Miskolc Mathematical Notes
Vol. 26 (2025), No. 1, pp. 305–319

HU e-ISSN 1787-2413
DOI: 10.18514/MMN.2025.4519

INVERSE TRIGONOMETRICALLY CONVEXITY AND BETTER APPROXIMATIONS

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Received 10 December, 2022

Abstract. In this paper, we introduce and study the concept of inverse trigonometrically convex functions and their some algebraic properties. We prove some Hermite-Hadamard type integral inequalities for the newly introduced class of functions. We also obtain some refinements of the Hermite-Hadamard inequality for functions whose first derivative in absolute value is inverse trigonometrically convex. Moreover, we proved that Hölder-İşcan and improved power-mean integral inequalities give a better approach than Hölder and power-mean inequalities.

2010 *Mathematics Subject Classification:* 26A51; 26D10; 26D15

Keywords: inverse trigonometrically convexity, Hölder-İşcan inequality, improved power-mean inequality, Hermite-Hadamard inequality

1. Introduction

Definition 1. $\varpi: I \to \mathbb{R}$ is said to be convex function if the inequality

$$\varpi(s\phi + (1-s)\chi) \leq s\varpi(\phi) + (1-s)\varpi(\chi)$$

is valid for all $\phi, \chi \in I$ and $s \in [0,1]$.

Convexity theory has become a deep research area in pure and applied sciences. In recent years, several extensions and generalizations of classical convexity have been studied by many researchers using novel methods and ideas. See articles [3, 6, 8, 10–13] and the references therein.

Let $\varpi: I \to \mathbb{R}$ be a convex function. Then

$$\varpi\left(\frac{\phi+\chi}{2}\right) \leq \frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x \leq \frac{\varpi(\phi)+\varpi(\chi)}{2}$$

holds for all $\phi, \chi \in I$ with $\phi < \chi$. This is well known as the Hermite Hadamard (H-H) integral inequality (see [4]). Some refinements of the Hermite-Hadamard inequality for convex functions have been obtained [2, 15]. This inequality is one of the most studied result in inequalities via convex functions. It provides a necessary and sufficient condition for a function to be convex.

Definition 2 ([3])**.** A non-negative $\varpi\colon I\to\mathbb{R}$ is said to be a P-function if the inequality

$$\varpi\left(r\phi+(1-r)\chi\right)\leq\varpi\left(\phi\right)+\varpi\left(\chi\right)$$

holds for all $\phi,\chi\in I$ and $r\in[0,1]$.

Definition 3 ([14, page 304, Definition 4])**.** Let $h\colon J\to\mathbb{R}$ be a non-negative function, $h\neq 0$. We say that $\varpi\colon I\to\mathbb{R}$ is an h-convex function, or that ϖ belongs to the class $SX\left(h,I\right)$, if ϖ is non-negative and for all $\phi,\chi\in I$, $r\in(0,1)$ we have

$$\varpi\left(r\phi+(1-r)\chi\right)\leq h(r)\varpi\left(\phi\right)+h(1-r)\varpi\left(\chi\right).$$

In [7], Kadakal gave the concept of trigonometrically convex function and Hermite-Hadamard type inequalities as follows:

Definition 4 ([7, page 20, Definition 4])**.** A non-negative function $\varpi\colon I\to\mathbb{R}$ is called trigonometrically convex if for every $\phi,\chi\in I$ and $r\in[0,1]$,

$$\varpi\left(r\phi+(1-r)\chi\right)\leq\left(\sin\frac{\pi r}{2}\right)\varpi\left(\phi\right)+\left(\cos\frac{\pi r}{2}\right)\varpi\left(\chi\right).$$

The class of all trigonometrically convex functions is denoted by $TC\left(I\right)$ on interval I. We note that, every trigonometrically convex function is a h-convex function for $h\left(t\right)=\sin\frac{\pi r}{2}$.

Theorem 1 ([7, page 23, Theorem 4])**.** *Let $\varpi\colon\ [\phi,\chi]\to\mathbb{R}$ be a trigonometrically convex function. If $\phi<\chi$ and $\varpi\in L[\phi,\chi]$, then the following inequality holds:*

$$\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\leq\frac{2}{\pi}\left[\varpi\left(\phi\right)+\varpi\left(\chi\right)\right].$$

Theorem 2. *Let the function $\varpi\colon\ [\phi,\chi]\to\mathbb{R}$, be a trigonometrically convex function. If $\phi<\chi$ and $\varpi\in L[\phi,\chi]$, then the following inequality holds:*

$$\varpi\left(\frac{\phi+\chi}{2}\right)\leq\frac{\sqrt{2}}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x.$$

An refinement of Hölder integral inequality better approach than Hölder integral inequality can be given as follows:

Theorem 3 (Hölder-İşcan Integral Inequality [5, page 2, Theorem 2.1])**.** *Let $p>1$ and $\frac{1}{p}+\frac{1}{q}=1$. If f and g are real functions defined on interval $[a,b]$ and if $|f|^{p}$, $|g|^{q}$*

are integrable functions on interval $[a,b]$ *then*

$$\int_a^b |f(x)g(x)|\,\mathrm{d}x \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_a^b (b-x)\,|g(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)\,|f(x)|^p\,\mathrm{d}x\right)^{\frac{1}{p}}\left(\int_a^b (x-a)\,|g(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\right\}.$$

An refinement of power-mean integral inequality better approach than power-mean inequality as a result of the Hölder-İşcan integral inequality can be given as follows:

Theorem 4 (Improved power-mean integral inequality [9, page 444, Theorem 2.1])**.** *Let* $q \ge 1$. *If* f *and* g *are real functions defined on interval* $[a,b]$ *and if* $|f|$, $|f|\,|g|^q$ *are integrable functions on* $[a,b]$ *then*

$$\int_a^b |f(x)g(x)|\,\mathrm{d}x \le \frac{1}{b-a}\left\{\left(\int_a^b (b-x)\,|f(x)|\,\mathrm{d}x\right)^{1-\frac{1}{q}} \times \left(\int_a^b (b-x)\,|f(x)|\,|g(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}} + \left(\int_a^b (x-a)\,|f(x)|\,\mathrm{d}x\right)^{1-\frac{1}{q}}\left(\int_a^b (x-a)\,|f(x)|\,|g(x)|^q\,\mathrm{d}x\right)^{\frac{1}{q}}\right\}.$$

2. Main Results

In this section, we introduce a new concept, which is called inverse trigonometrically convexity and we give some properties for the inverse trigonometrically convex functions, as follows:

Definition 5. A non-negative function $\varpi\colon I \to \mathbb{R}$ is called inverse trigonometrically convex function (or inverse trigonometrically convex) if for every $\phi,\chi \in I$ and $r \in [0,1]$,

$$\varpi\left(r\phi + (1-r)\chi\right) \le \left(\frac{2}{\pi}\arcsin r\right)\varpi(\phi) + \left(\frac{2}{\pi}\arccos r\right)\varpi(\chi).$$

We will denote by $IT\,(I)$ the class of all inverse trigonometrically convex functions on interval I.

We discuss some connections between the class of inverse trigonometrically convex functions and other classes of generalized convex functions.

Remark 1. Clearly, if $\varpi(x)$ is a nonnegative function, then every inverse trigonometric convex function is a P-function. Indeed, for every $\phi,\chi \in I$ and $r \in [0,1]$ we get

$$\varpi\left(r\phi+(1-r)\chi\right)\leq\left(\frac{2}{\pi}\arcsin r\right)\varpi(\phi)+\left(\frac{2}{\pi}\arccos r\right)\varpi(\chi)\leq\varpi(\phi)+\varpi(\chi).$$

Example 1. Every constant function is an inverse trigonometrically convex function.

Theorem 5. *Let* $\varpi,\omega\colon\ [\phi,\chi]\to\mathbb{R}$. *If* ϖ *and* ω *are inverse trigonometrically convex functions, then*

(i) $\varpi+\omega$ *is inverse trigonometrically convex function,*
(ii) For $c\in\mathbb{R}\quad(c\geq 0)$, $c\varpi$ *is inverse trigonometrically convex function.*

Proof.

(i) Let ϖ,ω be inverse trigonometrically convex functions, then

$$\begin{aligned}&(\varpi+\omega)\left(r\phi+(1-r)\chi\right)\\&\leq\left(\frac{2}{\pi}\arcsin r\right)\varpi(\phi)+\left(\frac{2}{\pi}\arccos r\right)\varpi(\chi)+\left(\frac{2}{\pi}\arcsin r\right)\omega(\phi)\\&\quad+\left(\frac{2}{\pi}\arccos r\right)\omega(\chi)\\&=\left(\frac{2}{\pi}\arcsin r\right)(\varpi+\omega)(\phi)+\left(\frac{2}{\pi}\arccos r\right)(\varpi+\omega)(\chi).\end{aligned}$$

(ii) Let ϖ be inverse trigonometrically convex function and $c\in\mathbb{R}\ (c\geq 0)$, then

$$\begin{aligned}(c\varpi)\left(r\phi+(1-r)\chi)\right)&\leq c\left[\left(\frac{2}{\pi}\arcsin r\right)\varpi(\phi)+\left(\frac{2}{\pi}\arccos r\right)\varpi(\chi)\right]\\&=\left(\frac{2}{\pi}\arcsin r\right)(c\varpi)(\phi)+\left(\frac{2}{\pi}\arccos r\right)(c\varpi)(\chi).\end{aligned}$$

□

Theorem 6. *Let* $\varpi,\omega\colon I\to\mathbb{R}$ *are both nonnegative and monotone increasing. If* ϖ *and* ω *are inverse trigonometrically convex functions, then* $\varpi\omega$ *is inverse trigonometrically convex function.*

Proof. If $\phi\leq\chi$, then $[\varpi(\phi)-\varpi(\chi)][\omega(\chi)-\omega(\phi)]\leq 0$ which implies

$$\varpi(\phi)\omega(\chi)+\varpi(\chi)\omega(\phi)\leq\varpi(\phi)\omega(\phi)+\varpi(\chi)\omega(\chi).\tag{2.1}$$

On the other hand for $\phi, \chi \in I$ and $r \in [0,1]$,

$$\begin{aligned}
&(\varpi\omega)(r\phi + (1-r)\chi) \\
&\leq \left[\left(\frac{2}{\pi}\arcsin r\right)\varpi(\phi) + \left(\frac{2}{\pi}\arccos r\right)\varpi(\chi)\right] \\
&\quad \times \left[\left(\frac{2}{\pi}\arcsin r\right)\omega(\phi) + \left(\frac{2}{\pi}\arccos r\right)\omega(\chi)\right] \\
&= \frac{4}{\pi^2}\left(\arcsin^2 r\right)\varpi(\phi)\omega(\phi) + \frac{4}{\pi^2}(\arcsin r)(\arccos r)\left[\varpi(\phi)\omega(\chi) + \varpi(\chi)\omega(\phi)\right] \\
&\quad + \frac{4}{\pi^2}\left(\arccos^2 r\right)\varpi(\chi)\omega(\chi).
\end{aligned}$$

Using (2.1) in the above inequality, get

$$\begin{aligned}
&(\varpi\omega)(r\phi + (1-r)\chi) \\
&\leq \frac{4}{\pi^2}\left(\arcsin^2 r\right)\varpi(\phi)\omega(\phi) + \frac{4}{\pi^2}(\arcsin r)(\arccos r)\left[\varpi(\phi)\omega(\chi) + \varpi(\chi)\omega(\phi)\right] \\
&\quad + \frac{4}{\pi^2}\left(\arccos^2 r\right)\varpi(\chi)\omega(\chi).
\end{aligned}$$

Since $\arcsin r + \arccos r = \frac{\pi}{2}$, we get

$$\begin{aligned}
(\varpi\omega)(r\phi + (1-r)\chi) &\leq \left(\frac{2}{\pi}\arcsin r\right)\varpi(\phi)\omega(\phi) + \left(\frac{2}{\pi}\arccos r\right)\varpi(\chi)\omega(\chi) \\
&= \left(\frac{2}{\pi}\arcsin r\right)(\varpi\omega)(\phi) + \left(\frac{2}{\pi}\arccos r\right)(\varpi\omega)(\chi).
\end{aligned}$$

□

Theorem 7. *If $\varpi : I \to J$ is convex and $\omega : J \to \mathbb{R}$ is inverse trigonometrically convex function and nondecreasing, then $\omega \circ \varpi : I \to \mathbb{R}$ is an inverse trigonometrically convex function.*

Proof. For $\phi, \chi \in I$ and $r \in [0,1]$, we get

$$\begin{aligned}
(\omega \circ \varpi)(r\phi + (1-r)\chi) &\leq \left(\frac{2}{\pi}\arcsin r\right)\omega(\varpi(\phi)) + \left(\frac{2}{\pi}\arccos r\right)\omega(\varpi(\chi)) \\
&= \left(\frac{2}{\pi}\arcsin r\right)(\omega \circ \varpi)(\phi) + \left(\frac{2}{\pi}\arccos r\right)(\omega \circ \varpi)(\chi).
\end{aligned}$$

□

Theorem 8. *If $\varpi : I \to J$ is an inverse trigonometrically convex and $\omega : J \to \mathbb{R}$ is a nonnegative, nondecreasing and trigonometrically convex function, then $\omega \circ \varpi : I \to \mathbb{R}$ is an P- function.*

Proof. For $\phi,\chi \in I$ and $r \in [0,1]$, we get

$$\begin{aligned}(\omega\circ\varpi)(r\phi+(1-r)\chi) &\leq \omega\left(\left(\frac{2}{\pi}\arcsin r\right)\varpi(\phi)+\left(\frac{2}{\pi}\arccos r\right)\varpi(\chi)\right)\\ &\leq \sin\frac{\pi}{2}\left(\frac{2}{\pi}\arcsin r\right)\omega(\varpi(\phi))\\ &\quad+\cos\frac{\pi}{2}\left(\frac{2}{\pi}\arccos r\right)\omega(\varpi(\chi))\\ &= r\left[(\omega\circ\varpi)(\phi)+(\omega\circ\varpi)(\chi)\right] \leq \left[(\omega\circ\varpi)(\phi)+(\omega\circ\varpi)(\chi)\right].\end{aligned}$$

□

Theorem 9. *Let $\chi > 0$ and $\varpi_\delta\colon [\phi,\chi] \to \mathbb{R}$ be an arbitrary family of inverse trigonometrically convex function and let $\varpi(x) = \sup_\delta \varpi_\delta(x)$. If $J = \{u \in [\phi,\chi] : \varpi(u) < \infty\}$ is nonempty, then J is an interval and ϖ is an inverse trigonometrically convex function on J.*

Proof. Let $r \in [0,1]$ and $\phi,\chi \in J$ be arbitrary. Then

$$\begin{aligned}\varpi(r\phi+(1-r)\chi) &\leq \sup_\delta\left[\left(\frac{2}{\pi}\arcsin r\right)\varpi_\delta(\phi)+\left(\frac{2}{\pi}\arccos r\right)\varpi_\delta(\chi)\right]\\ &\leq \left(\frac{2}{\pi}\arcsin r\right)\varpi(\phi)+\left(\frac{2}{\pi}\arccos r\right)\varpi(\chi) < \infty.\end{aligned}$$

This shows simultaneously that J is an interval, since it contains every point between any two of its points, and that ϖ is an inverse trigonometrically convex function on J. □

3. Hermite-Hadamard inequality for inverse trigonometrically convex functions

The goal of this paper is to establish some inequalities of Hermite-Hadamard type for inverse trigonometrically convex functions. In this section, we will denote by $L[\phi,\chi]$ the space of (Lebesgue) integrable functions on $[\phi,\chi]$.

Theorem 10. *Let $\varpi\colon [\phi,\chi] \to \mathbb{R}$ be an inverse trigonometrically convex function. If $\phi < \chi$ and $\varpi \in L[\phi,\chi]$, then*

$$\frac{1}{\chi-\phi}\int_\phi^\chi \varpi(x)\mathrm{d}x \leq \left(1-\frac{2}{\pi}\right)\varpi(\phi)+\frac{2}{\pi}\varpi(\chi).$$

Proof. By using the property of the inverse trigonometrically convex function of the function ϖ, if the variable is changed as $u = r\phi + (1-r)\chi$, then

$$\begin{aligned}\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x &= \int_0^1 \varpi\left(r\phi+(1-r)\chi\right)\mathrm{d}r\\ &\le \int_0^1\left\{\left(\frac{2}{\pi}\arcsin r\right)\varpi(\phi)+\left(\frac{2}{\pi}\arccos r\right)\varpi(\chi)\right\}\mathrm{d}r\\ &= \left(1-\frac{2}{\pi}\right)\varpi(\phi)+\frac{2}{\pi}\varpi(\chi).\end{aligned}$$

□

Theorem 11. *Let the function $\varpi\colon [\phi,\chi]\to\mathbb{R}$,be an inverse trigonometrically convex function. If $\phi<\chi$ and $\varpi\in L[\phi,\chi]$, then*

$$\varpi\left(\frac{\phi+\chi}{2}\right)\le\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x.$$

Proof. From the propery of the inverse trigonometrically convex function of ϖ, we get

$$\begin{aligned}\varpi\left(\frac{\phi+\chi}{2}\right) &= \varpi\left(\frac{[r\phi+(1-r)\chi]+[(1-r)\phi+r\chi]}{2}\right)\\ &= \varpi\left(\frac{1}{2}[r\phi+(1-r)\chi]+\frac{1}{2}[(1-r)\phi+r\chi]\right)\\ &\le \left(\frac{2}{\pi}\arcsin\frac{1}{2}\right)\varpi\left(r\phi+(1-r)\chi\right)+\left(\frac{2}{\pi}\arccos\frac{1}{2}\right)\varpi\left((1-r)\phi+r\chi\right)\\ &= \frac{\varpi\left(r\phi+(1-r)\chi\right)}{3}+\frac{2\varpi\left((1-r)\phi+r\chi\right)}{3}.\end{aligned}$$

Now, if we take integral in the last inequality with respect to $r\in[0,1]$, we deduce that

$$\begin{aligned}\varpi\left(\frac{\phi+\chi}{2}\right) &\le \left[\frac{1}{3}\int_0^1\varpi\left(r\phi+(1-r)\chi\right)dr+\frac{2}{3}\int_0^1\varpi\left((1-r)\phi+r\chi\right)\mathrm{d}r\right]\\ &= \frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x.\end{aligned}$$

□

4. Better approximations for inverse trigonometrically convex functions

The main purpose of this section is to establish new estimates that refine Hermite-Hadamard inequality for functions whose first derivative in absolute value, raised to a certain power which is greater than one, respectively at least one, is inverse trigonometrically convex function. Moreover, we proved that Hölder-İşcan and improved power-mean integral inequalities give a better approach than Hölder and power-mean inequalities. We will use the following lemma.

Lemma 1 ([1, page 91, Lemma 2.1]). *Let* $\varpi\colon I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}$ *be a differentiable mapping on* I°, $\phi, \chi \in I^{\circ}$ *with* $\phi < \chi$. *If* $\varpi' \in L[\phi,\chi]$, *then*

$$\frac{\varpi(\phi)+\varpi(\chi)}{2} - \frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x = \frac{\chi-\phi}{2}\int_0^1 (1-2r)\varpi'(r\phi+(1-r)\chi)\,\mathrm{d}r.$$

Theorem 12. *Let the function* $\varpi\colon I \to \mathbb{R}$ *be a differentiable on* I°, $\phi,\chi \in I^{\circ}$ *with* $\phi < \chi$ *and assume that* $\varpi' \in L[\phi,\chi]$. *If* $|\varpi'|$ *is an inverse trigonometrically convex function on interval* $[\phi,\chi]$, *then*

$$\begin{aligned}&\left|\frac{\varpi(\phi)+\varpi(\chi)}{2} - \frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right| \\ &\leq \frac{\chi-\phi}{\pi}\left[\left(\frac{3\sqrt{3}}{4}-1\right)\left|\varpi'(\phi)\right| + \left(\frac{\pi}{4}-\frac{3\sqrt{3}}{4}+1\right)\left|\varpi'(\chi)\right|\right]\end{aligned}$$

holds for $r \in [0,1]$.

Proof. Using Lemma 1 and the inequality

$$\left|\varpi'(r\phi+(1-r)\chi)\right| \leq \left(\frac{2}{\pi}\arcsin r\right)\left|\varpi'(\phi)\right| + \left(\frac{2}{\pi}\arccos r\right)\left|\varpi'(\chi)\right|,$$

we get

$$\begin{aligned}&\left|\frac{\varpi(\phi)+\varpi(\chi)}{2} - \frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right| \\ &\leq \frac{\chi-\phi}{2}\int_0^1 |1-2r|\left|\varpi'(r\phi+(1-r)\chi)\right|\mathrm{d}r \\ &\leq \frac{\chi-\phi}{2}\int_0^1 |1-2r|\left[\left(\frac{2}{\pi}\arcsin r\right)\left|\varpi'(\phi)\right| + \left(\frac{2}{\pi}\arccos r\right)\left|\varpi'(\chi)\right|\right]\mathrm{d}r \\ &= \frac{\chi-\phi}{\pi}\left[\left(\frac{3\sqrt{3}}{4}-1\right)\left|\varpi'(\phi)\right| + \left(\frac{\pi}{4}-\frac{3\sqrt{3}}{4}+1\right)\left|\varpi'(\chi)\right|\right],\end{aligned}$$

where

$$\begin{aligned}\int_0^1 |1-2r|\arcsin r\mathrm{d}r &= \frac{3\sqrt{3}}{4}-1, \\ \int_0^1 |1-2r|\arccos r\mathrm{d}r &= \frac{\pi}{4}-\frac{3\sqrt{3}}{4}+1.\end{aligned}$$

□

Theorem 13. *Let the function* $\varpi\colon I\to\mathbb{R}$ *be a differentiable on* I°, $\phi,\chi\in I^{\circ}$ *with* $\phi<\chi$, $q>1$, $\frac{1}{p}+\frac{1}{q}=1$ *and assume that* $\varpi'\in L[\phi,\chi]$. *If* $|\varpi'|^{q}$ *is an inverse trigonometrically convex function on interval* $[\phi,\chi]$, *then*

$$\begin{aligned}&\left|\frac{\varpi(\phi)+\varpi(\chi\varphi)}{2}-\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right|\\&\leq\frac{\chi-\phi}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(\pi-2)\left|\varpi'(\phi)\right|^{q}+2\left|\varpi'(\chi)\right|^{q}}{\pi}\right]^{\frac{1}{q}}\end{aligned}\tag{4.1}$$

holds for $r\in[0,1]$.

Proof. By considering Lemma 1, Hölder's integral inequality and the following inequality

$$\left|\varpi'(r\phi+(1-r)\chi)\right|^{q}\leq\left(\frac{2}{\pi}\arcsin t\right)\left|\varpi'(\phi)\right|^{q}+\left(\frac{2}{\pi}\arccos t\right)\left|\varpi'(\chi)\right|^{q},$$

which is the inverse trigonometrically convex function of $|\varpi'|^{q}$, we get

$$\begin{aligned}&\left|\frac{\varpi(\phi)+\varpi(\chi)}{2}-\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right|\\&\leq\frac{\chi-\phi}{2}\left(\int_{0}^{1}|1-2r|^{p}\,\mathrm{d}r\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left|\varpi'(r\phi+(1-r)\chi)\right|^{q}\mathrm{d}r\right)^{\frac{1}{q}}\\&\leq\frac{\chi-\phi}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\int_{0}^{1}\left[\left(\frac{2}{\pi}\arcsin r\right)\left|\varpi'(\phi)\right|^{q}+\left(\frac{2}{\pi}\arccos r\right)\left|\varpi'(\chi)\right|^{q}\right]\mathrm{d}r\right)^{\frac{1}{q}}\\&=\frac{\chi-\phi}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left(\frac{2\left|\varpi'(\phi)\right|^{q}}{\pi}\int_{0}^{1}\arcsin r\mathrm{d}r+\frac{2\left|\varpi'(\chi)\right|^{q}}{\pi}\int_{0}^{1}\arccos r\mathrm{d}r\right)^{\frac{1}{q}}\\&=\frac{\chi-\phi}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(\pi-2)\left|\varpi'(\phi)\right|^{q}+2\left|\varpi'(\chi)\right|^{q}}{\pi}\right]^{\frac{1}{q}},\end{aligned}$$

where

$$\int_{0}^{1}|1-2r|^{p}\,\mathrm{d}r=\frac{1}{p+1},$$
$$\int_{0}^{1}\arcsin r\mathrm{d}r=\frac{\pi}{2}-1,\qquad\int_{0}^{1}\arccos r\mathrm{d}r=1.$$

□

Theorem 14. *Let the function* $\varpi\colon I\subseteq\mathbb{R}\to\mathbb{R}$ *be a differentiable on* I°, $\phi,\chi\in I^{\circ}$ *with* $\phi<\chi$, $q\geq1$, $\frac{1}{p}+\frac{1}{q}=1$ *and assume that* $\varpi'\in L[\phi,\chi]$. *If* $|\varpi'|^{q}$ *is an inverse* *trigonometrically convex function on the interval* $[\phi,\chi]$*, then*

$$\left|\frac{\varpi(\phi)+\varpi(\chi)}{2}-\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right|$$
$$\leq\frac{\chi-\phi}{2^{2-\frac{1}{q}}}\left[\frac{2}{\pi}\left|\varpi'(\phi)\right|^{q}\left(\frac{3\sqrt{3}}{4}-1\right)+\frac{2}{\pi}\left|\varpi'(\chi)\right|^{q}\left(\frac{\pi}{4}-\frac{3\sqrt{3}}{4}+1\right)\right]^{\frac{1}{q}} \tag{4.2}$$

holds for $r\in[0,1]$.

Proof. Assume first that $q>1$. By considering Lemma 1, Hölder integral inequality and the property of the inverse trigonometrically convex function of $|\varpi'|^{q}$, we obtain

$$\left|\frac{\varpi(\phi)+\varpi(\chi)}{2}-\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right|$$
$$\leq\frac{\chi-\phi}{2}\left(\int_{0}^{1}|1-2r|\,dr\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}|1-2r|\left|\varpi'\left((r\phi+(1-r)\chi)\right)\right|^{q}\mathrm{d}r\right)^{\frac{1}{q}}$$
$$=\frac{\chi-\phi}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}|1-2r|\left[\left(\tfrac{2\arcsin r}{\pi}\right)\left|\varpi'(\phi)\right|^{q}+\left(\tfrac{2\arccos r}{\pi}\right)\left|\varpi'(\chi)\right|^{q}\right]\mathrm{d}r\right)^{\frac{1}{q}}$$
$$=\frac{\chi-\phi}{2}\left(\frac{1}{2}\right)^{1-\frac{1}{q}}\left[\frac{2\left|\varpi'(\phi)\right|^{q}}{\pi}\left(\frac{3\sqrt{3}}{4}-1\right)+\frac{2\left|\varpi'(\chi)\right|^{q}}{\pi}\left(\frac{\pi}{4}-\frac{3\sqrt{3}}{4}+1\right)\right]^{\frac{1}{q}}$$
$$=\frac{\chi-\phi}{2^{2-\frac{1}{q}}}\left[\frac{2\left|\varpi'(\phi)\right|^{q}}{\pi}\left(\frac{3\sqrt{3}}{4}-1\right)+\frac{2\left|\varpi'(\chi)\right|^{q}}{\pi}\left(\frac{\pi}{4}-\frac{3\sqrt{3}}{4}+1\right)\right]^{\frac{1}{q}}.$$

So, the desired result are obtained. For $q=1$ we use the estimates from the proof of Theorem 12, which also follow step by step the above estimates. □

Corollary 1. *Under the assumption of Theorem 14 with* $q=1$*, we get the conclusion of Theorem 12.*

Theorem 15. *Let the function* $\varpi\colon I\to\mathbb{R}$ *be a differentiable on* I°*,* $\phi,\chi\in I^{\circ}$ *with* $\phi<\chi$*,* $q>1$*,* $\frac{1}{p}+\frac{1}{q}=1$ *and assume that* $\varpi'\in L[\phi,\chi]$*. If* $|\varpi'|^{q}$ *is an inverse trigonometrically convex function on interval* $[\phi,\chi]$*, then*

$$\left|\frac{\varpi(\phi)+\varpi(\chi\varphi)}{2}-\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right|$$
$$\leq\frac{\chi-\phi}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{2\left|\varpi'(\phi)\right|^{q}}{\pi}\left[\frac{3\pi}{8}-1\right]+\frac{2\left|\varpi'(\chi)\right|^{q}}{\pi}\left[1-\frac{\pi}{8}\right]\right)^{\frac{1}{q}}$$
$$+\frac{\chi-\phi}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{2\left|\varpi'(\phi)\right|^{q}}{\pi}\frac{\pi}{8}+\frac{2\left|\varpi'(\chi)\right|^{q}}{\pi}\frac{\pi}{8}\right)^{\frac{1}{q}}, \tag{4.3}$$

holds for $r \in [0,1]$.

Proof. By considering Lemma 1, Hölder-İşcan integral inequality and the property of the inverse trigonometrically convex function of $|\varpi'|^q$, we get

$$\begin{aligned}
&\left|\frac{\varpi(\phi)+\varpi(\chi)}{2}-\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right| \\
&\leq \frac{\chi-\phi}{2}\left(\int_0^1(1-r)|1-2r|^p\,\mathrm{d}r\right)^{\frac{1}{p}}\left(\int_0^1(1-r)\left|\varpi'(r\phi+(1-r)\chi)\right|^q\mathrm{d}r\right)^{\frac{1}{q}} \\
&\quad+\frac{\chi-\phi}{2}\left(\int_0^1 r|1-2r|^p\,\mathrm{d}r\right)^{\frac{1}{p}}\left(\int_0^1 r\left|\varpi'(r\phi+(1-r)\chi)\right|^q\mathrm{d}r\right)^{\frac{1}{q}} \\
&\leq \frac{\chi-\phi}{2}\left(\int_0^1(1-r)|1-2r|^p\,\mathrm{d}r\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1(1-r)\left[\left(\frac{2}{\pi}\arcsin r\right)\left|\varpi'(\phi)\right|^q+\left(\frac{2}{\pi}\arccos r\right)\left|\varpi'(\chi)\right|^q\right]\mathrm{d}r\right)^{\frac{1}{q}} \\
&\quad+\frac{\chi-\phi}{2}\left(\int_0^1 r|1-2r|^p\,\mathrm{d}r\right)^{\frac{1}{p}} \\
&\quad\times\left(\int_0^1 r\left[\left(\frac{2}{\pi}\arcsin r\right)\left|\varpi'(\phi)\right|^q+\left(\frac{2}{\pi}\arccos r\right)\left|\varpi'(\chi)\right|^q\right]\mathrm{d}r\right)^{\frac{1}{q}} \\
&=\frac{\chi-\phi}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{2|\varpi'(\phi)|^q}{\pi}\int_0^1(1-r)\arcsin r\mathrm{d}r+\frac{2|\varpi'(\chi)|^q}{\pi}\int_0^1(1-r)\arccos r\mathrm{d}r\right)^{\frac{1}{q}} \\
&\quad+\frac{\chi-\phi}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}} \\
&\quad\times\left(\frac{2|\varpi'(\phi)|^q}{\pi}\int_0^1 r\arcsin r\mathrm{d}r+\frac{2|\varpi'(\chi)|^q}{\pi}\int_0^1 r\arccos r\mathrm{d}r\right)^{\frac{1}{q}} \\
&\quad+\frac{\chi-\phi}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{2|\varpi'(\phi)|^q}{\pi}\frac{\pi}{8}+\frac{2|\varpi'(\chi)|^q}{\pi}\frac{\pi}{8}\right)^{\frac{1}{q}}.
\end{aligned}$$

□

Remark 2. The inequality (4.3) is better than the inequality (4.1).

Proof. By using concavity of the function

$$h\colon [0,\infty)\to\mathbb{R},\quad h(x)=x^s,\quad 0<s\leq 1,$$

we can write the right hand-side of the inequality (4.3) as follow:

$$\begin{aligned}
&\frac{\chi-\phi}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{2|\varpi'(\phi)|^{q}}{\pi}\left[\frac{3\pi}{8}-1\right]+\frac{2|\varpi'(\chi)|^{q}}{\pi}\left[1-\frac{\pi}{8}\right]\right)^{\frac{1}{q}}\\
&+\frac{\chi-\phi}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left(\frac{2|\varpi'(\phi)|^{q}}{\pi}\frac{\pi}{8}+\frac{2|\varpi'(\chi)|^{q}}{\pi}\frac{\pi}{8}\right)^{\frac{1}{q}}\\
&\leq 2\frac{\chi-\phi}{2}\left(\frac{1}{2(p+1)}\right)^{\frac{1}{p}}\left[\frac{1}{2}\left(\frac{(\pi-2)|\varpi'(\phi)|^{q}}{\pi}+\frac{2|\varpi'(\chi)|^{q}}{\pi}\right)\right]\\
&=\frac{\chi-\phi}{2}\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(\pi-2)|\varpi'(\phi)|^{q}}{\pi}+\frac{2|\varpi'(\chi)|^{q}}{\pi}\right].
\end{aligned}$$

The last statement is the right hand side of (4.3). This completes the proof of Remark. □

Theorem 16. *Let the function $\varpi: I\subseteq\mathbb{R}\to\mathbb{R}$ be a differentiable on I°, $\phi,\chi\in I^{\circ}$ with $\phi<\chi$, $q\geq 1$, $\frac{1}{p}+\frac{1}{q}=1$ and assume that $\varpi'\in L[\phi,\chi]$. If $|\varpi'|^{q}$ is an inverse trigonometrically convex function on the interval $[\phi,\chi]$, then*

$$\begin{aligned}
&\left|\frac{\varpi(\phi)+\varpi(\chi)}{2}-\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right|\leq\frac{\chi-\phi}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}} \qquad (4.4)\\
&\times\left\{\left(\frac{2|\varpi'(\phi)|^{q}}{\pi}\left[-\frac{10\pi-81\sqrt{3}+104}{72}\right]+\frac{2|\varpi'(\chi)|^{q}}{\pi}\left[\frac{19\pi-81\sqrt{3}+104}{72}\right]\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\frac{2|\varpi'(\phi)|^{q}}{\pi}\left[\frac{10\pi-27\sqrt{3}+32}{72}\right]+\frac{2|\varpi'(\chi)|^{q}}{\pi}\left[-\frac{\pi-27\sqrt{3}+32}{72}\right]\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

holds for $r\in[0,1]$.

Proof. Assume first that $q>1$. By considering Lemma 1, improved power-mean integral inequality and the property of the inverse trigonometrically convex function of $|\varpi'|^q$, we obtain

$$\begin{aligned}
&\left|\frac{\varpi(\phi)+\varpi(\chi)}{2}-\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right| \\
&\leq \frac{\chi-\phi}{2}\left(\int_0^1(1-r)\,|1-2r|\,\mathrm{d}r\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\int_0^1(1-r)\,|1-2t|\,\left|\varpi'\left((r\phi+(1-r)\chi)\right)\right|^q\mathrm{d}r\right)^{\frac{1}{q}} \\
&\quad+\frac{\chi-\phi}{2}\left(\int_0^1 r\,|1-2r|\,\mathrm{d}r\right)^{1-\frac{1}{q}}\left(\int_0^1 r\,|1-2r|\,\left|\varpi'\left((r\phi+(1-r)\chi)\right)\right|^q\mathrm{d}r\right)^{\frac{1}{q}} \\
&\leq \frac{\chi-\phi}{2}\left(\int_0^1(1-r)\,|1-2r|\,\mathrm{d}r\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\frac{2\,|\varpi'(\phi)|^q}{\pi}\int_0^1(1-r)\,|1-2r|\arcsin r\mathrm{d}r\right. \\
&\qquad\left.+\frac{2\,|\varpi'(\chi)|^q}{\pi}\int_0^1(1-r)\,|1-2r|\arccos r\mathrm{d}r\right)^{\frac{1}{q}}+\frac{\chi-\phi}{2}\left(\int_0^1 r\,|1-2r|\,\mathrm{d}r\right)^{1-\frac{1}{q}} \\
&\quad\times\left(\frac{2\,|\varpi'(\phi)|^q}{\pi}\int_0^1 r\,|1-2r|\arcsin r\mathrm{d}r+\frac{2\,|\varpi'(\chi)|^q}{\pi}\int_0^1 r\,|1-2r|\arccos r\mathrm{d}r\right)^{\frac{1}{q}} \\
&=\frac{\chi-\phi}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2|\varpi'(\phi)|^q}{\pi}\left[-\frac{10\pi-81\sqrt{3}+104}{72}\right]+\frac{2|\varpi'(\chi)|^q}{\pi}\left[\frac{19\pi-81\sqrt{3}+104}{72}\right]\right)^{\frac{1}{q}}\right. \\
&\quad\left.+\left(\frac{2\,|\varpi'(\phi)|^q}{\pi}\left[\frac{10\pi-27\sqrt{3}+32}{72}\right]+\frac{2\,|\varpi'(\chi)|^q}{\pi}\left[-\frac{\pi-27\sqrt{3}+32}{72}\right]\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where

$$\begin{aligned}
\int_0^1(1-r)\,|1-2r|\arcsin r\mathrm{d}r&=-\frac{10\pi-81\sqrt{3}+104}{72}, \\
\int_0^1(1-r)\,|1-2r|\arccos r\mathrm{d}r&=\frac{19\pi-81\sqrt{3}+104}{72}, \\
\int_0^1 r\,|1-2r|\arcsin r\mathrm{d}r&=\frac{10\pi-27\sqrt{3}+32}{72}, \\
\int_0^1 r\,|1-2r|\arccos r\mathrm{d}r&=-\frac{\pi-27\sqrt{3}+32}{72}.
\end{aligned}$$

□

Corollary 2. *Under the assumption of Theorem 16 with $q=1$, we get the following inequality:*

$$\begin{aligned}&\left|\frac{\varpi(\phi)+\varpi(\chi)}{2}-\frac{1}{\chi-\phi}\int_{\phi}^{\chi}\varpi(x)\mathrm{d}x\right|\\&\leq\frac{\chi-\phi}{2}\left[\left|\varpi'(\phi)\right|\frac{54\sqrt{3}-72}{36\pi}+\left|\varpi'(\chi)\right|\frac{18\pi-54\sqrt{3}+72}{36\pi}\right].\end{aligned}$$

Remark 3. The inequality (4.4) gives better approximation than the inequality (4.2). Really, we can write the right hand side of the inequality (4.4) as

$$\begin{aligned}&\frac{\chi-\phi}{2}\left(\frac{1}{4}\right)^{1-\frac{1}{q}}\left\{\left(\frac{2|\varpi'(\phi)|^{q}}{\pi}\left[-\frac{10\pi-81\sqrt{3}+104}{72}\right]+\frac{2|\varpi'(\chi)|^{q}}{\pi}\left[\frac{19\pi-81\sqrt{3}+104}{72}\right]\right)^{\frac{1}{q}}\right.\\&\left.+\left(\frac{2\left|\varpi'(\phi)\right|^{q}}{\pi}\left[\frac{10\pi-27\sqrt{3}+32}{72}\right]+\frac{2\left|\varpi'(\chi)\right|^{q}}{\pi}\left[-\frac{\pi-27\sqrt{3}+32}{72}\right]\right)^{\frac{1}{q}}\right\}\\&\leq\frac{\chi-\phi}{2^{2-\frac{1}{q}}}\left[\frac{\left|\varpi'(\phi)\right|^{q}}{\pi}\left(\frac{3\sqrt{3}}{4}-1\right)+\frac{\left|\varpi'(\chi)\right|^{q}}{\pi}\left(\frac{\pi}{4}-\frac{3\sqrt{3}}{4}+1\right)\right]^{\frac{1}{q}}.\end{aligned}$$

The last inequality is the right hand side of the inequality (4.2). Therefore, this shows that (4.4) is better than (4.2). This completes the proof of Remark.

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