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DYNAMICS OF A DISCRETE-TIME PREDATOR-PREY SYSTEM WITH RATIO-DEPENDENT FUNCTIONAL RESPONSE

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Abstract. The aim of this article is to analyze the qualitative behavior of a discrete-time predator-prey system with ratio-dependent functional response. We express algebraically the conditions for the existence of a positive fixed point, and determining its stability.

We also prove that the system has a Neimark-Sacker bifurcation when some parametric conditions are satisfied. The Chaos control is nicely studied by applying the hybrid control method. Finally, we present some numerical simulations in order to verify the theoretical results we have shown.

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1. INTRODUCTION

In 1838, Verhulst introduced an equation to describe the growth of a population in an environment with limited resources. The curve that describes the evolution of this population is known as the logistic curve [29]. In 1887, S. A. Forbes concluded that the interactions between different species that make up an ecosystem are so strong that one cannot be altered without causing a change in the others [14]. However, in 1925, Lotka introduced a pair of differential equations to describe the dynamics of the interaction between two populations, namely, a prey and a predator. Furthermore, Volterra independently developed and generalized the Lotka model [24, 30] in 1927. The equations proposed by Volterra [30] present the existence of two species, one of which (the prey, whose population density is denoted by N) has a positive growth rate. The second species, the predator P, would tend to disappear if the population density of the prey is zero, due to the lack of food. However, if both species coexist and the second feeds on the first, both species can survive, and the system of differential equations that models their evolution can be written as

$$\begin{aligned} \frac{dN(\tau)}{d\tau} &= N(\tau)(\alpha - \omega P(\tau))\,, \\ \frac{dP(\tau)}{d\tau} &= P(\tau)(\omega\delta N(\tau) - \beta)\,. \end{aligned}$$

In this system, $N(\tau)$ and $P(\tau)$ denote the population densities of prey and predator, respectively, at time τ. The natural growth rate of the prey is denoted by $\alpha > 0$, and $\beta > 0$ is the natural death rate of the predator in the absence of prey. Moreover, $\omega > 0$ represents the effect of predation on the prey, and $\delta > 0$ is the conversion rate of prey into predator. This simple model was modified in order to introduce intra-specific competition of prey population as follows:

$$\begin{aligned} \frac{dN(\tau)}{d\tau} &= N(\tau)(\alpha - \nu N(\tau) - \omega P(\tau))\,, \\ \frac{dP(\tau)}{d\tau} &= P(\tau)(\omega\delta N(\tau) - \beta)\,, \end{aligned}$$

where $\nu > 0$. In order to have a better accuracy in the interactions between prey and predators, the so-called “trophic function” was introduced, which describes the amount of prey that a predator consumes per unit of time when the densities of populations of prey and predators are given by N and P, respectively. Most of recent works, such as [8, 10, 13, 22], have focused on analyzing how this function varies with the prey density. Arditi and Ginzburg [3] called “prey-dependence” to this kind of dependence. However, the effect of the predator density in the functional response has been recognized in the last decades [3, 5, 17]. In fact, Arditi and Ginzburg [3] suggested that the trophic function depends on the ratio of prey to predator abundances which is called ratio-dependence. The trophic function depends on the ratio N/P. The introduction of this new kind of dependency has managed to model the interactions between predators and prey in real situations in a much better way [2,4]. The general model introduced by Arditi and Ginzburg is given by the equations

$$\begin{aligned} \frac{dN}{d\tau} &= rN\phi(N) - g\left(\frac{N}{P}\right)P\,, \\ \frac{dP}{d\tau} &= ePg\left(\frac{N}{P}\right) - qP\,, \end{aligned} \tag{1.1}$$

where the trophic function is $g(x) = g(N/P)$ and

$$\phi(N) = 1 - \frac{N}{K}\,.$$

Arditi and Ginzburg assumed that

$$g(x) = \alpha\frac{x}{1+\alpha h x}\,,$$

so that

$$g\left(\frac{N}{P}\right) = \alpha\frac{N}{P+\alpha hN}\,.$$

With the use of this trophic function, Equation (1.1) can be rewritten as

$$\begin{aligned}\frac{dN}{d\tau} &= rN\left(1-\frac{N}{K}\right) - \alpha\frac{NP}{P+\alpha hN}\,,\\ \frac{dP}{d\tau} &= e\frac{NP}{P+\alpha hN} - qP\,,\end{aligned} \tag{1.2}$$

where the parameters r, K, α, h, e, q are positive. The change of variables $(N, P, \tau) \to (x, y, t)$ defined by

$$N = Kx\,, \quad P = K\alpha hy\,, \quad \tau = \frac{t}{r}\,,$$

together with the corresponding definitions of parameters given by

$$a = \frac{\alpha}{r}\,, \quad b = \frac{q}{r}\,, \quad c = \frac{e}{rh}\,,$$

converts Equations (1.2) into

$$\begin{aligned}\dot{x}(t) &= x(1-x) - \frac{axy}{x+y}\,,\\ \dot{y}(t) &= -by + \frac{cxy}{x+y}\,,\end{aligned} \tag{1.3}$$

where $a, b, c > 0$. In Equations (1.3), b denotes the death rate of the (new defined) predators y and the trophic function is described by parameters a and c, where a stands for the maximum asymptotic prey death rate due to predation, for a finite density of predators and c denotes the maximum asymptotic predator growth rate for an infinite density of prey.

The Lotka-Volterra model described by differential equations are appropriate when the different generations can coexist, while the difference equations work better when there is no overlap between different generations. Discrete models are a very powerful tool from a computational point of view in order to perform numerical simulations. Moreover, discrete systems present dynamics more complex and rich than the corresponding continuous system. In recent years, an increasing number of researchers have studied the existence of Neimark-Sacker bifurcations in different discrete Lotka-Volterra systems [1, 6, 8, 10, 13, 16–19, 21–23]. In this paper, we discretize Equations (1.3) using the piecewise constant arguments method [22, 26] to obtain the discrete system

$$\begin{aligned}x_{n+1} &= x_n \exp\left(1 - x_n - \frac{ay_n}{x_n+y_n}\right),\\ y_{n+1} &= y_n \exp\left(-b + \frac{cx_n}{x_n+y_n}\right),\end{aligned} \tag{1.4}$$

where $a,b,c>0$. We aim to discuss the qualitative study of this system.

The paper is organized as follows. In the Section 2, we calculate the fixed points of Equations (1.4) and establish their stability. In Section 3, we give the conditions under which a Neimark-Sacker bifurcation occurs. Chaos control is discussed in Section 4. Finally, in Section 5, we illustrate all the theoretical results we have obtained with a numerical simulation of the system.

2. STABILITY ANALYSIS

The non trivial equilibrium points of Equations (1.4) are

$$E=(1,0), \quad P=\left(\frac{ab+c-ac}{c}, \frac{(c-b)(ab+c-ac)}{bc}\right).$$

The positive coexistence equilibrium P exists if $c>b$ and $ab+c>ac$.

Lemma 1. *The extinction equilibrium $E=(1,0)$ of Equations (1.4) is locally asymptotically stable (sink) if $b>c$, unstable (saddle) if $b<c$, and a non-hyperbolic point if $b=c$.*

Proof. The Jacobian matrix of the Equations (1.4) at the point E is given by

$$J(E)=\begin{pmatrix} 0 & -a \\ 0 & e^{c-b} \end{pmatrix},$$

and its eigenvalues are $\mu_1=0<1$ and $\mu_2=e^{c-b}$. Thus, $E=(1,0)$ is locally asymptotically stable if $b>c$, unstable if $b<c$, and non-hyperbolic if $b=c$. □

In order to analyze the stability of the positive coexistence equilibrium point P, we apply the following Lemma [7,8,23].

Lemma 2. *Consider the polynomial $\rho(\lambda)=\lambda^2-T\lambda+D$, where $\rho(1)>0$, and λ_1 and λ_2 are the two roots of $\rho(\lambda)=0$. Then,*

(1) $|\lambda_1|<1$ *and* $|\lambda_2|<1$ *if and only if* $\rho(-1)>0$ *and* $\rho(0)<1$.
(2) $|\lambda_1|<1$ *and* $|\lambda_2|>1$ *(or* $|\lambda_1|>1$ *and* $|\lambda_2|<1$*) if and only if* $\rho(-1)<0$.
(3) $|\lambda_1|>1$ *and* $|\lambda_2|>1$ *if and only if* $\rho(-1)>0$ *and* $\rho(0)>1$.
(4) λ_1 *and* λ_2 *are complex numbers and* $|\lambda_1|=|\lambda_2|=1$ *if and only if* $T^2-4D<0$ *and* $\rho(0)=1$.

The Jacobian matrix of Equations (1.4) evaluated at the equilibrium P reads

$$J(P)=\begin{pmatrix} \dfrac{a(c^2-b^2)}{c^2} & -\dfrac{ab^2}{c^2} \\ \dfrac{(c-b)^2}{c} & 1-\dfrac{b(c-b)}{c} \end{pmatrix},$$

and its characteristic equation is given by

$$\rho(\mu)=\mu^2-\left(a+1-b-\frac{ab^2}{c^2}+\frac{b^2}{c}\right)\mu+\frac{a(c-b)(c+b-bc+b^2)}{c^2}. \tag{2.1}$$

From Equation (2.1), we can obtain

$$\rho(0) = \frac{a(c-b)(c+b-bc+b^2)}{c^2}, \quad \rho(1) = \frac{b(c^2-bc-a(c-b)^2)}{c^2}$$

and

$$\rho(-1) = 2 + \frac{2a(c^2-b^2)+b(bc-c^2-a(c-b)^2)}{c^2}.$$

Then, taking into account that $c^2 > bc + a(c-b)^2$, it follows that $\rho(1) > 0$ and, thus, we can apply Lemma 2 to state the following result:

Lemma 3. *Assume that $c > b$ and $a < \frac{c}{(c-b)}$. Then,*

$$P = \left(\frac{ab+c-ac}{c}, \frac{(c-b)(ab+c-ac)}{bc} \right),$$

is a positive coexistence equilibrium point of Equations (1.4) and $\rho(1) > 0$. Moreover,

(1) *P is locally asymptotically stable (sink stable) if and only if*

$$a < \frac{c}{c-b} \quad \text{for } c \geq b+1,$$

and

$$a < \frac{c^2}{(c-b)(c+b(1-c)+b^2)} \quad \text{for } a < c \leq b+1.$$

or

$$b+c-bc+b^2 \leq 0,$$

and

$$\rho(-1) = \frac{c^2(a+1)(2-b)+cb^2(2a+1)-ab(b^2+2b)}{c^2} > 0.$$

(2) *P is saddle point if and only if*

$$b+c-bc+b^2 \leq 0,$$

and

$$\rho(-1) = \frac{c^2(a+1)(2-b)+cb^2(2a+1)-ab(b^2+2b)}{c^2} < 0.$$

(3) *P is unstable (source) if and only if*

$$b < c < b+1,$$

and

$$a > \frac{c^2}{(c-b)(c+b-bc+b^2)}.$$

(4) *The roots of Equation (2.1) are complex conjugates with unit modulus if and only if*

$$b < c < b+1, \tag{2.2}$$

and

$$a = \frac{c^2}{(c-b)(c+b-bc+b^2)}. \tag{2.3}$$

3. NEIMARK-SACKER BIFURCATION

In this section, we study the existence of a Neimark-Sacker bifurcation for Equations (1.4). A Neimark-Sacker bifurcation occurs when a closed invariant curve emerges from an equilibrium point in a discrete dynamical system and, then, the stability of the equilibrium changes via a pair of complex eigenvalues with unit modulus [27, 28]. We now consider this system around the positive coexistence equilibrium P. According to Lemma 2, the characteristic equation of the linearization of Equations (1.4) at P has two conjugate complex roots with modulus one if the conditions (2.2) and (2.3) are satisfied. Thus, P presents a Neimark-Sacker bifurcation if the parameters (a,b,c) vary in a neighborhood of the set

$$\mathcal{B} = \left\{ (a,b,c) \in \mathbb{R}^3_+ : a = \frac{c^2}{(c-b)(c+b-bc+b^2)}, b < c < b+1 \right\}.$$

Let $(a,b,c) \in \mathcal{B}$ and assume that the change of variables is given by

$$u_n = x_n - x_0, \quad v_n = y_n - y_0,$$

with

$$x_0 = \frac{ab+c-ac}{c}, \quad y_0 = \frac{(c-b)(ab+c-ac)}{bc},$$

transforms the fixed point $P = (x_0, y_0)$ into the origin $O = (0,0)$. Equations (1.4) is also changed into

$$\begin{aligned} u_{n+1} &= (u_n + x_0)\exp\left(1 - u_n - x_0 - \frac{a(v_n+y_0)}{(u_n+v_n+x_0+y_0)}\right) - x_0, \\ v_{n+1} &= (v_n + y_0)\exp\left(-b + \frac{c(u_n+x_0)}{(u_n+v_n+x_0+y_0)}\right) - y_0, \end{aligned} \tag{3.1}$$

where $a,b,c > 0$.

Let $\bar{a}$ denote a small perturbation in a, with $|\bar{a}| \ll 1$, i.e., $\bar{a} = a - a_0$. Then, the perturbation of Equations (3.1) can be arranged as follows:

$$\begin{aligned} u_{n+1} &= (u_n + x_0)\exp\left(1 - u_n - x_0 - \frac{(\bar{a}+a_0)(v_n+y_0)}{(u_n+v_n+x_0+y_0)}\right) - x_0, \\ v_{n+1} &= (v_n + y_0)\exp\left(-b + \frac{c(u_n+x_0)}{(u_n+v_n+x_0+y_0)}\right) - y_0. \end{aligned} \tag{3.2}$$

The characteristic equation of the linearization of Equations (3.2) at the origin can be written as

$$\mu^2 - T(\bar{a})\mu + D(\bar{a}) = 0\,, \tag{3.3}$$

where

$$T(\bar{a}) = \left((\bar{a}+a_0)+1-b-\frac{(\bar{a}+a_0)b^2}{c^2}+\frac{b^2}{c}\right),$$

and

$$D(\bar{a}) = \frac{(\bar{a}+a_0)(c-b)(c+b-bc+b^2)}{c^2}\,.$$

It is easy to obtain that $T^2(\bar{a}) - 4D(\bar{a}) < 0$ and $(a,b,c) \in \mathcal{B}$. Thus, the complex conjugate roots with unit modulus of Equation (3.3) are

$$\mu_{1,2}(\bar{a}) = \frac{T(\bar{a})}{2} \pm \frac{i}{2}\sqrt{4D(\bar{a}) - T^2(\bar{a})}\,.$$

Next, we can calculate

$$(|\mu_{1,2}(\bar{a})|)|_{\bar{a}=0} = \sqrt{D(\bar{a})}|_{\bar{a}=0} = \sqrt{\frac{a_0(c-b)(c+b-bc+b^2)}{c^2}} = 1\,,$$

which implies that

$$\left(\frac{d|\mu_1|}{d\bar{a}}\right)_{\bar{a}=0} = \left(\frac{d|\mu_2|}{d\bar{a}}\right)_{\bar{a}=0} = \left(\frac{\sqrt{(c-b)(c+b-bc+b^2)}}{2c\sqrt{a_0}}\right) > 0\,.$$

The existence of a Neimark-Sacker bifurcation needs the following conditions to be satisfied:

$$\left(\frac{d|\mu_{1,2}(\bar{a})|}{d\bar{a}}\right)_{\bar{a}=0} \neq 0,$$

and

$$\mu_{1,2}(0)^n \neq 1, \quad n = 1,2,3,4.$$

One can easily calculate $T(0)$ as follows:

$$\begin{aligned} T(0) &= 1 + \frac{c^2+bc+b(b-c)(c+b-bc+b^2)}{c(c+b-bc+b^2)}, \\ &= \frac{c^2+bc+(b^2-bc+c)(c+b-bc+b^2)}{2c(c+b-bc+b^2)}, \end{aligned}$$

and $D(0) = 1$. Thus,

$$\mu_{1,2} = \frac{T(0)}{2} \pm \frac{i}{2}\sqrt{4 - T^2(0)}.$$

Then, we can state that $\mu_{1,2}(0)^n \neq 1$, for any $n = 1,2,3,4$. According to [31], all the conditions for the existence of a Neimark-Sacker bifurcation hold. In order to determine the normal form of Equations (3.2) [25], we compute its Taylor expansion to the third order around the origin as follows:

$$\begin{aligned} u_{n+1} &= g_{11}u_n + g_{12}v_n + F_1(u_n, v_n)\,, \\ v_{n+1} &= g_{21}u_n + g_{22}v_n + F_2(u_n, v_n)\,, \end{aligned} \tag{3.4}$$

where

$$\begin{aligned} F_1(u_n, v_n) &= \frac{1}{2}g_{13}u_n^2 + 2g_{14}u_nv_n + \frac{1}{2}g_{15}v_n^2 \\ &\quad + \frac{1}{6}g_{16}u_n^3 + \frac{1}{2}g_{17}u_n^2v_n + \frac{1}{2}g_{18}u_nv_n^2 + \frac{1}{6}g_{19}v_n^3 + \mathcal{R}_{F_1,4}(u_n, v_n)\,, \\ F_2(u_n, v_n) &= \frac{1}{2}g_{23}u_n^2 + 2g_{24}u_nv_n + \frac{1}{2}g_{25}v_n^2 \\ &\quad + \frac{1}{6}g_{26}u_n^3 + \frac{1}{2}g_{27}u_n^2v_n + \frac{1}{2}g_{28}u_nv_n^2 + \frac{1}{6}g_{29}v_n^3 + \mathcal{R}_{F_2,4}(u_n, v_n)\,. \end{aligned}$$

Here, $\mathcal{R}_{F_1,4}(u_n, v_n)$ and $\mathcal{R}_{F_2,4}(u_n, v_n)$ denote the terms of order larger than 3 in the Taylor expansion. Furthermore, we have

$$\begin{aligned} g_{11} &= \frac{a_0(c^2 - b^2)}{c^2}\,, \\ g_{12} &= -\frac{a_0b^2}{c^2}\,, \\ g_{13} &= -2 + \frac{2a_0(c-b)b^2}{c^2(a_0b + c - a_0c)} + \frac{a_0b + c - a_0c}{c} + \frac{a_0^2(c-b)^2b^2}{c^3(a_0b + c - a_0c)} - \frac{2(c-b)b}{c^2}\,, \\ g_{14} &= -\frac{2a_0(c-b)b^2}{c^2(a_0b + c - a_0c)} + \frac{a_0b^2}{c^2} - \frac{a_0^2(c-b)b^3}{c^3(a_0b + c - a_0c)}\,, \\ g_{15} &= \frac{a_0b^3(2c - a_0b)}{c^3(a_0b + c - a_0c)}\,, \\ g_{16} &= \frac{a_0(c-b)b^2}{c(a_0b + c - a_0c)^2}\left(2 + a_0 - \frac{6b}{c} - \frac{5a_0b}{c} + \frac{4a_0b^2}{c^2}\right) + \frac{4(c-b)b^2}{c^2(a_0b + c - a_0c)^2} \\ &\quad + \left(-1 + \frac{2a_0(c-b)b^2}{c^2(a_0b + c - a_0c)} - \frac{a_0(c-b)}{c} + \frac{a_0(c-b)^2b^2}{c^2(a_0b + c - a_0c)^2}\right. \\ &\qquad \left. - \frac{2(c-b)b}{c^2}\right) \times \left(-1 + \frac{a_0(c-b)b}{c(a_0b + c - a_0c)}\right) + 1 - \frac{2(c-b)b}{c^2}\,, \\ g_{17} &= \frac{a_0(c-b)b^2}{c(a_0b + c - a_0c)}\left(4 - \frac{2a_0}{c} - \frac{2}{a_0b + c - a_0c} + \frac{6b}{c(a_0b + c - a_0c)}\right) \\ &\quad + \left(-1 + \frac{a_0(c-b)b}{c(a_0b + c - a_0c)}\right)\left(\frac{2a_0}{c^2} - \frac{2a_0(c-b)b^2}{c(a_0b + c - a_0c)}\right. \end{aligned}$$

$$-\frac{a_0^2(c-b)b^3}{c^3(a_0b+c-a_0c)}\Bigg),$$

$$g_{18}=\frac{a_0b^3}{c(a_0b+c-a_0c)^2}\left(4-\frac{6b}{c}+\frac{3a_0b}{c}-\frac{4a_0b^2}{c^2}+\frac{1}{c}\right.$$
$$\left.\left(2+\frac{a_0b}{c}\right)\left(-1+\frac{a_0(c-b)b}{c^2(a_0b+c-a_0c)}\right)\right),$$

$$g_{19}=\frac{a_0b^2}{(a_0b+c-a_0c)^2}\left(-6-\frac{6a_0b^3}{c^3}-\frac{a_0^2b^4}{c^3}\right),$$

$$g_{23}=\frac{(c-b)^2b}{a_0b+c-a_0c}\left(c-b-\frac{2}{c}\right),$$

$$g_{24}=\frac{(c-b)b^2(2-c+b)}{c(a_0b+c-a_0c)},$$

$$g_{25}=\frac{b^3(c-b-2)}{c(a_0b+c-a_0c)},$$

$$g_{26}=\frac{(c-b)^2b^2(-6c+6b+(c-b)^2)}{c(a_0b+c-a_0c)^2},$$

$$g_{27}=\frac{(c-b)b^2}{a_0b+c-a_0c}\left(3b-c+2-\frac{2b(2b+3)}{c}\right)$$
$$+\frac{(c-b)^2b^2}{c^2}\left(c-b+\frac{2b}{a_0b+c-a_0c}\right),$$

$$g_{28}=\frac{b^3(10b-8c-3(c-b)^2}{c(a_0b+c-a_0c)^2},$$

$$g_{29}=\frac{9-3(c-b)b^3-(c-b)b^5}{c(a_0b+c-a_0c)^2}.$$

We now assume that $J(P)$ is the Jacobian matrix of Equations (3.4) about the point P. Hence,

$$J(P)=\begin{pmatrix} g_{11} & g_{12}\\ g_{21} & g_{22}\end{pmatrix}=\begin{pmatrix}\frac{a(c^2-b^2)}{c^2} & -\frac{ab^2}{c^2}\\ \frac{(c-b)^2}{c} & 1-\frac{b(c-b)}{c}\end{pmatrix}.$$

The eigenvalues of $J(P)$ are given by

$$\mu_{1,2}=\xi\pm i\pi,$$

where $\xi=T(0)/2$ and $\pi=\sqrt{4D(0)-T^2(0)}/2$. Now, we define the transformation

$$\begin{pmatrix} u\\ v\end{pmatrix}=T\begin{pmatrix} w\\ z\end{pmatrix}, \tag{3.5}$$

where

$$T = \begin{pmatrix} -\dfrac{ab^2}{c^2} & 0 \\ \xi - \dfrac{a(c^2-b^2)}{c^2} & -\pi \end{pmatrix}.$$

If we carry out the transformation (3.5) into Equations (3.4), we obtain

$$\begin{aligned} w_{n+1} &= \xi w_n - \pi z_n + \tilde{P}(w,z)\,, \\ z_{n+1} &= \pi w_n + \xi z_n + \tilde{Q}(w,z)\,, \end{aligned}$$

where

$$\begin{aligned} \tilde{P}(w,z) = &\frac{g_{13}}{2g_{12}}u^2 + \frac{2g_{14}}{g_{12}}uv + \frac{g_{15}}{2g_{12}}v^2 + \frac{g_{16}}{6g_{12}}u^3 + \frac{g_{17}}{2g_{12}}u^2 v \\ &+ \frac{g_{18}}{2g_{12}}uv^2 + \frac{g_{19}}{6g_{12}}v^3 + \mathcal{R}_{\tilde{P},4}(u,v), \end{aligned}$$

and

$$\begin{aligned} \tilde{Q}(w,z) = &\left(\frac{g_{13}(\xi-g_{11})}{2g_{12}} - \frac{g_{23}}{2\pi}\right)u^2 + \left(\frac{2g_{14}(\xi-g_{11})}{g_{12}} - \frac{2g_{24}}{\pi}\right)uv \\ &+ \left(\frac{g_{15}(\xi-g_{11})}{2g_{12}} - \frac{g_{25}}{2\pi}\right)v^2 + \left(\frac{g_{16}(\xi-g_{11})}{6g_{12}} - \frac{g_{26}}{6\pi}\right)u^3 \\ &+ \left(\frac{g_{17}(\xi-g_{11})}{2g_{12}} - \frac{g_{27}}{2\pi}\right)u^2 v + \left(\frac{g_{18}(\xi-g_{11})}{2g_{12}} - \frac{g_{28}}{2\pi}\right)uv^2 \\ &+ \left(\frac{g_{19}(\xi-g_{11})}{6g_{12}} - \frac{g_{29}}{6\pi}\right)v^3 + \mathcal{R}_{\tilde{Q},4}(u,v)\,. \end{aligned}$$

Here, $\mathcal{R}_{\tilde{P},4}(u,v)$ and $\mathcal{R}_{\tilde{Q},4}(u,v)$ present the terms of order larger than 3 in the Taylor expansion of $\tilde{P}$ and $\tilde{Q}$, respectively. Moreover, we have

$$u = g_{12}w\,, \quad v = (\xi - g_{11})w - \pi z\,.$$

Next, in order to determine the direction of the appearance of the invariant curve in a system exhibiting a Neimark–Sacker bifurcation, we consider the first Lyapunov coefficients at the point $(w,z) = (0,0)$ which is given by

$$L = \left(\mathrm{Re}(\mu_2\tau_{21}) - \mathrm{Re}\left(\frac{(1-2\mu_1)\mu_2^2}{1-\mu_1}\tau_{20}\tau_{11}\right) - \frac{1}{2}|\tau_{11}|^2 - |\tau_{02}|^2\right)_{\tilde{a}=0}, \tag{3.6}$$

where

$$\begin{aligned} \tau_{20} &= \frac{1}{8}\left[\tilde{P}_{ww} - \tilde{P}_{zz} + 2\tilde{Q}_{wz} + i\left(\tilde{Q}_{ww} - \tilde{Q}_{zz} - 2\tilde{P}_{wz}\right)\right], \\ \tau_{11} &= \frac{1}{4}\left[\tilde{P}_{ww} + \tilde{P}_{zz} + i\left(\tilde{Q}_{ww} + \tilde{Q}_{zz}\right)\right], \\ \tau_{02} &= \frac{1}{8}\left[\tilde{P}_{ww} - \tilde{P}_{zz} - 2\tilde{Q}_{wz} + i\left(\tilde{Q}_{ww} - \tilde{Q}_{zz} + 2\tilde{P}_{wz}\right)\right], \end{aligned}$$

$$\tau_{21} = \frac{1}{16}\left[\tilde{P}_{www} + \tilde{P}_{wzz} + \tilde{Q}_{wwz} + \tilde{Q}_{zzz} + i\left(\tilde{Q}_{www} + \tilde{Q}_{wzz} - \tilde{P}_{wwz} - \tilde{P}_{zzz}\right)\right].$$

According to the computations described above, we can summarize the results of the existence of a Neimark-Sacker bifurcation in the following theorem [9, 11, 20, 27, 28, 31].

Theorem 1. *If $c > b$, $a < \frac{c}{c-b}$ and condition (3.6) hold with $L \neq 0$. Then, Equations (1.4) has a Neimark-Sacker bifurcation at the positive coexistence equilibrium point*

$$P = \left(\frac{ab+c-ac}{c}, \frac{(c-b)(ab+c-ac)}{bc}\right),$$

when the parameter a varies in a small neighborhood of

$$a_0 = \frac{c^2}{(c-b)(c+b-bc+b^2)}.$$

If $L < 0$, the equilibrium point bifurcates in an attracting invariant closed curve for $a > a_0$. If $L > 0$, a repelling invariant closed curve bifurcates from the equilibrium point when $a < a_0$.

4. Chaos Control

In order to control chaotic behavior, small perturbations must be added, and in result, the randomness change into order. Therefore, chaos control techniques improve the predictability and stability of chaotic orbits [12, 15]. The stabilization of a perturbed system needs to employ chaos control methods as a means to reduce fluctuation and unpredictability.

Here,we discuss the Chaos control of Equations (1.4) using the hybrid control method. In order to apply the hybrid control method to Equations (1.4), we write the corresponding control model in the following from:

$$\begin{aligned} x_{n+1} &= \delta x_n \exp\left(1 - x_n - \frac{a y_n}{x_n + y_n}\right) + (1-\delta)x_n, \\ y_{n+1} &= \delta y_n \exp\left(-b + \frac{c x_n}{x_n + y_n}\right) + (1-\delta)y_n, \end{aligned} \tag{4.1}$$

where $0 < \delta < 1$ is a control parameter for the hybrid control method. The Jacobian matrix of Equations (4.1) evaluated at the positive coexistence equilibrium point P is given by

$$J(P) = \begin{pmatrix} 1 + \delta\left(\dfrac{a(c^2-b^2)-c^2}{c^2}\right) & -\dfrac{a\delta b^2}{c^2} \\ \dfrac{\delta(c-b)^2}{c} & 1 - \dfrac{\delta b(c-b)}{c} \end{pmatrix}, \tag{4.2}$$

Hence, the characteristic equation of the matrix (4.2) is

$$\rho(\mu) = \mu^2 - T\mu + D,$$

where

$$T = 2 + \delta\left(a - 1 - b - \frac{ab^2}{c^2} + \frac{b^2}{c}\right),$$

$$D = \left(1 + \delta\left(\frac{a(c^2-b^2)-c^2}{c^2}\right)\right)\left(1 - \frac{\delta b(c-b)}{c}\right) + \frac{\delta^2 ab(c-b)}{c^3}.$$

Lemma 4. *Let $c > b$ and $ab + c > ac$. Then, the unique positive coexistence equilibrium point $P = \left(\frac{ab+c-ac}{c}, \frac{(c-b)(ab+c-ac)}{bc}\right)$ of Equations (4.1) is locally asymptotically stable if the following condition satisfies*

$$|T| < 1 + D < 2.$$

5. Numerical Simulations

This section presents some numerical examples to verify the obtained theoretical results.

Example 1. Consider the particular case of Equations (1.4) defined by the following values of the parameters:

$$b = 0.7, \quad c = 1.5, \quad a \in [1.7, 1.75],$$

where a the bifurcation parameter. When $(a,b,c) = (1.714939, 0.7, 1.5)$, the Jacobian matrix of Equations (1.4) at coexistence equilibrium P is given by

$$J(P) = \begin{pmatrix} 1.341 & -0.373 \\ 0.426 & 0.626 \end{pmatrix},$$

whose characteristic equation is

$$\rho(\mu) = \mu^2 - 1.9676\mu + 0.99869 = 0.$$

The eigenvalues are given by

$$\mu_{1,2} = 0.98509 \pm 0.175808i,$$

with unit modulus. Furthermore, it is easy to see that

$$\rho(1) = 0.031877 > 0, \quad \left(\frac{d|\mu_{1,2}|}{d\bar{a}}\right)_{\bar{a}=0} = 0.291558 > 0, \quad T(0) = 1.9676 \neq 0, 1,$$

which leads to $\mu_{1,2}^n \neq 1$, for any $n = 1,2,3,4$. In this example, F_1 and F_2 are given by

$$F_1(x,y) = 2.19x^2 - 11.577xy + 2.857y^2 - 4.224x^3 + 2.541x^2y + 36.659xy^2 - 21.278y^3 + \mathcal{R}_{F_1,4}(x,y),$$

and

$$F_2(x,y) = -1.400x^2 + 4.898xy - 1.070y^2 + 3.913x^3 + 4.092x^2y - 16.130xy^2 + 8.359y^3 + \mathcal{R}_{F_2,4}(x,y).$$

Furthermore, the first Lyapunov exponent for these parameters is given by $L \leq -0.0001898$, which proves the correctness of Theorem 1. The bifurcation diagrams for x_n and y_n are depicted in Figure 1, and the maximum Lyapunov exponent is plotted in Figure 2. It is easy to observe that the positive coexistence equilibrium P of Equations (1.4) is locally asymptotically stable for $1.7 \leq a < 1.714939$ (see Figures 3, 4, 5). For $a = 1.714939$, the point P loses the property of local asymptotic stability and a closed invariant curve Γ_s appears around P (see Figures 7, 8 and 9). Thus, a Neimark-Sacker bifurcation occurs at $a = 1.714939$. In Figures 7, 8 and 9, the orbit with initial condition $(0.089, 0.097)$, colored in green, leaves the unstable equilibrium point P and tends asymptotically to the stable invariant curve Γ_s. The orbit colored in red, with initial condition $(0.17, 0.22)$, also tends asymptotically to Γ_s. In Figure 9 we have depicted also an orbit, colored in cyan, with initial condition $(0.3, 0.25)$. From all the above, we can conclude that all the orbits with initial conditions within the invariant curve, except the equilibrium point P, tends asymptotically from the inside towards Γ_s, and all the orbits starting outside the invariant curve also tends asymptotic to Γ_s. The invariant curve Γ_s tends to expand as the value of the parameter $a > a_0$ grows (see Figures 2 (right panel)), and it tends to narrow when we decrease the value of a. For values of the parameter smaller or equal than a_0, the invariant curve has collapsed into the equilibrium point P.

Example 2. In order to assess the performance of the hybrid control method in improving chaotic (unstable) Equations (1.4), we take the same parameter values as given in Example 1 ($b = 0.7$, $c = 1.5$) with $a = 1.7167$. Then, it shows that the coexistence equilibrium point P of Equations (1.4) is unstable (see Figure 10 (right)). However, this equilibrium point is stable for the control Equations (4.1) if $0 < \delta < 0.965$ (see Figure 10 (left)) and unstable if $0.965 < \delta < 1$. This is confirmed by the bifurcation diagrams of x_n and y_n in Figure 11.

6. Concluding remarks

This paper focuses on a qualitative analysis ofa discrete-time Lotka-Volterra system with ratio-dependence, given by Equations (1.4). This system depends on three parameters, a, b and c, and has one positive equilibrium P. We have analyzed the stability of P in Lemma 3. In this Lemma, we have presented some conditions under which the point P is sink, unstable saddle, source, or non-hyperbolic. It is clear that there is a unique positive coexistence equilibrium P of Equations (1.4) with $c > b$ and $ab + c > ac$. We have also proved the existence of a Neimark-Sacker bifurcation

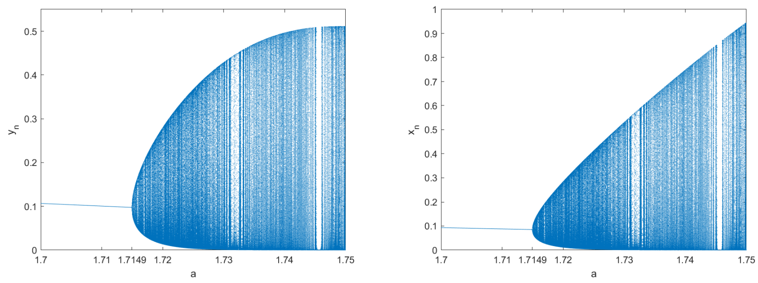

FIGURE 1. Bifurcation diagram for x_n (right) and y_n (left), for the initial condition $(0.089, 0.097)$, and $a \in [1.7, 1.75]$.

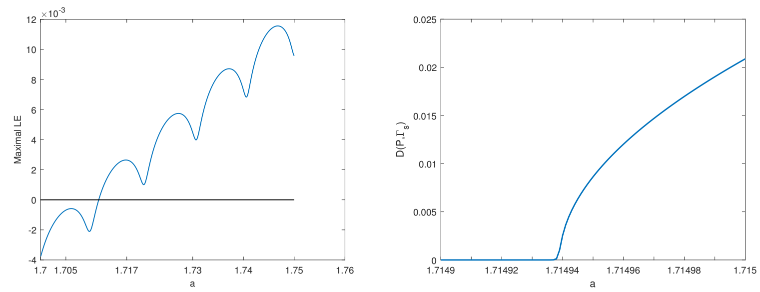

FIGURE 2. In the left panel, maximum Lyapunov exponent for the system given in Equations (1.4), with $a \in [1.7, 1.75]$, $(b, c) = (0.7, 1.5)$ and initial conditions $(0.0853, 0.09756)$. In the right panel, evolution of the distance between the equilibrium point P and the invariant closed curve Γ_s in terms of the bifurcation parameter a.

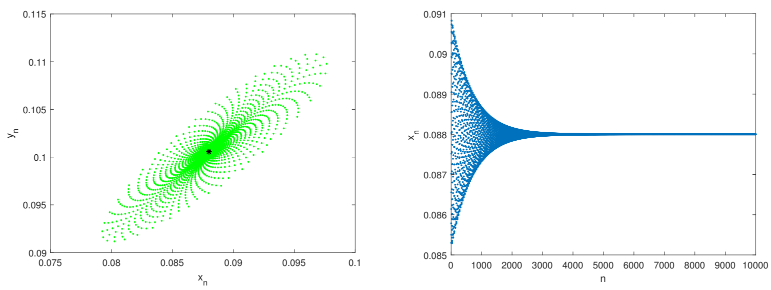

FIGURE 3. Phase portrait for $(a, b, c) = (1.71, 0.7, 1.5)$ (left panel) and evolution of x_n for $a = 1.71$ (right panel).

at the coexistence equilibrium P through the analysis of the normal form of the system, concluding this type of bifurcation occurs when the parameters (a, b, c) vary on

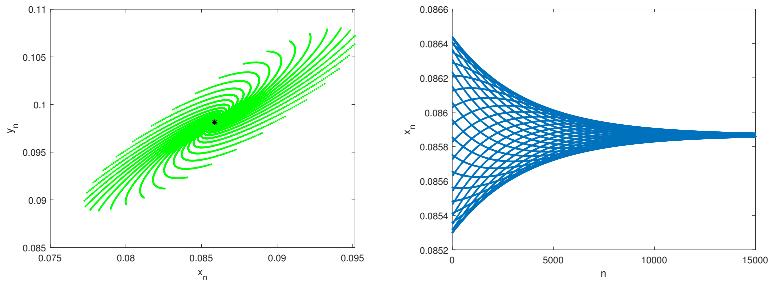

FIGURE 4. Phase portrait for $(a,b,c) = (1.714, 0.7, 1.5)$ (left panel), and evolution of x_n for $a = 1.714$ (right panel).

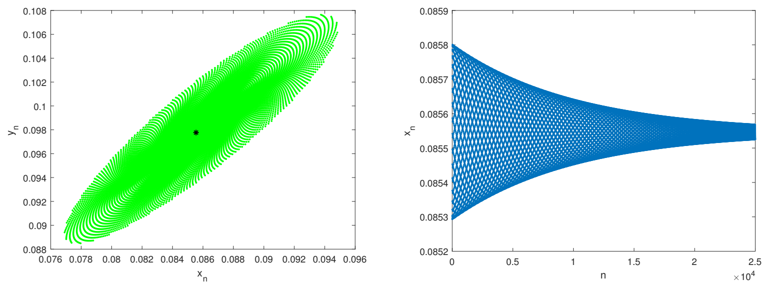

FIGURE 5. Phase portrait for $(a,b,c) = (1.7146, 0.7, 1.5)$ (left panel), and evolution of x_n for $a = 1.7146$ (right panel).

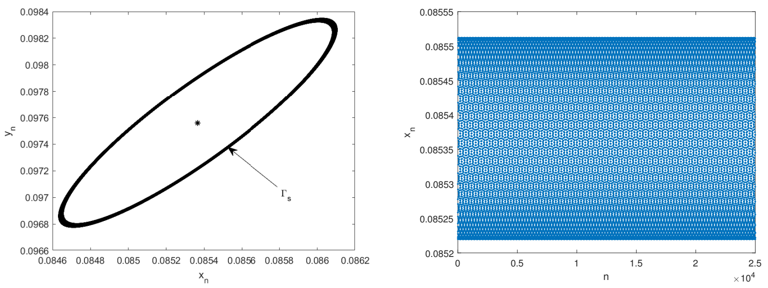

FIGURE 6. Invariant closed curve Γ_s (left panel) and evolution of x_n (right panel) for $(a,b,c) = (1.714939, 0.7, 1.5)$.

the neighborhood $\mathcal{B}$. In the context of biology, Equations (1.4) can be viewed as a predator-prey system interaction. In terms of the latter, the existence of a Neimark-Sacker bifurcation in Equations (1.4) implies that both the prey and predator populations can oscillate around some mean values, and these oscillations will continue

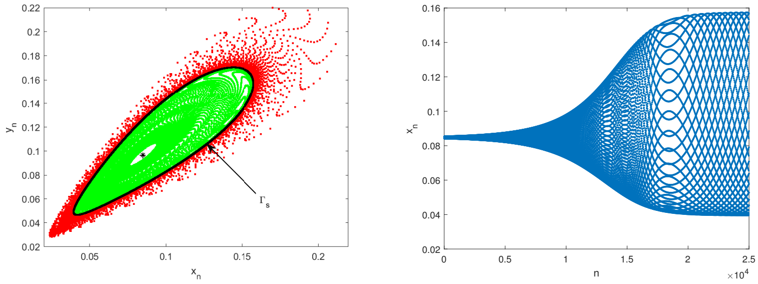

FIGURE 7. Phase portrait (left panel) and evolution of x_n (right panel) for $a = 1.71599$.

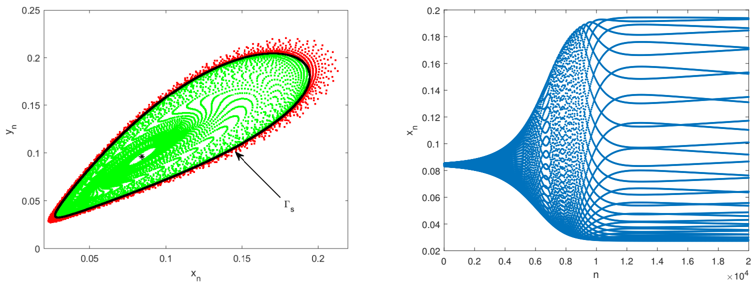

FIGURE 8. Phase portrait (left panel) and evolution of x_n (right panel) for $a = 1.71699$.

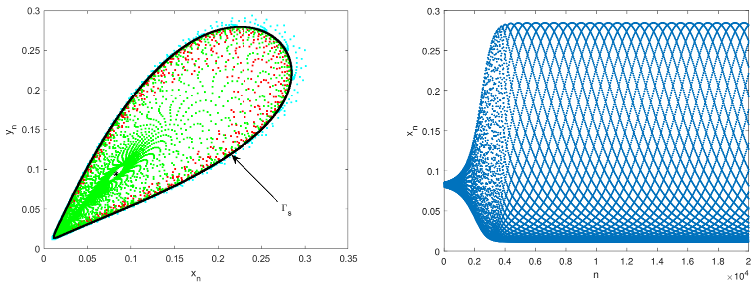

FIGURE 9. Phase portrait (left panel) and evolution of x_n (right panel) for $a = 1.72$.

indefinitely under suitable conditions.These results show far richer dynamics of the discrete model compared to the continuous model. The chaos control of Equations (1.4) has been successfully explored using the hybrid control method. Finally, we have illustrated this theoretical results with the help of a numerical examples.

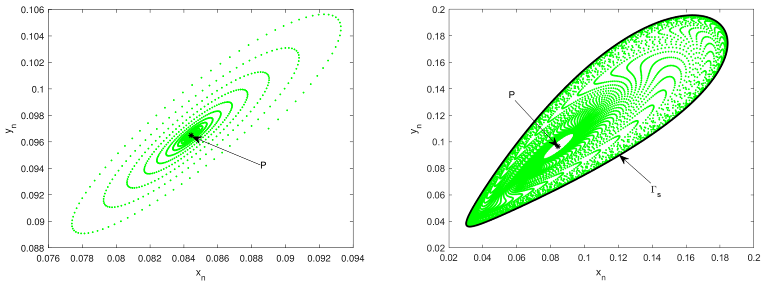

FIGURE 10. Phase portrait of controlled Equations (4.1) for $a = 1.7167$, $\delta = 0.4$ (left) and phase portrait of Equations (1.4) for $a = 1.7167$ (right).

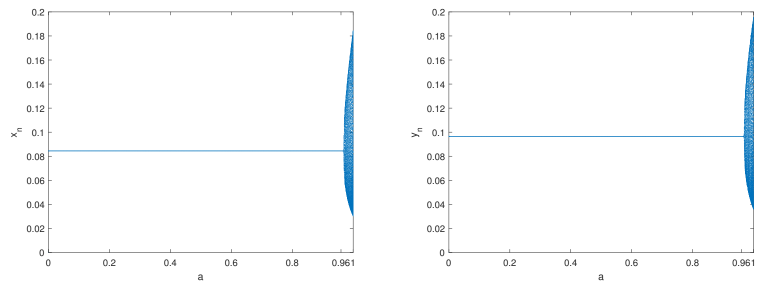

FIGURE 11. Bifurcation diagram of x_n (left) and y_n (right) for controlled Equations (4.1) with $a = 1.7167$, $b = 0.7$, $c = 1.5$ and $\delta \in [0,1]$.

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