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THE AUSLANDER-REITEN CORRESPONDENCE ABOUT ∞-COTILTING OBJECTS IN EXTRIANGULATED CATEGORIES

ZHEN ZHANG AND SHANCE WANG

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Abstract. In this paper, we introduce ∞-cotilting objects in an extriangulated category. The definition given unifies those of Wakamatsu cotilting modules and of cotilting objects in an extriangulated category. We give a similar Bazzoni characterization and a partial Auslander-Reiten correspondence between ∞-cotilting objects and resolving subcategories in an extriangulated category, which recovers several different results from the literature. More importantly, extensions of the known results for Wakamatsu cotilting modules and for cotilting objects are very natural, but nontrivial, and we use a new method to prove them. At the same time, we give a dual version respect to the ∞-tilting objects.

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1. INTRODUCTION

Let R be an artin algebra. A finitely generated (left) R-module W is called (n-)cotilting if $Ext_R^i(W,W)=0$ for all $i \geq 1$, the injective dimension of W is at most n, and there is a long exact sequence $W_n \to \cdots \to W_1 \to W_0 \to DR \to 0$ with each W_i in $addW$ (the closure of W under finite direct sums and summands). The celebrated 'Auslander-Reiten correspondence' of [1] establishes a very natural bijection between (basic) cotilting modules and contravariantly finite resolving subcategories of the category of finitely generated R-modules. Note that there is also a dual version of this correspondence for tilting modules and covariantly finite coresolving subcategories.

One can also define (n-)tilting objects in the (big) module category of an arbitrary ring R. One of the many celebrated results in this direction is due to Bazzoni [2,

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Theorem 3.11], who showed that an R-module W is (n-)cotilting if and only if the following are equivalent for all R-modules M:

(1) There is an exact sequence $0 \to M \to W^0 \to \cdots \to W^n$;
(2) $\mathrm{Ext}^{\mathrm{i}}_{\mathrm{R}}(\mathrm{M},\mathrm{W}) = 0$ for all $i \geq 1$.

In the decades since they first appeared, there have been many successful attempts to extend the 'Auslander-Reiten correspondence' and/or 'Bazzoni characterization' to various new contexts and levels of generality. One example are the Wakamatsu cotilting modules, where the injective dimension of W and resolution of injective cogenerator E by $addW$ are both allowed to be infinite, see [5, Theorem 2.10]. Another are cotilting objects in triangulated categories, see [3, Theorems 3.7 and 3.15].

In the recent seminal work [6], Nakaoka and Palu unified the axioms of exact (so as a special case abelian) and triangulated categories to define extriangulated categories. The theory they developed has inspired many efforts at unifying constructions which have been made independently in exact (or abelian) and triangulated categories. One example has been tilting objects (and more generally subcategories), for which a generalized definition, an Auslander-Reiten correspondence, and a Bazzoni characterization are all successfully given in [8, Theorems 1and 2].

Motivated by this idea, in this paper, we takes this a step further to introduce ∞-cotilting (and dually ∞-tilting) objects in extriangulated categories. The definition given unifies those of Wakamatsu cotilting modules and of cotilting objects in an extriangulated category. Our main aim is to establish an Auslander-Reiten correspondence and Bazzoni characterization for these new objects.

The definition and main results in this paper are very natural, but nontrivial, extensions of the known results for Wakamatsu cotilting modules and for cotilting objects in extriangulated categories. In particular, the proof techniques are new.

2. Preliminaries

Throughout the article, $\mathscr{C}$ denotes an additive category. All subcategories considered are full additive subcategories closed under isomorphisms. We denote by $\mathscr{C}(A,B)$ the set of morphisms from A to B in $\mathscr{C}$. If $f \in \mathscr{C}(A,B)$, $g \in \mathscr{C}(B,C)$, we denote composition of f and g by gf. We recall the definition and some basic properties of extriangulated categories from [6] and [8].

Suppose that $\mathscr{C}$ is equipped with a biadditive functor $\mathbb{E} : \mathscr{C}^{op} \times \mathscr{C} \to Ab$, where Ab is the category of abelian groups. For any pair of objects $A,C \in \mathscr{C}$, an element $\delta \in \mathbb{E}(C,A)$ is called an $\mathbb{E}$-*extension*. Zero element $\delta \in \mathbb{E}(C,A)$ is called the spilt $\mathbb{E}$-extension.

For any $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$, since $\mathscr{C}$ and $\mathbb{E}$ are additive, we can define the $\mathbb{E}$-extension

$$\delta \oplus \delta' \in \mathbb{E}(C \oplus C', A \oplus A')$$

Since $\mathbb{E}$ is a bifunctor, for any $a \in \mathscr{C}(A,A')$ and $c \in \mathscr{C}(C',C)$, we have $\mathbb{E}$-extensions

$$\mathbb{E}(C,a)(\delta) \in \mathbb{E}(C,A'), \qquad \mathbb{E}(c,A)(\delta) \in \mathbb{E}(C',A.$$

We abbreviate $\mathbb{E}(C,a)(\delta)$ and $\mathbb{E}(c,A)(\delta)$ to $a_*\delta$ and $c^*\delta$, respectively.

Definition 2.1. *[6, Definition 2.3] A morphism $(a,c) : \delta \to \delta'$ of $\mathbb{E}$-extensions $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$ is a pair of morphisms $a \in \mathscr{C}(A,A')$ and $c \in \mathscr{C}(C,C')$ satisfying $a_*\delta = c^*\delta'$.*

Two sequences of morphisms $A \xrightarrow{x} B \xrightarrow{y} C$ and $A \xrightarrow{x'} B' \xrightarrow{y'} C$ in $\mathscr{C}$ are said to be equivalent if there exists an isomorphism $b \in \mathscr{C}(B,B')$ which makes the following diagram commutative.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \| & & \cong \big\downarrow b & & \| \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C \end{array}$$

We denote the equivalence class of $A \xrightarrow{x} B \xrightarrow{y} C$ by $[A \xrightarrow{x} B \xrightarrow{y} C]$.

For any $A,C \in \mathscr{C}$, we denote as $0 = [A \xrightarrow{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]} A \oplus C \xrightarrow{[0\ 1]} C]$.

Definition 2.2. *[6, Definition 2.9] Let $\mathfrak{s}$ be a correspondence which associates an equivalence class $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ to any $\mathbb{E}$-extension $\delta \in \mathbb{E}(C,A)$. This $\mathfrak{s}$ is called a* realization *of $\mathbb{E}$ if it satisfies the following condition.*

Let $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$ be any two $\mathbb{E}$-extension with $\mathfrak{s}(\delta) = [A \xrightarrow{x} B \xrightarrow{y} C]$ and $\mathfrak{s}(\delta') = [A' \xrightarrow{x'} B' \xrightarrow{y'} C']$. Then, for any morphism $(a,c) : \delta \to \delta'$, there exists a morphism $b \in \mathscr{C}(B,B')$ which makes the following diagram commutative.

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \big\downarrow a & & \big\downarrow b & & c \big\downarrow \\ A & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

In this case, we say the sequence $A \xrightarrow{x} B \xrightarrow{y} C$ realizes δ.

Remark that this condition does not depend on the choices of the representatives of the equivalence classes. In the above situation, we say the triplet (a,b,c) realizes (a,c).

Definition 2.3. *[6, Definition 2.10] Let $\mathscr{C}$, $\mathbb{E}$ be as above. A realization $\mathfrak{s}$ of $\mathbb{E}$ is said to be additive, if it satisfies the following conditions.*

(1) *for any $A,C \in \mathscr{C}$, the split $\mathbb{E}$-extension $0 \in \mathbb{E}(C,A)$ satisfies $\mathfrak{s}(0) = 0$;*
(2) *for any pair of $\mathbb{E}$-extension $\delta \in \mathbb{E}(C,A)$ and $\delta' \in \mathbb{E}(C',A')$, $\mathfrak{s}(\delta \oplus \delta') = \mathfrak{s}(\delta) \oplus \mathfrak{s}(\delta')$.*

Definition 2.4. *[6, Definition 2.12] We call the triplet $(\mathscr{C},\mathbb{E},\mathfrak{s})$ an externally triangulated category, or for short, extriangulated category if it satisfies the following conditions:*

(ET1) *$\mathbb{E}:\mathscr{C}^{op}\times\mathscr{C}\to Ab$ is a biadditive functor.*

(ET2) *$\mathfrak{s}$ is an additive realization of $\mathbb{E}$.*

(ET3) *Let $\delta\in\mathbb{E}(C,A)$ and $\delta'\in\mathbb{E}(C',A')$ be any pair of $\mathbb{E}$-extensions, realized as $\mathfrak{s}(\delta)=[A\xrightarrow{x}B\xrightarrow{y}C]$ and $\mathfrak{s}(\delta')=[A'\xrightarrow{x'}B'\xrightarrow{y'}C']$. For any commutative square in $\mathscr{C}$,*

$$\begin{array}{ccccc} A & \xrightarrow{x} & B & \xrightarrow{y} & C \\ \downarrow{\scriptstyle a} & & \downarrow{\scriptstyle b} & & \\ A' & \xrightarrow{x'} & B' & \xrightarrow{y'} & C' \end{array}$$

there exists a morphism $(a,c):\delta\to\delta'$ which is realized by (a,b,c).

(ET3)^{op} *Dual of $(ET3)$.*

(ET4) *Let (A,δ,D) and (B,δ',F) be two $\mathbb{E}$-extensions realized by $A\xrightarrow{f}B\xrightarrow{f'}D$ and $B\xrightarrow{g}C\xrightarrow{g'}F$, respectively. Then there exist an object $\mathbb{E}\in\mathscr{C}$, a commutative diagram*

$$\begin{array}{ccccc} A & \xrightarrow{f} & B & \xrightarrow{f'} & D \\ \| & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle d} \\ A & \xrightarrow{h} & C & \xrightarrow{h'} & E \\ & & \downarrow{\scriptstyle g'} & & \downarrow{\scriptstyle e} \\ & & F & = & F \end{array}$$

in $\mathscr{C}$ and an $\mathbb{E}$-extension $\delta''\in\mathscr{C}(E,A)$ realized by $A\xrightarrow{h}C\xrightarrow{h'}E$, which satisfy the following compatibilities:

(i) *$D\xrightarrow{d}E\xrightarrow{e}F$ realizes $\mathbb{E}(F,f')(\delta')$,*

(ii) *$\mathbb{E}(d,A)(\delta'')=\delta$,*

(iii) *$\mathbb{E}(E,f)(\delta'')=\mathbb{E}(e,B)(\delta')$.*

$(ET4^{op})$ *Dual of $(ET4)$.*

For an extriangulated category $\mathscr{C}$, we use the following notation [6, 8].

- A sequence $A\xrightarrow{a}B\xrightarrow{b}C$ is called *conflation* if it realizes some $\mathbb{E}$-extension $\delta\in\mathbb{E}(C,A)$, in which case, the morphism a is called an *inflation*, the morphism b is called an *deflation* and we call $A\xrightarrow{a}B\xrightarrow{b}C\overset{\delta}{\dashrightarrow}$ an $\mathbb{E}$-*triangle*.
- Let $A\xrightarrow{a}B\xrightarrow{b}C\overset{\delta}{\dashrightarrow}$ be an $\mathbb{E}$-triangle, A is called the *CoCone* of the deflation b: $B\to C$, and denote it by CoCone(b); C is called the *Cone* of the inflation

$a\colon A \to B$, and denote it by Cone(a). Note that the *CoCone* of a deflation and the *Cone* of an inflation are well-defined by [6, Remark 3.10].

Remark 2.5. *Let $\mathscr{C}$ be an extriangulated category.*

(1) *[6, Remark 2.16] Both inflations and deflations are closed under composition.*
(2) *A subcategory $\mathscr{T}$ of $\mathscr{C}$ is called extension-closed if for any $\mathbb{E}$-triangle $A \xrightarrow{a} B \xrightarrow{b} C \overset{\delta}{\dashrightarrow}$ with A, $C \in \mathscr{T}$ we have $B \in \mathscr{T}$.*

The following condition is analogous to the weak idempotent completeness in exact category (see [6, Condition 5.8]).

(Condition (WIC)) Consider the following conditions.

(1) Let $f \in \mathscr{C}(A,B)$, $g \in \mathscr{C}(B,C)$ be any composable pair of morphisms. If gf is an inflation, then so is f.
(2) Let $f \in \mathscr{C}(A,B)$, $g \in \mathscr{C}(B,C)$ be any composable pair of morphisms. If gf is an deflation, then so is g.

Definition 2.6. *[6, Definition 3.23] Let $\mathscr{C}$ be an extriangulated category. An object I is called* injective *if for any $\mathbb{E}$-triangle $A \xrightarrow{x} B \xrightarrow{y} C \overset{\delta}{\dashrightarrow}$ and any morphism $c \in \mathscr{C}(A,I)$, there exists a morphism $b \in \mathscr{C}(B,I)$, satisfying $b \circ x = c$.*

The subcategory consisting of injective (resp., projective) objects in $\mathscr{C}$ is denoted by $\mathscr{I}$ (resp., $\mathscr{P}$).

Definition 2.7. *Let $\mathscr{X}$ be a subcategories of $\mathscr{C}$ and E an object of $\mathscr{X}$. We call E is* an $\mathbb{E}$-injective cogenerator *of $\mathscr{X}$ if $\mathbb{E}(\mathscr{X},E) = 0$ and for any object $X \in \mathscr{X}$, there is an $\mathbb{E}$-triangle $X \to E^0 \to X^1 \dashrightarrow$ with $X^1 \in \mathscr{X}$ and $E^0 \in addE$.*

The $\mathbb{E}$-projective generator *can be defined dually.*

In particular, we know that an $\mathbb{E}$-injective cogenerator of $\mathscr{C}$ is an injective object and an $\mathbb{E}$-projective generator of $\mathscr{C}$ is an projective object.

It is easy to show $\mathbb{E}$-injective cogenerator (resp., $\mathbb{E}$-projective generator) of $\mathscr{X}$ is unique under isomorphism.

Definition 2.8. *A subcategory $\mathscr{X} \subset \mathscr{C}$ is called* coresolving *if it contains $\mathscr{I}$, closed under extensions and Cones of inflations.* Resolving subcategory *can be defined dually.*

Definition 2.9. *[6, Definition 3.25] Let $(\mathscr{C},\mathbb{E},\mathfrak{s})$ be an extriangulated category. If for any object $A \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle $A \to I \to A_1 \overset{\delta}{\dashrightarrow}$, with $I \in \mathscr{I}$, then we say the extriangulated category $(\mathscr{C},\mathbb{E},\mathfrak{s})$ has enough injectives. Dually, if for any object $C \in \mathscr{C}$, there exists an $\mathbb{E}$-triangle $C_1 \to P \to C \overset{\sigma}{\dashrightarrow}$, with $P \in \mathscr{P}$, then we say the extriangulated category $(\mathscr{C},\mathbb{E},\mathfrak{s})$ has enough projectives.*

Liu and Nakaoka [4, 5.1 and 5.2] defined the higher extension groups in an extriangulated category having enough projectives and injectives. They showed the following result.

Lemma 2.10. *[4, Proposition 5.2] Let* $A \xrightarrow{f} B \xrightarrow{g} C \overset{\delta}{\dashrightarrow}$ *be an* $\mathbb{E}$*-triangle. For any object* $X \in \mathscr{C}$*, there are long exact sequences*

$$\cdots \to \mathbb{E}^i(X,A) \xrightarrow{f_*} \mathbb{E}^i(X,B) \xrightarrow{g_*} \mathbb{E}^i(X,C) \to \mathbb{E}^{i+1}(X,A) \xrightarrow{f_*} \mathbb{E}^{i+1}(X,B) \xrightarrow{g_*} \mathbb{E}^{i+1}(X,C) \to \cdots$$

$$\cdots \to \mathbb{E}^i(C,X) \xrightarrow{f^*} \mathbb{E}^i(B,X) \xrightarrow{g^*} \mathbb{E}^i(A,X) \to \mathbb{E}^{i+1}(C,X) \xrightarrow{f^*} \mathbb{E}^{i+1}(B,X) \xrightarrow{g^*} \mathbb{E}^{i+1}(C,X) \to \cdots$$

In particularly, there exist long exact sequences

$$\mathscr{C}(X,A) \xrightarrow{\mathscr{C}(X,f)} \mathscr{C}(X,B) \xrightarrow{\mathscr{C}(X,g)} \mathscr{C}(X,C) \xrightarrow{(\delta_\sharp)_X} \mathbb{E}(X,A) \xrightarrow{f_*} \mathbb{E}(X,B) \xrightarrow{g_*} \mathbb{E}(X,C) \to \cdots$$

$$\mathscr{C}(C,X) \xrightarrow{\mathscr{C}(g,X)} \mathscr{C}(B,X) \xrightarrow{\mathscr{C}(f,X)} \mathscr{C}(A,X) \xrightarrow{(\delta^\sharp)_X} \mathbb{E}(C,X) \xrightarrow{g^*} \mathbb{E}(B,X) \xrightarrow{f^*} \mathbb{E}(A,X) \to \cdots$$

For a subcategory $\mathscr{X} \subseteq \mathscr{C}$, we define

$$\mathscr{X}^\perp = \{Y \in \mathscr{C} \mid \mathbb{E}^i(X,Y) = 0, \forall i \geq 1,\ X \in \mathscr{X}\}.$$

and

$$^\perp\mathscr{X} = \{Y \in \mathscr{C} \mid \mathbb{E}^i(Y,X) = 0, \forall i \geq 1,\ X \in \mathscr{X}\}.$$

Definition 2.11. *A subcategory* $\mathscr{X} \subseteq \mathscr{C}$ *is called self-orthogonal provided that*

$$\mathscr{X} \subseteq \mathscr{X}^\perp$$

Definition 2.12. *[8, lemma 2] An* $\mathbb{E}$*-triangle sequence in* $\mathscr{C}$ *is defined as a sequence*

$$\cdots \to X_{n+1} \xrightarrow{d_{n+1}} X_n \xrightarrow{d_n} X_{n-1} \to \cdots$$

such that for any n, there are $\mathbb{E}$*-triangles* $K_{n+1} \xrightarrow{g_n} X_n \xrightarrow{f_n} K_n \overset{\delta_n}{\dashrightarrow}$ *and the differential* $d_n = g_{n-1}f_n$*, where* g_n *is an inflation and* f_n *is a deflation.*

The symbol $\hat{\mathscr{X}}_n$ (resp., $\check{\mathscr{X}}_n$) denotes the subcategory of objects $A \in \mathscr{C}$ such that there exists an E-triangle sequence

$$X_n \to X_{n-1} \to \cdots \to X_0 \to A (\text{resp.}, A \to X^0 \to X^1 \to \cdots \to X^n)$$

with each X_i (resp., X^i) is contained in $\mathcal{X}$.

We denote the union of all $\hat{\mathscr{X}}_n$ (resp., $\check{\mathscr{X}}_n$) for all nonnegative n by $\hat{\mathscr{X}}$ (resp., $\check{\mathscr{X}}$).

3. ∞-COTILTING (RESP., TILTING) OBJECTS

In this section, we always assume that

(1) $\mathscr{C} = (\mathscr{C}, \mathbb{E}, \mathfrak{s})$ is an extriangulated category which admits products and direct sums;
(2) $\mathscr{C}$ admits an $\mathbb{E}$-injective cogenerator E and an $\mathbb{E}$-projective generator Q, see Definition 2.7;
(3) $\mathscr{C}$ satisfies Condition(WIC).

Let W be an object. Denote $addW$ by the closure of W under finite direct sums and summands. Firstly, we introduce two important subcategories which are contained in ${}^{\perp}W$ and $W^{\perp}$, respectively. Denote by ${}_{W}\mathcal{X}$ the subcategory of all objects $A \in W^{\perp}$ such that there is an infinite $\mathbb{E}$-triangle sequence in $\mathscr{C}$

$$\cdots \to W_2 \overset{d_2}{\to} W_1 \overset{d_1}{\to} W_0 \overset{d_0}{\to} A$$

with $W_i \in addW$ and $\mathrm{CoCone}(f_i) \in W^{\perp}$,for $i \geq 0$, where $f_0 = d_0$ and f_i is defined in Definition 2.12. Dually, denote by $\mathcal{X}_W$ the subcategory of all objects $A \in {}^{\perp}W$ such that there is an infinite $\mathbb{E}$-triangle sequence in $\mathscr{C}$

$$A \overset{d^0}{\to} W^0 \overset{d^1}{\to} W^1 \overset{d^2}{\to} W^2 \to \cdots$$

with $W^i \in addW$ and $\mathrm{Cone}(g^i) \in {}^{\perp}W$ for $i \geq 0$, where $g^0 = d^0$ and g^i is defined in Definition 2.12.

Lemma 3.1. *If W is a self-orthogonal object. Then*

(1) *the subcategory ${}_{W}\mathcal{X}$ is closed under extensions, direct summands , and Cones of inflations;*
(2) *the subcategory $\mathcal{X}_W$ is closed under extensions, direct summands , and CoCones of deflations.*

Proof. It follows from [8, Lemma 8]. □

Now, we give the definition of ∞-cotilting (resp., tilting) object in an extriangulated category.

Definition 3.2. *Let W be a non-zero object and E an $\mathbb{E}$-injective cogenerator of $\mathscr{C}$. We call W an ∞-cotilting object if the following conditions are satisfied:*

(1) *W is a self-orthogonal object in $\mathscr{C}$;*
(2) *$E \in {}_{W}\mathcal{X}$.*

If, moreover, W has injective dimension n for some positive integer n and $E \in (\widehat{addW})_n$, W is called *n-cotilting object*.

Definition 3.3. *Let T be a non-zero object and Q an $\mathbb{E}$-projective generator of $\mathscr{C}$. We call T an ∞-tilting object if the following conditions are satisfied:*

(1) *T is a self-orthogonal object in $\mathscr{C}$;*

(2) $Q \in \mathcal{X}_T$.

If, moreover, T has projective dimension n for some positive integer n and $Q \in (add\check{d}T)_n$, T is called *n-tilting object*.

Example 3.4. (1) *Let $\mathscr{C}$ be a triangulated category and W a n-cotilting (resp., n-tilting) object, then W is obvilusely an ∞-cotilting (resp. ∞-tilting) object with finite injective (resp., projective) dimension in our sense.*

(2) *Let $\mathscr{C}$ be an extriangulated category. If $addW$ is a cotilting (resp.,tilting) subcategory in the sense of [8, Definitions 8 and 7], then W is an ∞-cotilting (resp. ∞-tilting) object object with finite projective (resp., injective) dimension in our sense.*

(3) *Let Λ be an artin algebra. Consider the finitely generated left Λ-module category. Then W is Wakamatsu cotilting (resp., Wakamatsu tilting) in the sense of [5, section 2] if and only if W is ∞-cotilting (resp., ∞-tilting) object in our sense.*

Proof. We only prove (3). Let W be an ∞-cotilting object in our sense. Since $D\Lambda$ is an injective module, $D\Lambda \in {}_W\mathcal{X}$ by Definition 3.2. So W is a Wakamatsu cotilting module. Conversely, let W be a Wakamatsu cotilting module. By [5, Propositions 2.1 and 2.2], we have ${}_W\mathcal{X} = S^{\perp}$ for some left Λ module S. So for any injective Λ module E, we have $E \in S^{\perp} = {}_W\mathcal{X}$. Hence W is an ∞-cotilting object in our sense.

Dually, let W be an ∞-tilting object in our sense. Clearly $\Lambda \in \mathcal{X}_W$. So So W is a Wakamatsu tilting module. Conversely, let W be a Wakamatsu tilting module. Since $\mathscr{P} \subseteq add\Lambda$, $\mathscr{P} \subseteq \mathcal{X}_W$ by lemma 3.1. So W is an ∞-tilting object in our sense. □

By Definition 3.2, if W is an ∞-cotilting object, there exists an $\mathbb{E}$-triangle sequence

$$\cdots \to W_1 \xrightarrow{d_1} W_0 \xrightarrow{d_0} E \qquad (\dagger)$$

with $W_i \in addW$.

By Definition 2.12 we can break it into several $\mathbb{E}$-triangle:

$$K_i \xrightarrow{g_i} W_i \xrightarrow{f_i} K_{i-1} \dashrightarrow$$

where $d_i = g_{i-1}f_i$ for all $i \geq 1$.

In the following we always denote $\mathrm{CoCone}(d_i)$ by K_i for each $i \geq 1$ and denote $\mathrm{CoCone}(d_0)$ by K_0, then $K_i \in W^{\perp}$ for all $i \geq 0$. Dually, if T is an ∞-tilting object, then there exists an $\mathbb{E}$-triangle sequence

$$Q \xrightarrow{d^0} T^0 \xrightarrow{d^1} T^1 \to \cdots \qquad (\ddagger)$$

with $T^i \in addT$.

In the following, we always denote $\mathrm{Cone}(d^i)$ by L^{i+1} for each $i \geq 0$ and denote $\mathrm{Cone}(d^0)$ by L^0, then $L^i \in {}^{\perp}T$ for all $i \geq 0$. Let $\mathscr{X}$ be a subcategory of $\mathscr{C}$ and A an object, a morphism $f\colon X_A \to A$ is called *right $\mathscr{X}$-approximation* of A if $X_A \in \mathscr{X}$ and the morphism $\mathscr{C}(X,f): \mathscr{C}(X,X_A) \to \mathscr{C}(X,A)$ is epic for any object $X \in \mathscr{X}$. A *left $\mathscr{X}$-approximation* of A is defined dually.

From here to the end of this section, we always assume that any $M \in {}^{\perp}W$ admits a left $addW$-approximation and any $M \in W^{\perp}$ admits a right $addW$-approximation.

Lemma 3.5. *Let W be an object in $\mathscr{C}$.*

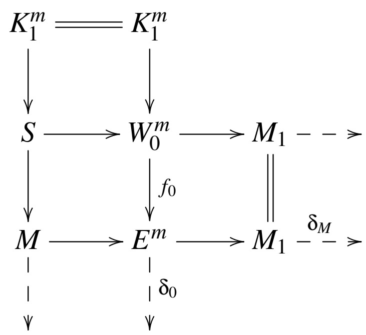
(1) *If W is an ∞-cotilting object. Then ${}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W) \subseteq \mathcal{X}_W$ for $i \geq 0$.*
(2) *If W is an ∞-tilting object. Then $(\bigoplus_{i\in\mathbb{N}} L^i \oplus W)^{\perp} \subseteq {}_W\mathcal{X}$ for $i \geq 0$.*

Proof. We only prove (1) and the proof of (2) is dually. Let $M \in {}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W)$, then $M \in {}^{\perp}W$ and $M \in {}^{\perp}K_i$ for each $i \geq 0$. Since E is the injective cogenerator for $\mathscr{C}$, there exists an $\mathbb{E}$-triangle $M \to E^m \to M_1 \overset{\delta_M}{\dashrightarrow}$ for some nonnegative integer m by Definition 2.7. Since W is an ∞-tilting object, $E \in {}_W\mathcal{X}$. So there exist an $\mathbb{E}$-triangle sequence following from the $\mathbb{E}$-triangle sequence $(\dagger)$:

$$\cdots \to W_1^m \to W_0^m \to E^m$$

with $W_i \in addW$.

So by ET(3) we have the following commutative diagram:



Since $M \in {}^{\perp}K_1$, the $\mathbb{E}$-triangle in the first column splits. So there exists an inflation $M \to S$. By the Condition (WIC)), there exists an inflation $y: M \to W_0^m$. By the assumption, M admits a left $addW$-approximation. Let $x: M \to X$ be a left $addW$-approximation. By [4, Proposition 1.20], we have an $\mathbb{E}$-triangle $M \overset{\binom{-x}{y}}{\to} X \oplus W_0^m \to M_1 \dashrightarrow$. Since x is a left $addW$-approximation, we can easily show $\binom{-x}{y}$ is a left $addW$-approximation of M. So $M_1 \in {}^{\perp_1}W$. Since $M \in {}^{\perp}W$ and $X \oplus W^m \in addW$, $M_1 \in {}^{\perp}W$. Applying the functor $\mathscr{C}(M_1,-)$ to the $\mathbb{E}$-triangles $K_{i+1} \to W_i \to K_i$ which is following from $(\dagger)$ for each $i \geq 1$, respectively, we get

$$(1) \qquad \mathbb{E}^k(M_1,K_i) \cong \mathbb{E}^{k+1}(M_1,K_{i+1})$$

for any $k \geq 1$ and $i \geq 0$ by Lemma 2.10. Applying the functor $\mathscr{C}(-,K_{i+1})$ to the $\mathbb{E}$-triangle $M \to X \oplus W_0^m \to M_1 \dashrightarrow$. Since $K_i \in W^{\perp}$ and $X \oplus W_0^m \in addW$, we get

$$(2) \qquad \mathbb{E}^{k+1}(M_1,K_{i+1}) \cong \mathbb{E}^k(M,K_{i+1})$$

for any $k \geq 1$ and $i \geq 0$. Hence $\mathbb{E}^k(M_1, K_i) \cong \mathbb{E}^k(M, K_{i+1}) = 0$ for each $i \geq 0$ and $k \geq 1$. So $M_1 \in {}^{\perp}K_i$ and $M_1 \in {}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W)$. Repeating the same argument for M_1 in the $\mathbb{E}$-triangle $M \to X \oplus W_0^m \to M_1 \dashrightarrow$ and so on, we obtain that $M \in \mathcal{X}_W$. □

By Lemma 3.5, we can get that ∞-cotilting object actually generalize at the same time both cotilting and tilting objects.

Corollary 3.6. *W is an ∞-cotilting object if and only if it is an ∞-tilting object.*

Proof. If W is an ∞-cotilting object, then W is self-orthogonal and ${}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W) \subseteq \mathcal{X}_W$ by Lemma 3.5. But $\mathscr{P} \subseteq {}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W)$. So $\mathscr{P} \subseteq \mathcal{X}_W$ and thus W is ∞-tilting. Conversely, when W is an ∞-tilting object, we have W is self-orthogonal and $(\bigoplus_{i\in\mathbb{N}} L^i \oplus W)^{\perp} \subseteq {}_W\mathcal{X}$. But $\mathscr{I} \subseteq (\bigoplus_{i\in\mathbb{N}} L^i \oplus W)^{\perp}$. So $\mathscr{I} \subseteq {}_W\mathcal{X}$ and thus W is ∞-cotilting. □

When W is an n-cotilting object for some positive integer, there exist an $\mathbb{E}$-triangle sequence $W_n \to W_{n-1} \to \cdots \to W_0 \to E$ with $W_i \in addW$. Applying $\mathscr{C}(M,-)$ for any $M \in {}^{\perp}W$ to every $\mathbb{E}$-triangle $K_{i+1} \to W_{i+1} \to K_i \dashrightarrow$ for $0 \leq i \leq n-1$ and $K_n = W_n$. We get $\mathbb{E}^j(M, K_i) = 0$ for any $0 \leq i \leq n$ and $j \geq 1$. So ${}^{\perp}(\prod_{i=0}^{n} K_i \oplus W) = {}^{\perp}W$.

Hence it is reasonable to use ${}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W)$ instead of ${}^{\perp}W$ when W is an ∞-cotilting object. The dual statement holds for n-tilting objects.

So we can prove the similar Bazzoni characterization of ∞-cotilting (tilting) objects in an extriangulated category.

Theorem 3.7. *Let W be a self-orthogonal object. Then*

(1) *W is an ∞-cotilting object if and only if $\mathcal{X}_W = {}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W)$;*

(2) *T is an ∞-tilting object if and only if $(\bigoplus_{i\in\mathbb{N}} L^i \oplus T)^{\perp} = {}_T\mathcal{X}$.*

Proof. We only prove (1) and the proof of (2) is dually. For the if part, clearly $\mathscr{P} \subseteq {}^{\perp}(\prod_{i\in\mathbb{N}} K_i \prod W)$. Hence $\mathscr{P} \subseteq \mathcal{X}_W$. So W is an ∞-tilting object by Definition 3.3. And thus W is ∞-cotilting by Corollary 3.6.

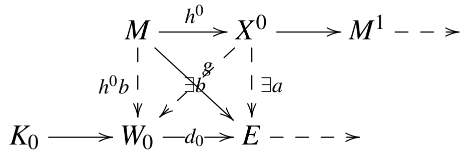
For the only if part, since W is an ∞-cotilting object, ${}^{\perp}(\prod_{i\in\mathbb{N}} K_i \oplus W) \subseteq \mathcal{X}_W$ by Lemma 3.5. Let $M \in \mathcal{X}_W$, then $M \in {}^{\perp}W$ and there exists an $\mathbb{E}$-triangle $M \overset{h^0}{\to} X^0 \to M^1 \dashrightarrow$ with $X^0 \in addW$ and $M^1 \in \mathcal{X}_W$ by the argument at the beginning of Section 3. We only need to show $M \in {}^{\perp}K_i$ for each $i \geq 0$.

Since W is ∞-cotilting, there exist an $\mathbb{E}$-triangle sequence $\cdots \to W_1 \overset{d_1}{\to} W_0 \overset{d_0}{\to} E$ with $W_i \in addW$ and $K_i = \mathrm{CoCone}(d_i) \in W^{\perp}$ for any $i \geq 0$ by Definition 3.1. For any morphism $g \in \mathscr{C}(M,E)$. Since E is an $\mathbb{E}$-injective object, there exists a morphism $a \in \mathscr{C}(X^0,E)$ such that $g = ah^0$. Applying the functor $\mathscr{C}(X^0,-)$ to the $\mathbb{E}$-triangle $K_0 \to W_0 \to E \dashrightarrow$, we get an exact sequence

$$\mathscr{C}(X^0, W_0) \overset{\mathscr{C}(X^0,d_0)}{\longrightarrow} \mathscr{C}(X^0, E) \to \mathbb{E}(X^0, K_0).$$

Since $K_0 \in W^{\perp}$, the morphism $\mathscr{C}(X^0,d_0)$ is epic. Thus there exists a morphism $b \in \mathscr{C}(X^0, W_0)$ such that $a = d_0 b$. So we get a morphism $h^0 b \in \mathscr{C}(M, W^0)$ such that

$g = (d_0 b)h^0$. Thus $\mathscr{C}(M, d_0)$ is epic.



Applying the functor $\mathscr{C}(M, -)$ to the $\mathbb{E}$-triangle $K_0 \to W_0 \to E \dashrightarrow$, we get an exact sequence

$$\mathscr{C}(M, W_0) \overset{\mathscr{C}(M,d_0)}{\longrightarrow} \mathscr{C}(M, E) \to \mathbb{E}(M, K_0) \to \mathbb{E}(M, W_0).$$

Since $M \in {}^{\perp}W$, we get $\mathbb{E}(M, K_0) = 0$, i.e. $M \in {}^{\perp_1}K_0$. Thus $M \in {}^{\perp}K_0$. Since $M^1 \in \mathcal{X}_W \subseteq {}^{\perp}W$, we can use the same argument to M_1. So we get $M_1 \in {}^{\perp}K_0$. Moreover, by applying the functor $\mathscr{C}(-, K_{t+1})$ to the $\mathbb{E}$-triangle $M \to X^0 \to M^1 \dashrightarrow$ and the functor $\mathscr{C}(M^1, -)$ to the $\mathbb{E}$-triangle $K_{t+1} \to W_{t+1} \to K_t \dashrightarrow$, we get

$$\mathbb{E}^j(M^1, K_t) \cong \mathbb{E}^{j+1}(M^1, K_{t+1}) \cong \mathbb{E}^j(M, K_{t+1})$$

for any $t \geq 0$ and $j \geq 1$. So $\mathbb{E}^j(M^1, K_t) \cong \mathbb{E}^j(M, K_{t+1})$ for any $t \geq 0$ and $j \geq 1$. By the induction, we conclude that $\mathbb{E}^j(M, K_t) = 0$ for any $t \geq 0$ and $j \geq 1$. Hence $M \in {}^{\perp}K_t$. So $M \in {}^{\perp}(\prod_{i \geq 0} K_i)$. And the result follows. □

In [3, Theorem 3.15], the authors obtained the Auslander-Reiten correspondence for tilting objects in an triangulated category. In that setting, $\mathscr{X}$ is a coresolving subcategory with an $\mathbb{E}$-projective generator such that $\mathscr{C} = \check{\mathscr{X}}_n$ and the latter condition is actually essential to prove the Auslander-Reiten correspondence. In the tilting case, the equality $\mathscr{C} = \check{\mathscr{T}}_n$ follows from the fact that $\mathscr{T}$ has finite projective dimension, which fails in general for the case of the projective dimension is infinite, see [7, Example 3.1] in mod$R$, where R is an artin algebra.

Conscious of this central difference between the two contexts, we can only prove a partial Auslander-Reiten correspondence for ∞-tilting (cotilting) objects in an ex-triangulated category.

Proposition 3.8. *Assume that W is an object of $\mathscr{C}$. Then the association $\phi : W \longrightarrow \mathcal{X}_W$ is an injective map between the isomorphism classes of ∞-cotilting objects and resolving subcategories with an $\mathbb{E}$-injective cogenerator.*

Proof. By the definition of the notation $\mathcal{X}_W$ at the beginning of Section 2 and Definition 2.7, we know W is an $\mathbb{E}$-injective cogenerator of $\mathcal{X}_W$. By Lemma 3.5, $\mathscr{P} \subseteq \mathcal{X}_W$. Since W is an ∞-cotilting object, $\mathcal{X}_W$ is closed under extensions and CoCone of deflations by Lemma 3.1. So $\mathcal{X}_W$ is a resolving subcategory by Definition 2.8. Since $\mathbb{E}$-injective cogenerator is unique under isomorphism, ϕ is injective. □

Proposition 3.9. *There exists a correspondence ψ between the class of resolving subcategories $\mathscr{X}$ with an $\mathbb{E}$-injective cogenerator and the isomorphism classes of ∞-cotilting objects, given by $\psi : \mathscr{X} \longrightarrow W$, where add$W = \mathscr{X} \cap \mathscr{X}^{\perp}$.*

Proof. Assume W_1 is an $\mathbb{E}$-injective cogenerator of $\mathscr{X}$. Then $\mathbb{E}^i(X,W_1)=0$ for any $X\in\mathscr{X}$ and $i\geq 1$. Thus $addW_1\subseteq\mathscr{X}\cap\mathscr{X}^{\perp}$. Conversely, let $A\in\mathscr{X}\cap\mathscr{X}^{\perp}$. Then there exists an $\mathbb{E}$-triangle $A\to W_1^0\to B\overset{\delta}{\dashrightarrow}$ with $W_1^0\in addW_1$ and $B\in\mathscr{X}$. Since $A\in\mathscr{X}^{\perp}$, $\mathbb{E}(B,A)=0$. So $A\in addW_1$. Hence $addW_1=\mathscr{X}\cap\mathscr{X}^{\perp}=addW$. Thus W is an $\mathbb{E}$-injective cogenerator of $\mathscr{X}$. So $\mathscr{X}\subseteq\mathcal{X}_W$. Since $\mathscr{X}$ is a resolving subcategory, $\mathscr{P}\subseteq\mathscr{X}$. And thus $\mathscr{P}\subseteq\mathcal{X}_W$. Also W is self-orthogonal, W is an ∞-tilting objects by Definition 3.3. So W is an ∞-cotilting object by Corollary 3.6. □

We can collect the results in Propositions 3.8 and 3.9 and obtain the following partial Auslander-Reiten correspondence for ∞-cotilting (tilting) objects in an extriangulated category.

Theorem 3.10.

(1) *There is an inverse bijection between classes of ∞-cotilting objects W and resolving subcategories $\mathscr{X}$ with an $\mathbb{E}$-injective cogenerator, maximal among those with the same $\mathbb{E}$-injective cogenerator, and the assignments are ϕ: $W\mapsto\mathcal{X}_W$ and $\psi:\mathscr{X}\mapsto W$ with $addW=\mathscr{X}\cap\mathscr{X}^{\perp}$.*

(2) *There is an inverse bijection between classes of ∞-tilting objects T and coresolving subcategories $\mathscr{Y}$ with an $\mathbb{E}$-projective generator, maximal among those with the same $\mathbb{E}$-projective generator, and the assignments are ϕ: $T\mapsto {}_T\mathcal{X}$ and ψ: $\mathscr{Y}\mapsto T$ with $addT={}^{\perp}\mathscr{Y}\cap\mathscr{Y}$.*

Proof. We only prove (1) and the proof of (2) is dually.

Let $\mathscr{X}$ be any resolving subcategory with an $\mathbb{E}$-injective cogenerator W, then $\mathscr{X}\subseteq\mathcal{X}_W=(\phi\circ\psi)(\mathscr{X})$ by Propositions 3.8 and 3.9. Thus, for any ∞-cotilting objects W, $\phi(W)$ is maximal among those resolving subcategories with the same $\mathbb{E}$-injective cogenerator W.

Conversely, if $\mathscr{X}$ is a subcategory maximal among those with the previous properties, then W with $addW={}^{\perp}\mathscr{X}\cap\mathscr{X}$ is an $\mathbb{E}$-injective cogenerator of $\mathscr{X}$ and $\psi(\mathscr{X})=W$ is an ∞-cotilting object by Proposition 3.9. So W is an $\mathbb{E}$-injective cogenerator of $\mathcal{X}_W$. Hence $\mathcal{X}_W\subseteq\mathscr{X}$ and $\mathscr{X}=(\phi\circ\psi)(\mathscr{X})$. □

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Authors' addresses

Zhen Zhang

Qilu Normal university, Department of Mathematics, Ji Nan 250200, China

E-mail address: zhangzhenbiye@163.com

Shance Wang

(**Corresponding author**) Qilu Normal university, Department of Mathematics, Ji Nan 250200, China

E-mail address: 157642043@qq.com