},$$

$$X_2=\left\{\frac{m}{n}: m=1,2,4,6,8,\dots;\ n=2,5,8,\dots,3k+2,\dots\right\},$$

where $k \in \mathbb{N}$ and let $X = X_1 \cup X_2$. Let us define $\mathrm{F}\colon X \to X$ by

$$\mathrm{F}(x) = \begin{cases} \left\{\frac{2}{3}x, \frac{6}{7}x\right\}, & x \in X_1, \\ \frac{1}{5}x, & x \in X_2. \end{cases}$$

The mapping F is a multi-valued Ćirić type operator with $q = \dfrac{6}{7}$. If both x and y are in X_1 or in X_2, then

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq \frac{6}{7}\mathrm{d}(x,y).$$

If we take $x \in X_1$ and $y \in X_2$, then we have that

$$x \geq \frac{7}{30}y \text{ implies } \mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) = \frac{6}{7}\left(x - \frac{7}{30}y\right) \leq \frac{6}{7}\left(x - \frac{1}{5}y\right) = \frac{6}{7}\mathrm{D}(x, \mathrm{F}(y)),$$

$$x < \frac{7}{30}y \text{ implies } \mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) = \frac{6}{7}\left(\frac{7}{30}y - x\right) \leq \frac{6}{7}(y - x) = \frac{6}{7}\mathrm{d}(x,y).$$

Therefore, we have that F satisfies the following condition:

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq \frac{6}{7}\max\left\{\mathrm{d}(x,y), \mathrm{D}(x, \mathrm{F}(y)), \mathrm{D}(y, \mathrm{F}(x))\right\},$$

and hence, it is a multi-valued Ćirić type operator.

In the following step, we show that F is not a multi-valued generalized contraction on X. Let $x = 1$ and $y = \frac{1}{2}$. Then we have that

$$\begin{aligned} p \cdot \mathrm{d}(x,y) + q \cdot [\mathrm{D}(x, \mathrm{F}(x)) + \mathrm{D}(y, \mathrm{F}(y))] + r \cdot [\mathrm{D}(x, \mathrm{F}(y)) + \mathrm{D}(y, \mathrm{F}(x))] = \\ = \frac{1}{2}p + \frac{4}{10}q + \frac{88}{70}r < (p + 2q + 2r)\frac{88}{140} < \\ < \frac{88}{140} < \frac{53}{70} = \mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)), \end{aligned}$$

as $p + 2q + 2r < 1$. Thus, we can see that F is not a multi-valued generalized contraction.

If (X, d) is a metric space and $\mathrm{F}\colon X \to P(X)$ is a multi-valued operator, then a sequence $(x_n)_{n\in\mathbb{N}}$ from X is called a sequence of Picard type starting from $(x,y) \in \mathrm{Graph}(F)$ if $x_0 = x, x_1 = y$ and $x_n \in \mathrm{F}(x_{n-1}), n \in \mathbb{N}^*$.

The following lemma is useful for our following results.

Lemma 1 (Cauchy's Lemma)**.** *Let $(a_n), (b_n)$ be two sequences of positive numbers such that $\sum_{n\geq 0} a_n < \infty$ and $\lim_{n\to\infty} b_n = 0$. Then $\lim_{n\to\infty}\left(\sum_{k=0}^{n} a_{n-k}b_k\right) = 0$.*

Theorem 4 ([2, A. Amini-Harandi Theorem 2.2])**.** *Let (X,d) be a complete metric space. Let* $\mathrm{F}\colon X \to P_{b,cl}(X)$ *be a multi-valued Ćirić type operator with constant* $k < \frac{1}{2}$*. Then,* F *has a fixed point.*

Here we will give a constructive proof of this theorem, as well as some data dependence and stability results for the fixed point problem $x \in \mathrm{F}(x)$.

Theorem 5. *Let (X,d) be a complete metric space. Let* $\mathrm{F}\colon X \to P_{cl}(X)$ *be a multi-valued Ćirić type operator with constant* $k < \frac{1}{2}$*. Then:*

(i) $\mathrm{Fix}(F) \neq \varnothing$ *and for every* $(x,y) \in \mathrm{Graph}(F)$ *there exists a sequence* $(x_n)_{n\in\mathbb{N}}$ *of Picard type starting from* $x_0 := x$, $x_1 := y$ *which converge to a fixed point* x^* *of* F*;*

(ii) the fixed point equation $x \in \mathrm{F}(x)$ *has the data dependence property, i.e., for any* $x^* \in \mathrm{Fix}(F)$ *and any* $G\colon X \to P(X)$ *such that* $\mathrm{Fix}(G) \neq \varnothing$ *and the inequality* $\mathrm{H}(\mathrm{F}(x),\mathrm{G}(x)) \leq \eta$ *holds for all* $x \in X$ *and some* $\eta > 0$*, there is* $u^* \in \mathrm{Fix}(\mathrm{G})$ *such that*

$$\mathrm{d}(x^*,u^*) \leq \frac{(1+k)q}{1-k}\eta,$$

where $1 < q < \dfrac{1}{2k}$*;*

(iii) the fixed point equation is well-posed, i.e., for every sequence $(u_n)_{n\in\mathbb{N}} \subset X$ *such that*

$$\mathrm{D}(u_n,\mathrm{F}(u_n)) \longrightarrow 0,$$

as $n \longrightarrow \infty$*, we have that* $u_n \longrightarrow x^*$*, as* $n \longrightarrow \infty$*.*

(iv) if $q < \frac{1}{2}$*, then the fixed point equation has the Ostrowski stability property, i.e., for any sequence* $(u_n)_{n\in\mathbb{N}} \subset X$ *with* $\mathrm{D}(u_{n+1},\mathrm{F}(u_n)) \longrightarrow 0$ *as* $n \longrightarrow \infty$*, we have that* $u_n \longrightarrow x^*$*;*

Proof. In order to prove (i), let $x_0 \in X$ and we construct the sequence $(x_n)_{n\in\mathbb{N}}$ of Picard type starting from $x_0 := x$ having the general term $x_n \in \mathrm{F}(x_{n-1}), n \in \mathbb{N}^*$. We prove that this sequence is Cauchy.

Let $x_1 \in \mathrm{F}(x_0)$ and $1 < q < \frac{1}{2k}$. Then, there exists $x_2 \in \mathrm{F}(x_1)$ such that $\mathrm{d}(x_1,x_2) \leq q\mathrm{H}(\mathrm{F}(x_0),\mathrm{F}(x_1))$. Then, we have:

$$\begin{aligned}
\mathrm{d}(x_1,x_2) &\leq qk\cdot\max\{\mathrm{d}(x_0,x_1),\mathrm{D}(x_0,\mathrm{F}(x_0)),\mathrm{D}(x_1,\mathrm{F}(x_1)),\\
&\qquad \mathrm{D}(x_0,\mathrm{F}(x_1)),\mathrm{D}(x_1,\mathrm{F}(x_0))\}\\
&\leq qk\cdot\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{D}(x_0,\mathrm{F}(x_1))\}\\
&\leq qk\cdot\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_2))\}\\
&\leq qk\cdot\max\{\mathrm{d}(x_0,x_1),\mathrm{d}(x_1,x_2),\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)\}\\
&\leq qk(\mathrm{d}(x_0,x_1)+\mathrm{d}(x_1,x_2)).
\end{aligned}$$

Hence,

$$\mathrm{d}(x_1,x_2) \le \frac{qk}{1-qk}\mathrm{d}(x_0,x_1).$$

We denote $\beta := \dfrac{qk}{1-qk} < 1$. Then $\mathrm{d}(x_1,x_2) \le \beta \mathrm{d}(x_0,x_1)$. Using mathematical induction we get that:

$$\mathrm{d}(x_n,x_{n+1}) \le \beta^n \mathrm{d}(x_0,x_1).$$

and

$$\begin{aligned}\mathrm{d}(x_m,x_{m+n}) &\le \mathrm{d}(x_m,x_{m+1}) + \cdots + \mathrm{d}(x_{m+n+1},x_{m+n})\\ &\le \beta^m \mathrm{d}(x_0,x_1) + \cdots + \beta^{m+n+1}\mathrm{d}(x_0,x_1) = \beta^m \frac{1-\beta^n}{1-\beta}\mathrm{d}(x_0,x_1).\end{aligned}$$

It follows that

$$\mathrm{d}(x_m,x_{m+n}) \le \frac{\beta^m}{1-\beta}\mathrm{d}(x_0,x_1).$$

Due to the fact that the series $\sum \beta^m$ is convergent, we get the sequence $(x_n)_{n\in\mathbb{N}}$ is Cauchy. Since (X,d) is a complete metric space, the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent to an element $x^* \in X$. We show first that $x^* \in \mathrm{Fix}(F)$. Indeed, we have

$$\begin{aligned}0 \le \mathrm{D}(x^*,\mathrm{F}(x^*)) &\le \mathrm{d}(x^*,x_{n+1}) + \mathrm{D}(x_{n+1},\mathrm{F}(x^*))\\ &\le \mathrm{d}(x^*,x_{n+1}) + \mathrm{H}(\mathrm{F}(x_n),\mathrm{F}(x^*))\\ &\le \mathrm{d}(x^*,x_{n+1}) + k\cdot\max\{\mathrm{d}(x_n,x^*),\mathrm{D}(x_n,\mathrm{F}(x_n),\mathrm{D}(x^*,\mathrm{F}(x^*)),\\ &\qquad \mathrm{D}(x_n,\mathrm{F}(x^*)),\mathrm{D}(x^*,\mathrm{F}(x_n))\}\\ &\le \mathrm{d}(x^*,x_{n+1}) + k\cdot\max\{\mathrm{d}(x_n,x^*),\mathrm{d}(x_n,x_{n+1}),\mathrm{D}(x^*,\mathrm{F}(x^*)),\\ &\qquad \mathrm{D}(x_n,\mathrm{F}(x^*)),\mathrm{d}(x^*,x_{n+1})\}.\end{aligned}$$

In the above inequality if we let $n \longrightarrow \infty$, then we get that

$$0 \le \mathrm{D}(x^*,\mathrm{F}(x^*)) \le k\mathrm{D}(x^*,\mathrm{F}(x^*)).$$

Thus $\mathrm{D}(x^*,\mathrm{F}(x^*)) = 0$ and so $x^* \in \mathrm{Fix}(F)$.

For proving (ii), let $x^* \in \mathrm{Fix}(F)$ and $1 < q < \frac{1}{2k}$. Then, there exists $u^* \in G(u^*)$ such that

$$\begin{aligned}\mathrm{d}(x^*,u^*) &\le q\mathrm{H}(\mathrm{F}(x^*),\mathrm{G}(u^*))\\ &\le q\mathrm{H}(\mathrm{F}(x^*),\mathrm{F}(u^*)) + q\eta\\ &\le qk\cdot\max\{\mathrm{d}(x^*,u^*),\mathrm{D}(u^*,\mathrm{F}(u^*)),\mathrm{D}(x^*,\mathrm{F}(u^*)),\mathrm{D}(u^*,\mathrm{F}(x^*))\} + q\eta\\ &\le qk\cdot\max\{\mathrm{d}(x^*,u^*),\mathrm{D}(u^*,\mathrm{G}(u^*))+\eta,\mathrm{D}(x^*,\mathrm{G}(u^*))+\eta\} + q\eta\\ &\le qk\cdot\max\{\mathrm{d}(x^*,u^*),\eta,\mathrm{d}(x^*,u^*)+\eta\} + q\eta\\ &\le qk(\mathrm{d}(x^*,u^*)+\eta) + q\eta,\end{aligned}$$

thus we obtain

$$d(x^*,u^*) \leq \frac{(1+k)q}{1-kq}\eta.$$

Thus, the fixed point equation with a multi-valued Ćirić type operator has the data dependence property.

Concerning conclusion (iii), in order to prove that the fixed point equation for F is well-posed, we take the sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that $D(u_n,F(u_n)) \longrightarrow 0$, as $n \longrightarrow \infty$. Then, we have $d(u_n,x^*) \leq D(u_n,F(u_n)) + H(F(u_n),F(x^*))$. Furthermore, we can write:

$$\begin{aligned} d(u_n,x^*) &\leq D(u_n,F(u_n)) + k\cdot\max\{d(u_n,x^*),D(u_n,F(u_n)),D(x^*,F(x^*)),\\ &\qquad\qquad D(x^*,F(u_n)),D(u_n,F(x^*))\}\\ &\leq D(u_n,F(u_n)) + k\cdot\max\{d(u_n,x^*),d(u_n,x^*)+D(x^*,F(u_n)),\\ &\qquad\qquad D(x^*,F(u_n)),D(u_n,F(x^*))\}\\ &\leq D(u_n,F(u_n)) + k(d(u_n,x^*)+D(x^*,F(u_n)))\\ &\leq D(u_n,F(u_n)) + k(2d(u_n,x^*)+D(u_n,F(u_n))), \end{aligned}$$

implying

$$d(u_n,x^*) \leq \frac{1+k}{1-2k}D(u_n,F(u_n)) \longrightarrow 0, n \longrightarrow \infty.$$

Regarding (iv), we will show that the operator $F\colon X \to P(X)$ has the Ostrowski property. Let us take the sequence $(u_n)_{n\in\mathbb{N}} \subset X$ such that

$$d(u_{n+1},x^*) \leq D(u_{n+1},F(u_n)) + D(F(u_n),x^*). \tag{3.1}$$

We take separately $D(F(u_n),x^*)$ from the above inequality and we have that

$$\begin{aligned} D(F(u_n),x^*) = H(F(u_n),F(x^*)) &\leq k\cdot\max\{d(u_n,x^*),D(u_n,F(u_n)),D(x^*,F(x^*)),\\ &\qquad D(x^*,F(u_n)),D(u_n,F(x^*))\}\\ &\leq k(d(u_n,x^*)+D(x^*,F(u_n))). \end{aligned}$$

Thus $D(F(u_n),x^*) \leq \frac{k}{1-k}d(u_n,x^*)$ and denote $\alpha := \frac{k}{1-k} < 1$. We replace this result in the relation (3.1) and it follows that

$$\begin{aligned} d(u_{n+1},x^*) &\leq D(u_{n+1},F(u_n)) + \alpha d(u_n,x^*)\\ &\leq D(u_{n+1},F(u_n)) + \alpha D(u_n,F(u_{n-1})) + \alpha^2 d(u_{n-1},x^*)\\ &\leq \cdots \leq D(u_{n+1},F(u_n)) + \alpha D(u_n,F(u_{n-1})) + \alpha^2 d(u_{n-1},x^*) + \cdots\\ &\quad + \alpha^n D(u_1,F(u_0)) + \alpha^{n+1} d(u_0,x^*)\\ &= \sum_{k=0}^{n} \alpha^{n-k} D(u_{k+1},F(u_k)) + \alpha^{n+1} d(u_0,x^*) \end{aligned}$$

Since $\alpha < 1$, using Cauchy's lemma (see 1), we get $\mathrm{d}(u_{n+1}, x^*) \longrightarrow 0$. □

We will now give a theorem that shows that, under an additional condition, the fixed point set and the strict fixed point set of a multi-valued Ćirić type operator coincide.

Theorem 6. *Let* (X, d) *be a complete metric space. Let* $\mathrm{F}\colon X \to P_{cl}(X)$ *be a multi-valued Ćirić type operator with constant* $k < 1$. *Suppose that* $\mathrm{SFix}(\mathrm{F}) \neq \varnothing$. *Then* $\mathrm{Fix}(\mathrm{F}) = \mathrm{SFix}(\mathrm{F}) = \{x^*\}$.

Proof. We will prove that F has a unique fixed point in X. Since $\mathrm{SFix}(\mathrm{F}) \neq \varnothing$ we know that there exists $x^* \in X$ such that $\mathrm{F}(x^*) = \{x^*\}$. We suppose that there exists $z \in \mathrm{Fix}(\mathrm{F})$ such that $z \neq x^*$. We have

$$\begin{aligned} \mathrm{d}(x^*, z) &\leq \mathrm{H}(\mathrm{F}(x^*), \mathrm{F}(z)) \\ &\leq k \max\{\mathrm{d}(z, x^*), \mathrm{D}(z, \mathrm{F}(z)), \mathrm{D}(x^*, \mathrm{F}(x^*)), \mathrm{D}(x^*, \mathrm{F}(z)), \mathrm{D}(z, \mathrm{F}(x^*))\} \\ &\leq k\mathrm{d}(z, x^*). \end{aligned}$$

This is a contradiction for $k < 1$. Therefore $\mathrm{SFix}(F) = \mathrm{Fix}(F) = \{x^*\}$. □

Now we will prove a local fixed point theorem.

Theorem 7. *Let* (X, d) *be a complete metric space,* $x_0 \in X$ *and* $r > 0$. *We consider the multi-valued operator* $\mathrm{F}\colon \tilde{B}(x_0; r) \to P_{cl}(X)$ *such that there exists* $k \in \left(0, \frac{1}{2}\right)$ *with*

$$\begin{aligned} \mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq k \max\{&\mathrm{d}(x, y), \mathrm{D}(x, \mathrm{F}(x)), \mathrm{D}(y, \mathrm{F}(y)), \mathrm{D}(x, \mathrm{F}(y)), \\ &\mathrm{D}(y, \mathrm{F}(x))\}, \text{ for all } x, y \in \tilde{B}(x_0; r). \end{aligned}$$

We also suppose that

$$\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} r.$$

Then, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ *of Picard iterates starting from* x_0 *which converges to a fixed point of* F.

Proof. Since $\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} r$ we get there exists $x_1 \in \mathrm{F}(x_0)$ such that

$$\mathrm{d}(x_0, x_1) < \frac{1-2k}{1-k} r.$$

Moreover,

$$\begin{aligned} \mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1)) &\leq k \max\{\mathrm{d}(x_0, x_1), \mathrm{D}(x_0, \mathrm{F}(x_0)), \mathrm{D}(x_1, \mathrm{F}(x_1)), \mathrm{D}(x_0, \mathrm{F}(x_1)), \\ &\qquad \mathrm{D}(x_1, \mathrm{F}(x_0))\} \\ &= k \max\{\mathrm{d}(x_0, x_1), \mathrm{D}(x_0, \mathrm{F}(x_0)), \mathrm{D}(x_1, \mathrm{F}(x_1)), \mathrm{D}(x_0, \mathrm{F}(x_1))\} \\ &\leq k \max\{\mathrm{d}(x_0, x_1), \mathrm{D}(x_1, \mathrm{F}(x_1)), \mathrm{d}(x_0, x_1) + \mathrm{D}(x_1, \mathrm{F}(x_1)))\} \\ &\leq k \max\{\mathrm{d}(x_0, x_1), \mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1)), \mathrm{d}(x_0, x_1) + \mathrm{H}(\mathrm{F}(x_0), \mathrm{F}(x_1))\} \end{aligned}$$

$$= k \max\left(\mathrm{d}(x_0,x_1) + \mathrm{H}(\mathrm{F}(x_0),\mathrm{F}(x_1))\right),$$

and thus

$$\mathrm{H}(\mathrm{F}(x_0),\mathrm{F}(x_1)) \le \frac{k}{1-k}\mathrm{d}(x_0,x_1) < \frac{k}{1-k}\frac{1-2k}{1-k}r.$$

We will now denote $h := \frac{k}{1-k}$, which immediately implies $\frac{1-2k}{1-k} = 1-h$, with $h \in (0,1)$. Hence,

$$\mathrm{H}(\mathrm{F}(x_0),\mathrm{F}(x_1)) < h(1-h)r.$$

Thus, there exists $x_2 \in \mathrm{F}(x_1)$ such that $\mathrm{d}(x_1,x_2) < h(1-h)r$. We assume

$$p(n): \text{ there exists } x_n \in \mathrm{F}(x_{n-1}) \text{ such that } \mathrm{d}(x_{n-1},x_n) < h^{n-1}(1-h)r,$$

and compute

$$\begin{aligned}
\mathrm{H}(\mathrm{F}(x_{n-1}),\mathrm{F}(x_n)) &\le k \max\{\mathrm{d}(x_{n-1},x_n), \mathrm{D}(x_{n-1},\mathrm{F}(x_{n-1})), \mathrm{D}(x_n,\mathrm{F}(x_n)),\\
&\qquad \mathrm{D}(x_{n-1},\mathrm{F}(x_n)), \mathrm{D}(x_n,\mathrm{F}(x_{n-1}))\}\\
&\le k \max\{\mathrm{d}(x_{n-1},x_n), \mathrm{D}(x_n,\mathrm{F}(x_n)), \mathrm{D}(x_{n-1},\mathrm{F}(x_n))\}\\
&\le k \max\{\mathrm{d}(x_{n-1},x_n), \mathrm{D}(x_n,\mathrm{F}(x_n)), \mathrm{d}(x_{n-1},x_n) + \mathrm{D}(x_n,\mathrm{F}(x_n))\}\\
&\le k(\mathrm{d}(x_{n-1},x_n) + \mathrm{H}(\mathrm{F}(x_{n-1}),\mathrm{F}(x_n))),
\end{aligned}$$

which implies

$$\mathrm{H}(\mathrm{F}(x_{n-1}),\mathrm{F}(x_n)) \le h\mathrm{d}(x_{n-1},x_n) < h^n(1-h)r.$$

Using the latter inequality, we get the existence of a point $x_{n+1} \in \mathrm{F}(x_n)$ such that the relation $p(n+1)$ holds, and therefore we proved $p(n)$ by mathematical induction. Again, by means of mathematical induction, one can easily prove the assumption

$$t(n): \mathrm{d}(x_0,x_n) < (1-h^n)r,$$

which shows that all the elements of the sequence $(x_n)_{n\in\mathbb{N}}$ are in the closed ball $\tilde{B}(x_0;r)$. Due to the following inequality

$$\begin{aligned}
\mathrm{d}(x_m,x_{m+n}) &\le \mathrm{d}(x_m,x_{m+1}) + \cdots + \mathrm{d}(x_{m+n-1},x_{m+n})\\
&\le h^m(1-h)(1+\cdots+h^{n-1})r \le h^m(1-h)\frac{1-h^n}{1-h}r \le h^m r,
\end{aligned}$$

the sequence $(x_n)_{n\in\mathbb{N}} \subset B(x_0;s)$ is Cauchy, thus convergent to a point $x^* \in \tilde{B}(x_0;r)$.

We finish the proof with showing $x^* \in \mathrm{Fix}(F)$, for which we compute

$$\begin{aligned}
\mathrm{D}(x^*,\mathrm{F}(x^*)) &\le \mathrm{d}(x^*,x_{n+1}) + \mathrm{H}(\mathrm{F}(x_n),\mathrm{F}(x^*))\\
&\le \mathrm{d}(x^*,x_{n+1}) + k\max\{\mathrm{d}(x_n,x^*), \mathrm{D}(x_n,\mathrm{F}(x_n)), \mathrm{D}(x^*,\mathrm{F}(x^*)),\\
&\qquad \mathrm{D}(x_n,\mathrm{F}(x^*)), \mathrm{D}(x^*,\mathrm{F}(x_n))\}\\
&\le \mathrm{d}(x^*,x_{n+1}) + k\max\{\mathrm{d}(x_n,x^*) + \mathrm{D}(x_n,\mathrm{F}(x_n)),\\
&\qquad \mathrm{d}(x_n,x^*) + \mathrm{D}(x^*,\mathrm{F}(x^*))\}
\end{aligned}$$

$$\leq \mathrm{d}(x^*, x_{n+1}) + k\mathrm{d}(x_n, x^*) + k\mathrm{d}(x_n, x_{n+1}) + k\mathrm{D}(x^*, \mathrm{F}(x^*)).$$

By considering $n \longrightarrow \infty$, we get the desired conclusion. □

By the above proof, we immediately get the following result.

Theorem 8. *Let* (X, d) *be a complete metric space,* $x_0 \in X$ *and* $r > 0$*. We consider the multi-valued operator* $\mathrm{F}\colon B(x_0; r) \to P_{cl}(X)$ *such that there exists* $k \in \left(0, \frac{1}{2}\right)$ *with*

$$\mathrm{H}(\mathrm{F}(x), \mathrm{F}(y)) \leq k \max\{\mathrm{d}(x,y), \mathrm{D}(x, \mathrm{F}(x)), \mathrm{D}(y, \mathrm{F}(y)), \mathrm{D}(x, \mathrm{F}(y)), \mathrm{D}(y, \mathrm{F}(x))\}, \text{ for all } x, y \in B(x_0; r).$$

We also suppose that

$$\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} r.$$

Then, there exists a sequence $(x_n)_{n \in \mathbb{N}}$ *of Picard iterates starting from* x_0 *which converges to a fixed point of* F*.*

Proof. Let $s \in (0, r)$ such that

$$\mathrm{D}(x_0, \mathrm{F}(x_0)) < \frac{1-2k}{1-k} s < \frac{1-2k}{1-k} r.$$

For our conclusion, we follow the approach given in the above proof for the operator $\mathrm{F}\colon \tilde{B}(x_0, s) \to P(X)$. □

Remark 1. It is an open question to obtain, by the above approach, a local fixed point theorem and related stability results for a multi-valued Ćirić type operators with constant $k \in (0, 1)$. For a different approach and a general existence result, see [7].

We now introduce the notion of a family of multi-valued Ćirić type operators with constant $k \in (0, 1)$.

Definition 6. Let (X, d) be a metric space. Then, the family $(\mathrm{F}_t)_{t \in [0,1]}$ (where $\mathrm{F}_t\colon Y \subseteq X \to P(X)$, for each $t \in [0, 1]$) is a family of multi-valued Ćirić type operators with constant k if $k \in (0, 1)$ and the following conditions are satisfied:

(i)

$$\mathrm{H}(\mathrm{F}_t(x_1), \mathrm{F}_t(x_2)) \leq k \max\{\mathrm{d}(x_1, x_2), \mathrm{D}(x_1, \mathrm{F}_t(x_1)), \mathrm{D}(x_2, \mathrm{F}_t(x_2)), \mathrm{D}(x_1, \mathrm{F}_t(x_2)), \mathrm{D}(x_2, \mathrm{F}_t(x_1))\}, \text{ for all } x_1, x_2 \in Y, t \in [0, 1].$$

(ii) $\mathrm{H}(\mathrm{F}_t(x), \mathrm{F}_s(x)) \leq |\phi(t) - \phi(s)|$, for all $t, s \in [0, 1]$ and $x \in Y$,

where $\phi\colon [0, 1] \to \mathbb{R}$ is strictly increasing and continuous.

Using the previous definitions, we can state, as an application of the multi-valued local fixed point theorem, a homotopy principle for multi-valued Ćirić type operators. The result generalizes a similar theorem given for multi-valued contraction, given by Frigon and Granas, see [5].

Theorem 9. *Let (X,d) be a complete metric space, $U \subset X$ be an open set and* $\mathrm{F}\colon [0,1]\times\overline{U} \to P_{cl}(X)$ *be a multi-valued operator with closed graph. We denote* $\mathrm{F}_t := \mathrm{F}(t,\cdot)$*, for* $t \in [0,1]$*. We suppose:*

- *(i)* $(\mathrm{F}_t)_{t\in[0,1]}$ *is a family of multi-valued Ćirić type operators with a constant* $k \in \left(0,\frac{1}{2}\right)$*;*
- *(ii)* $x \notin \mathrm{F}_t(x)$*. for all* $(t,x) \in [0,1]\times\partial U$*.*

Then F_0 *has a fixed point if and only if* F_1 *has a fixed point.*

Proof. Let $x^* \in U$ such that $x^* \in \mathrm{F}_0(x^*)$. We define the set

$$Q = \{(t,x) \in [0,1]\times U : x \in \mathrm{Fix}(\mathrm{F}_t)\}.$$

We observe that Q is nonempty, since $(0,x^*) \in Q$. Next, we consider the following partial order relation on Q

$$(t,x) \le (s,y) \text{ if and only if } t \le s \text{ and } \mathrm{d}(x,y) \le \frac{2\,(1-k)\,(\phi(s)-\phi(t))}{1-2k},$$

where ϕ is the function associated to the family $(\mathrm{F}_t)_{t\in[0,1]}$ of multi-valued Ćirić type operators with constant $k \in (0,1)$. We will use for Q the Kuratowski-Zorn Lemma (saying that if a partially ordered set Q has the property that every chain P in Q has an upper bound in Q, then the set Q contains at least one maximal element.)

We consider $P \subset Q$ a totally ordered subset (a chain in Q) and define

$$t^* = \sup\{t\colon (t,x) \in P\}.$$

We also consider a sequence $\{(t_n,x_n)\}$ in P such that

$$(t_n,x_n) \le (t_{n+1},x_{n+1}) \text{ and } t_n \longrightarrow t^*.$$

Then, taking into consideration the partial order relation on Q, we obtain that

$$\mathrm{d}(x_m,x_n) \le \frac{2(1-k)\,(\phi(t_m)-\phi(t_n))}{1-2k}, \text{ for all } m > n.$$

As a consequence, the sequence (x_n) is Cauchy, therefore it converges to an element $x^* \in \overline{U}$. Since F has closed graph, and it is fixed point free on the boundary of U, we get that $(t^*,x^*) \in Q$. Moreover, we have $(t,x) \le (t^*,x^*)$ for every $(t,x) \in P$, proving that (t^*,x^*) is an upper bound of P. Due to the Kuratowski-Zorn lemma, Q admits a maximal element $(t_0,x_0) \in Q$. Thus, x_0 is a fixed point of $\mathrm{F}_{t_0}(x_0)$.

We will show now, by contradiction, that $t_0 = 1$. We assume that $t_0 \neq 1$. Hence, there exist $t_1 \in (t_0,1]$ and $r > 0$ such that

$$0 < \frac{(1-k)(\phi(t_1)-\phi(t_0))}{1-2k} < r$$

and $B(x_0;r) \subset U$. We also have the following inequality

$$\mathrm{D}(x_0,\mathrm{F}_{t_1}(x_0)) \le \mathrm{D}(x_0,\mathrm{F}_{t_0}(x_0)) + \mathrm{H}(\mathrm{F}_{t_0}(x_0),\mathrm{F}_{t_1}(x_0)) \le |\phi(t_1)-\phi(t_0)|.$$

This implies

$$\mathrm{D}(x_0,\mathrm{F}_{t_1}(x_0)) < \frac{1-2k}{1-k}r.$$

Using the local fixed point theorem for multi-valued Ćirić type operators, we obtain that there exists a fixed point x_1 of F_{t_1} such that $\mathrm{d}(x_0,x_1) \le r$. Hence, (t_1,x_1) belongs to Q and $(t_0,x_0) < (t_1,x_1)$, which contradicts the maximality of (t_0,x_0).

Conversely, if $\mathrm{F}(1,\cdot)$ has a fixed point, then taking $t := 1-t$ in the previous approach, we get that $\mathrm{F}(0,\cdot)$ has a fixed point. The proof is complete. □

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