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RINGS CHARACTERIZED BY THE EXTENDING PROPERTY FOR FINITELY GENERATED SUBMODULES

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Abstract. A module M is called ef-extending if every closed submodule which contains essentially a finitely generated submodule is a direct summand of M. In this paper, we prove some properties of rings via ef-extending modules and essentially finite injective modules. It is shown that a module M is an ef-extending module and whenever $M = H \oplus K$ with H essentially finite, then H is essentially finite K-injective if and only if for essentially finite submodules N_1, N_2 of M with $N_1 \cap N_2 = 0$, there exist submodules M_1, M_2 of M such that N_i is essential in M_i ($i = 1,2$) and $M_1 \oplus M_2$ is a direct summand of M. A ring R is right co-Harada if and only if R is right (or left) perfect with ACC on right annihilators and $R \oplus R$ is ef-extending as a right R-module, iff R is right (or left) perfect and $R_R^{(\mathbb{N})}$ is an ef-extending module. Some properties of ef-extending modules over excellent extension rings are considered.

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1. INTRODUCTION

The class of injective modules and their generalizations have been researched and developed (in [4, 13–15]). The structure of rings through the properties of the classes of these modules has been considered. One of the important generalizations of the injective module class is the extending (or C1, CS) module class. It can be said that this class of modules has been strongly researched and developed over the past decades. Many authors have given many important properties and characteristics of this class of modules. In addition, QF-rings and co-Harada rings have been studied via extending modules ([3, 5, 6, 16]). In 1991, Thuyet and Wisbauer ([17]) introduced the concept of ef-extending modules. This is a generalization of the class of extending modules. Then the authors also gave many properties of this class of modules. Continuing this work, the authors Quynh, Thuyet, Chien ([2, 16]) have obtained more properties of the class of ef-extending modules.

In this paper, we give some other properties of ef-extending modules and the relatively injective of modules.

Firstly, we introduce the essentially finite N-injective modules. A module M is called *essentially finite N-injective* if any homomorphism from every essentially finite submodule of N to M can be extended to a homomorphism from N to M. We study the property of modules that whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2, then M_2 is essentially finite M_1-injective. It is shown that an essentially finite ef-extending module M satisfies C3 if and only if, whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1 and M_2, then M_2 is essentially finite M_1-injective (Proposition 4). We also prove that a module M is an ef-extending module and whenever $M = H \oplus K$ with H essentially finite, then H is essentially finite K-injective if and only if for essentially finite submodules N_1, N_2 of M with $N_1 \cap N_2 = 0$, there exist submodules M_1, M_2 of M such that N_i is essential in M_i $(i = 1,2)$ and $M_1 \oplus M_2$ is a direct summand of M (Proposition 5).

Secondly, we characterize the structure of co-Harada rings and show that a ring R is right co-Harada if and only if R is right (or left) perfect with ACC on right annihilators and $R \oplus R$ is ef-extending as a right R-module, iff R is right (or left) perfect and $R_R^{(\mathbb{N})}$ is an ef-extending module (Theorem 1). The structure of rings in which the direct sum of any two ef-extending right R-modules is ef-extending are considered. It is shown that if R has property the direct sum of any two ef-extending right R-modules is ef-extending and $E(R_R) = \bigoplus_{i \in I} E_i$ where E_i is indecomposable for all $i \in I$, then R is a right Artinian ring whose uniform right R-modules have length at most two (Theorem 8).

Finally, we study ef-extendingmodules over excellent extension rings and show that if M is a right S-module and M_R is an ef-extending module, then M_S is an ef-extending module (Theorem 2). On the other hand, if M is a right R-module and $(M \otimes_R S)_S$ is an ef-extending module then M_R is an ef-extending module (Theorem 3).

Throughout this paper, R will denote an associative ring with unit and Mod-R the category of unital right R-modules. We write M_R (resp., ${}_RM$) to denote that M is a right (resp., left) R-module. Unless otherwise mentioned, by a module we will mean a right R-module. We denote the Jacobson radical of a ring R by $J(R)$ and the injective hull of M by $E(M)$. If A is a submodule of M, we denote by $A \leq M$. A submodule K of M is *essential* in M if $K \cap L \neq 0$ for every non-zero submodule L of M. In this case, M is called an *essential extension* of K and we write $K \leq^e M$. A submodule C of M is *closed* in M if C has no proper essential extension in M. A module M is called uniform if $M \neq 0$ and every non-zero submodule of M is essential in M. A submodule N of M is called *small* in M, denote by $N \ll M$ whenever for any submodule L of M, $N + L = M$ implies $L = M$. A module M is called a *small module* if M is small in $E(M)$. If M is not a small module, we say that M is *non-small*. We denote the radical of a right R-module by $Rad(M)$. We refer to [1, 4, 10] for any undefined notion used in the text.

2. Direct sum of EF-extending modules are EF-extending

A module M is called *essentially finitely generated* (*essentially finite* for short) if M contains a finitely generated submodule that it is essential in M ([4, page 60]). A submodule N of M is called an *essentially finite submodule* if N is an essential finite right R-module.

Lemma 1. *Every direct summand of an essentially finite module is essentially finite.*

Proof. Let $M = N \oplus N'$ be an essentially finite module. Call $H = m_1R + m_2R + \cdots + m_kR$ an essential submodule of M. Write $m_i = n_i + n'_i$ with $n_i \in N$ and $n'_i \in N'$ for all $i = 1, 2, \ldots, k$. This shows that $K = n_1R + n_2R + \cdots + n_kR$ is an essential submodule of N. In fact, for every nonzero element y of N, there is x in R such that yx is nonzero in H, and so $yx = m_1r_1 + m_2r_2 + \cdots + m_kr_k$ for some r_i in R. It follows that $yx = n_1r_1 + n_2r_2 + \cdots + n_kr_k$ is nonzero in K. We deduce that K is essential in N. □

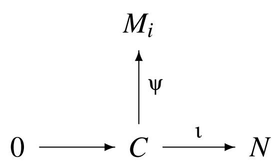
Let M and N be right R-modules. We define M to be *essentially finite N-injective* if any homomorphism from every essentially finite submodule of N to M can be extended to a homomorphism from N to M. We say to a module M is *essentially finite quasi-injective* if M is essentially finite M-injective.

Proposition 1. *Let M and N be right R modules.*

(1) *Any direct product $\prod_I M_i$ is essentially finite N-injective if and only if every M_i is essentially finite N-injective.*
(2) *If M is essentially finite N-injective, then M is essentially finite K-injective and essentially finite N/A-injective for all submodules A and K of N with A finitely generated.*

Proof. (1) Let $M = \prod_I M_i$. For each $i \in I$, let $\pi_i \colon M \to M_i$ be the canonical projection and $\iota_i \colon M_i \to M$ be the inclusion map.

Assume that M is essentially finite N-injective. Let C be an essentially finite submodule of N and $\psi \colon C \to M_i$ be a homomorphism. Let $\iota \colon C \to N$ be the inclusion map.



Since M is essentially finite N-injective, there is a homomorphism $\alpha\colon N \to M$ such that $\alpha\iota = \iota_i\psi$. Call $\gamma = \pi_i\alpha$. It follows that $\gamma\iota = \psi$. Thus, M_i is essentially finite N-injective.

Conversely, assume that M_i is essentially finite N-injective for all $i \in I$. Let C be an essentially finite submodule of N, $\iota\colon C \to N$ be the inclusion map and $\psi\colon C \to M$ be a homomorphism. For each $i \in I$, since M_i is essentially finite N-injective, there exists a homomorphism $h_i\colon M \to M_i$ such that $h_i\iota = \pi_i\psi$. Consider the homomorphism $h\colon M \to \prod_{i\in I} M_i$ defined by $h(m) = (h_i(m))_{i\in I}$ for all $m \in M$. Then, $h\iota = \psi$. We deduce that M is essentially finite N-injective.

(2) Let $f\colon U \to M$ be a homomorphism from an essentially finite submodule U of K to M. Since M is essentially finite N-injective, f is extended to a homomorphism $g\colon N \to M$. It follows that $g|_K\colon K \to M$ is an extension of f. Thus, M is essentially finite K-injective.

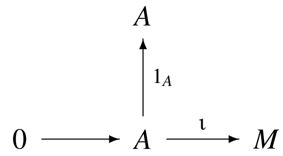
Next, we show that M is essentially finite N/A-injective with each finitely generated submodule A of N. Let $f\colon X/A \to M$ be a homomorphism from an essentially finite submodule X/A of N/A to M. Call $A_1 = (a_1 + A)R + (a_2 + A)R + \cdots + (a_k + A)R$ an essential submodule of X/A. One can check that $a_1R + a_2R + \cdots + a_kR + A$ is essential in X and so X is an essentially finite submodule of N. Let $\pi\colon N \to N/A$ be the natural projection. Since M is essentially finite N-injective, there is a homomorphism $g\colon N \to M$ such that g is an extension of $f \circ \pi|_X$. Then, $g(A) = (f \circ \pi|_X)(A) = 0$ so that there exists a homomorphism $h\colon N/A \to M$ such that $h \circ \pi = g$. For each $x \in X$, we have

$$h(x + A) = (h \circ \pi)(x) = g(x) = (f \circ \pi|_X)(x) = f(x).$$

It is shown that h is an extension of f. We deduce that M is essentially finite N/A-injective. □

Corollary 1. *Let M be an essentially finite quasi-injective right R-module. Then, every essentially finite submodule of M that is isomorphic to a direct summand of M is itself a direct summand of M.*

Proof. Let A be an essentially finite submodule of M that is isomorphic to a direct summand B of M. Since M is essentially finite M-injective, B is essentially finite M-injective by Proposition 1(1) and so A is essentially finite M-injective.



It follows that there is a homomorphism $f\colon M \to A$ such that $g \circ \iota = 1_A$ with $\iota\colon A \to M$ the inclusion map. Therefore, ι splits and so A is a direct summand of M. □

A nonzero module M does not contain an infinite direct sum of nonzero submodules if and only if there exists a positive integer n such that M contains an essential submodule of the form $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ for some uniform submodules U_i of M. Furthermore, the integer n is an invariant of the module M called the *uniform dimension* or *Goldie dimension* and denoted by $\dim(M)$. If $M = 0$ we write $\dim(M) = 0$.

Proposition 2. *Let M be a right R-module.*

(1) *If M is a right R-module with finite Goldie dimension, then M is quasi-injective if and only if M is essentially finite quasi-injective.*
(2) *If $M = M_1 \oplus M_2$ is essentially finite quasi-injective then M_1 is essentially finite M_2-injective.*

Proof. (1) Assume that M is a right R-module with finite Goldie dimension. Let $f : A \to M$ be a homomorphism from a submodule A of M to M. It is well-known that every submodule of a right R-module with finite Goldie dimension has finite Goldie dimension. Then, A is an essentially finite submodule of M. We have that M is quasi-injective and obtain that f is extended to an endomorphism of M. We deduce that M is quasi-injective.

(2) Assume that $M = M_1 \oplus M_2$ is essentially finite quasi-injective. Then, M_1 is essentially finite M-injective by Proposition 1(1). Again by Proposition 1(2) we have that M_1 is essentially finite M_2-injective. □

Corollary 2. *Let M be a right R-module. If M is a Noetherian module, then M is quasi-injective if and only if M is essentially finite quasi-injective.*

Proof. Since every Noetherian right R-module has finite Goldie dimension, M is quasi-injective if M is essentially finite quasi-injective by Proposition 2. □

Recall that a right R-module M is called *Zelmanowitz regular* if, every finitely generated submodule of M is projective and is a direct summand of M ([18]). Note that R_R is a Zelmanowitz regular module if and only if R is a von Neumann regular ring.

Lemma 2. *If M is a Zelmanowitz regular module, then M is essentially finite quasi-injective.*

Proof. Let H be an essentially finite submodule of M. Then, there is a finitely generated submodule I of M such that I is essential in H. We have that M is Zelmanowitz regular and obtain that I is a direct summand of M and so, it is a direct summand of H. This implies that $H = I$ is a direct summand of M. We deduce that every homomorphism from H to M can be extended to an edomorphism of M. Thus, M is essentially finite quasi-injective. □

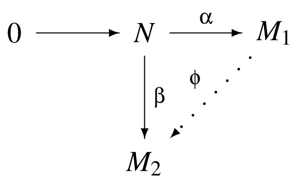
From Lemma 2, we have the following example of an essentially finite quasi-injective module which is not quasi-injective.

Example 1. Let D be a division ring and ${}_DV$ be a left D-vector space of infinite dimension. Take $R = \mathrm{End}_D(V)$. It is well-known that R is von Neumann regular but not right self-injective. By Lemma 2, R_R is essentially finite quasi-injective.

Lemma 3. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2. Then, the following conditions are equivalent:*

(1) *M_2 is essentially finite M_1-injective.*
(2) *For each essentially finite submodule N of M with $N \cap M_2 = 0$, there exists a submodule M' of M containing N such that $M = M' \oplus M_2$.*

Proof. $(1) \Rightarrow (2)$. For $i = 1, 2$, let $\pi_i \colon M \longrightarrow M_i$ denote the canonical projection. Consider the following diagram with N an essentially finite submodule of M:



where $\alpha = \pi_1|_N$, $\beta = \pi_2|_N$. It is easy to see that α is a monomorphism, and so $N \cong \alpha(N) \leq M_1$. By (1), there exists a homomorphism $\phi \colon M_1 \longrightarrow M_2$ such that $\phi\alpha = \beta$. Let $M' = \{x + \phi(x) | x \in M_1\}$. One can check that $M = M' \oplus M_2$ and N is contained in M'.

$(2) \Rightarrow (1)$. Let K be an essentially finite submodule of M_1, and $f \colon K \longrightarrow M_2$ a homomorphism. Put $L = \{y - f(y) \mid y \in K\}$. Since K is essentially finite, L is also an essentially finite submodule of M with $L \cap M_2 = 0$. By (2), $M = L' \oplus M_2$ for some submodule L' of M containing L. Let $\pi \colon M \longrightarrow M_2$ denote the canonical projection (for the direct sum $M = L' \oplus M_2$). Let $\bar{f} = \pi|_{M_1} \colon M_1 \longrightarrow M_2$ and, for any $y \in K$, we have $\bar{f}(y) = \bar{f}(y - f(y) + f(y)) = f(y)$. It means that $\bar{f}$ is an extension of f and so M_2 is essentially finite M_1-injective. □

Corollary 3. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1, M_2. If M_2 is essentially finite M_1-injective, then for every essentially finite closed submodule N of M such that $N \cap M_1$ is essential in M_1 and $N \cap M_2 = 0, M = N \oplus M_2$.*

Proof. Let K be an essentially finite closed submodule of M such that $K \cap M_1$ is essential in M_1 and $K \cap M_2 = 0$. We have $K \cap M_1 = K \cap ((K \cap M_1) \oplus M_2)$. It follows that $K \cap M_1$ is essential in K. By Lemma 3, there exists a submodule M' of M such that $M = M' \oplus M_2$ and $K \leq M'$. Next, we show that K is essential in M'. In fact, let $x = m_1 + m_2$ be a nonzero element of M with $m_i \in M_i$. If m_1 is nonzero, there exists r in R such that $m_1 r \in K \cap M_1$ is nonzero. It follows that $xr = m_1 r + m_2 r \in K \oplus M_2$ is nonzero. From this, we deduce that there exists $y \in R$ such that $xy \in K \oplus M_2$ is nonzero. It is shown that $K \oplus M_2$ is essential in M. On the other hand, we have that $K = M' \cap (K \oplus M_2)$ and obtain that K is essential in M'. But, K is closed in M, and so $K = M'$. □

Recall that a module M is *extending* (resp., *ef-extending*) if every closed (resp., essentially finite closed) submodule of M is a direct summand of M (see [4, 17]).

Example 2. Let D be a division ring and V a left D-vector space of infinite dimension. Take $S = \mathrm{End}_D(V)$ and $R = \begin{pmatrix} S & S \\ S & S \end{pmatrix}$. It is well-known that S is a von Neumann regular ring. One can check that R_R is ef-extending but not extending.

It is well-known that a module M is extending if every submodule of M is essential in a direct summand of M. We have a similar situation for ef-extending modules.

Proposition 3. *The following conditions are equivalent for a right R-module M:*

(1) *M is ef-extending.*
(2) *Every essentially finite submodule of M is essential in a direct summand of M.*

Proof. $(1) \Rightarrow (2)$. Let N be an essentially finite submodule of M. Then, N contains a finitely generated submodule which is essential in N. Call H a maximal essential extension of N in M. One can check that H is an essentially finite closed submodule of M. By (1), H is a direct summand of M.

$(2) \Rightarrow (1)$. Let C be an essentially finite closed submodule of M. By (2), there is a direct summand K of M such that C is essential in K. We have that C is closed and obtain that $C = K$ is a direct summand of M. □

Corollary 4. *Let M be a right R-module with finite Goldie dimension or finitely cogenerated. Then, M is an extending module if and only if M is an ef-extending module.*

Corollary 5. *Let M be a finitely generated ef-extending right R-module. Assume that*

(1) *every local direct summand of M is a direct summand, or*
(2) *$End(M)$ does not contain an infinite set of orthogonal idempotents.*

Then M is an extending module.

Proof. Assume that every local direct summand of M is a direct summand, or $End(M)$ does not contain an infinite set of orthogonal idempotents. Then, M is a direct sum of uniform submodules by [17, Corollary 2.3]. Moreover, we have that M is finitely generated and obtain that M has finite Goldie dimension. We deduce that M is extending by Corollary 4. □

Recall that a module M satisfies $C3$ if, $M_1 \oplus M_2$ is a direct summand of M for any two direct summands M_1 and M_2 of M with $M_1 \cap M_2 = 0$.

Lemma 4 ([5, Lemma 6])**.** *The following statements are equivalent for a right R-module M.*

(1) *M satisfies $C3$.*

(2) *For all direct summands P and Q of M with $P \cap Q = 0$, there exists a submodule P' of M such that $M = P \oplus P'$ and $Q \leq P'$.*

Proposition 4. *An essentially finite ef-extending module M satisfies C3 if and only if whenever $M = M_1 \oplus M_2$ is a direct sum of submodules M_1, M_2, then M_2 is essentially finite M_1-injective.*

Proof. $(\Rightarrow)$. Assume that M is ef-extending satisfying C3. Let N be an essentially finite submodule N of M with $N \cap M_2 = 0$. Since M is ef-extending, there exists a direct summand N' of M such that N is essential in N'. Clearly $N' \cap M_2 = 0$. By Lemma 4, $M = M' \oplus M_2$ for some submodule M' such that $N' \leq M'$, and so $N \leq M'$. Thus M_2 is essentially finite M_1-injective by Lemma 3.

$(\Leftarrow)$ Assume that M_2 is essentially finite M_1-injective whenever $M = M_1 \oplus M_2$. By Lemma 3 and Lemma 4, M satisfies C3. □

Corollary 6. *If $M = M_1 \oplus M_2$ is an essentially finite ef-extending right R-module satisfying C3, then M_i is essentially finite M_j-injective for all $i, j \in \{1,2\}$, $i \neq j$.*

Proposition 5. *The following conditions are equivalent for a right R-module M:*

(1) *For essentially finite submodules N_1, N_2 of M with $N_1 \cap N_2 = 0$, there exist submodules M_1 and M_2 of M such that N_i is essential in M_i ($i = 1,2$) and $M_1 \oplus M_2$ is a direct summand of M.*

(2) *M is an ef-extending module and whenever $M = H \oplus K$ with H essentially finite, then H is essentially finite K-injective.*

Proof. $(1) \Rightarrow (2)$ Let N be an essentially finite submodule of M. Call N' a complement of N in M. Then, $N \cap N' = 0$. Take K a finitely generated submodule of N' and so $N \cap K = 0$. By (1), there are submodules M_1 and M_2 of M such that N is essential in M_1, K is essential in M_2 and $M_1 \oplus M_2$ is a direct summand of M. It follows that M is an ef-extending module. Next, we show that if M has a decomposition $M = H \oplus K$ with H essentially finite, then H is essentially finite K-injective. In fact, let L be an essentially finite submodule of M with $L \cap H = 0$. By (1), there are submodules P and Q of M such that L is essential in P, H is essential in Q and $P \oplus Q$ is a direct summand of M. We have that H is closed and obtain that $H = Q$. Write $M = P \oplus Q \oplus W$. Then, $M = H \oplus (P \oplus W)$ and $L \leq P \oplus W$. By Lemma 3, H is essentially finite K-injective.

$(2) \Rightarrow (1)$ Let N_1 and N_2 be essentially finite submodules of M with $N_1 \cap N_2 = 0$. Call L a finitely generated submodule of M such that L is essential in N_1. Let C be a complement of L in M, and so $N_1 \cap C = 0$. By Zorn's lemma, take D a maximal essential extension of N_1 in M. One can check that $D \cap C = 0$, D is closed and L is essential in D. By (2), $M = D \oplus D'$ for some submodule D' of M. Moreover, from the essentiality of N_1 in D, it immediately infers that $N_2 \cap D = 0$. Again, by (2) and Lemma 3, we have a decomposition $M = D \oplus D$" with D" a direct summand of M containing N_2. Since M is an ef-extending module, D" is also an ef-extending module. By Proposition 3, N_2 is essential in a direct summand W of D". Write $D" = W \oplus W'$. Then, $M = D \oplus W \oplus W'$ with N_1 essential in D and N_2 essential in W. □

Note that the direct sum of two direct summands which are ef-extending is not necessarily ef-extending.

Example 3. Let p be a prime number. Then $\mathbb{Z}$-modules $\mathbb{Z}/p\mathbb{Z}$, $\mathbb{Z}/p^3\mathbb{Z}$ are ef-extending. But $\mathbb{Z}$-module $M = \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p^3\mathbb{Z}$ is not ef-extending because $(1+p\mathbb{Z}, p+p^3\mathbb{Z})\mathbb{Z}$ is a closed submodule of M (which contains a finitely generated, essential submodule) and it is not direct summand of M.

Proposition 6. *Let $M = M_1 \oplus M_2$ be a direct sum of submodules M_1 and M_2. Assume that the following conditions hold:*

(1) *every closed submodule K of M with $K \cap M_1$ essential in K is a direct summand of M;*
(2) *every essentially finite closed submodule L of M with $L \cap M_1 = 0$ is a direct summand of M.*

Then M is ef-extending.

Proof. Let N be an essentially finite closed submodule of M. If $N \cap M_1 = 0$, we are done. Otherwise, there exists a closed submodule H of N such that $N \cap M_1$ is essential in H. Then, H is closed in M and $H \cap M_1 = N \cap M_1$ is essential in H. By (1), H is a direct summand of M. Take $M = H \oplus H'$, and so $N = H \oplus (N \cap H')$. It follows that $N \cap H'$ is a closed submodule of N, and so it is closed in M. By Lemma 1, $N \cap H'$ is an essentially finite closed submodule of M. We have $(N \cap H') \cap M_1 = 0$ and obtain, from (2), that $N \cap H'$ is a direct summand of M. We write $H' = (N \cap H') \oplus L$. Thus, $M = H \oplus H' = H \oplus (N \cap H') \oplus L = N \oplus L$. □

Corollary 7. *Assume that $M = M_1 \oplus M_2$ with M_1 is extending and M_2 is ef-extending. If one of the following conditions holds, then M is ef-extending.*

(1) *M_2 is M_1-injective and M_1 is essentially finite M_2-injective.*
(2) *M_2 is essentially finite M_1- injective and every closed submodule K of M such that $K \cap M_2 = 0$, is a direct summand of M.*

Proof. Assume that M_2 is M_1-injective and M_1 is essentially finite M_2-injective.

Firstly, we show that the condition (1) in Proposition 6 is satisfied. Let K be a closed submodule of M with $K \cap M_1$ an essential submodule of K. Then, $K \cap M_2 = 0$. Sine M_2 is M_1-injective, there exists a decomposition $M = L \oplus M_2$ such that L contains K. It follows that K is closed in L and $L \cong M_1$ an extending module. We deduce that K is a direct summand L, and so it is a direct summand of M.

Next, we show that the condition (1) in Proposition 6 is satisfied. Let N be an essentially finite closed submodule of M with $N \cap M_1 = 0$. Since M_1 is essentially finite M_2-injective, there is a submodule H of M such that $M = H \oplus M_1$ and $N \leq H$. Then, N is an essentially finite closed submodule of H and $H \cong M_2$ is an ef-extending module. We deduce that N is a direct summand of H. Thus, N is a direct summand of M. From Proposition 6, we deduce that M is an ef-extending module.

Case (2) has a similar proof. □

Lemma 5. *Let A and B be uniform modules with local endomorphism rings such that $M = A \oplus B$ is ef-extending. Let C be a submodule of A and $f: C \to B$ a homomorphism. Then the following hold.*

(1) *If f cannot be extended to a homomorphism from A to B, then f is a monomorphism and B is embedded in A.*
(2) *If any monomorphism $B \to A$ is an isomorphism, then B is A-injective.*
(3) *If B is not embedded in A, then B is A-injective.*

Proof. (1). Suppose f cannot be extended to A. Let

$$U = \{x - f(x) | x \in C\} \leq A \oplus B.$$

Then $U \cong C$ is a uniform submodule of M and clearly $U \cap B = 0$. Hence there is a direct summand U^* of M such that U is essential in U^*. By the Krull-Schmidt-Azumaya Theorem ([1, Corollary 12.7]), we have $M = A \oplus U^*$ or $M = U^* \oplus B$. Suppose that $M = B \oplus U^*$. Let $\pi: B \oplus U^* \to B$ be the projection. For every $c \in C$, we have $c = f(c) + c - f(c)$ and so $(\pi|_A)(c) = \pi(f(c) + c - f(c)) = f(c)$. It is shown that $\pi|_A: A \to B$ is an extension of $f: C \to B$, a contradiction to our assumption. Thus $M = A \oplus U^*$ which implies that $f(x) \neq 0$ for all $x \neq 0$, i.e., f is a monomorphism. We have $U^* \cap B = 0$ and obtain that B is embedded in A.

(2). As in the proof of (1), given any homomorphism $f: C \to B$ with C a submodule of A. Suppose that $M = A \oplus U^*$. Let $\psi: A \oplus U^* \to A$ be the projection. Then, clearly $\psi|_B$ is a monomorphism (because U is essential in U^*, hence an isomorphism by the hypothesis). It follows easily that $M = B \oplus U^*$, so that, as in (1) f can be extended to a homomorphism from A to B. It follows that B is A-injective.

(3). It is clear by (1). □

A module M is called a *non-cosmall* module if M is a homomorphic image of a projective module P whose kernel is not essential in P. A ring R is called right *co-Harada* (or co-H) if it satisfies ACC on right annihilators and every non-cosmall right R-module contains a non-zero projective direct summand ([6]). We have a characterization of right co-Harada via ef-extending.

Theorem 1. *Let R be a ring. Then, the following statements are equivalent.*

(1) *R is right co-Harada.*
(2) *R is right (or left) perfect with ACC on right annihilators and $R \oplus R$ is ef-extending as a right R-module.*
(3) *R is right (or left) perfect and $R_R^{(\mathbb{N})}$ is an ef-extending module.*

Proof. $(1) \Rightarrow (2), (3)$ are clear.

$(2) \Rightarrow (1)$. Let R be a right perfect ring. We need only show that $E(R_R)$ is Σ-injective (i.e., $E(R_R)^{(I)}$ is injective for every index set I). Since R has ACC on right annihilators, it suffices to show that $E(R_R)$ is projective. Let e be any primitive idempotent of R. Suppose that eR is non-small. Then for any primitive idempotent f of R, assume that $h\colon eR \to fR$ is a monomorphism, if $eR \cong A = h(eR) < fR$. Since fR has only maximal, small submodule in fR. Thus $A \ll fR$ and so $A \ll E(A)$. Moreover, there exists $E(A) \cong E(eR)$. It implies that $eR \ll E(eR)$. This is a contradiction. It means that $h(eR) = fR$ and so h is an isomorphism. Note that eR and fR are uniform modules. Since $(R \oplus R)_R$ is ef-extending, $eR \oplus fR$ is ef-extending, too. By Lemma 5, eR is fR- injective. It is shown that eR is injective as a right R-module. Now suppose that eR is small. Since R is right perfect, there is a projective cover $\varphi\colon P \to E(eR)$, where $P = \oplus_I P_i$ each P_i, is indecomposable projective. If P_j is small for some $j \in I$, then $\varphi(P_j)$ is small in $E(eR)$ by [10, Lemma 4.2]. Let $P = P_j \oplus Q$, then $E(eR) = \varphi(Q)$. But $Ker(\varphi)$ is small in P, so it follows that $Q = P$, a contradiction. Thus each P_i is non-small, hence injective. It is easy to see that, because eR is projective, eR can be embedded into a finite direct sum of the P_i. Therefore $E(eR)$ is projective. We write $R_R = e_1R \oplus \cdots \oplus e_kR \oplus f_1R \oplus \cdots \oplus f_nR$, where $\{e_i\}_{i=1}^k \cup \{f_j\}_{j=1}^n$ is a complete set of orthogonal primitive idempotents of R, in which each e_iR is a non-small module $(i = 1,\ldots,k)$, each f_jR is a small module $(j = 1,\ldots,n)$. Therefore $E(R_R)$ is projective. Thus R is semiprimary and $Soc(R_R)$ is essential in R_R by [6, Theorem 6]. This implies that $R \oplus R$ is extending as a right R-module by Corollary 4. Thus R is right co-Harada by [3, Theorem II].

Now, we assume that R is left perfect. It follows that $Soc(R_R)$ is essential in R_R, and so R_R is finitely cogenerated. It follows that $(R \oplus R)_R$ is extending by Corollary 4. From [15, Theorem 3.7], it infers that R is right co-Harada.

(3) $\Rightarrow$ (2). By (3), R^n is ef-extending for each $n \in \mathbb{N}$. We will prove that R has ACC on right annihilators. Now we claim that $E(R_R)^{(\mathbb{N})}$ is injective. Since R is right perfect, $R_R = e_1R \oplus \cdots \oplus e_kR \oplus f_1R \oplus \cdots \oplus f_nR$, where $\{e_i\}_{i=1}^k \cup \{f_j\}_{j=1}^n$ is a complete set of orthogonal primitive idempotents of R such that:

- e_iR is non-small for all $i = 1,2,\ldots,k$.
- f_jR is small for all $j = 1,2,\ldots,n$.

By proving the same (2) $\Rightarrow$ (1), we have e_iR is injective for all $i = 1,2,\ldots,k$ and obtain that $E(R_R)$ is projective. For any $j = 1,2,\ldots,n$, since R is right ef-extending, f_jR is uniform and so $E(f_jR)$ is indecomposable. Moreover $E(f_jR)$ is projective, there is $i \in \{1,2,\ldots,k\}$ such that $E(f_jR) \cong e_iR$. From this, it is easy to see that

$$E(R_R) = (\oplus_{I_1} e_1R) \oplus \cdots \oplus (\oplus_{I_k} e_kR),$$

where I_i are finite sets. Then, we have

$$E(R_R)^{(\mathbb{N})} \cong ((\oplus_{I_1} e_1R) \oplus \cdots \oplus (\oplus_{I_k} e_kR))^{(\mathbb{N})} \cong (e_1R \oplus e_2R \oplus \cdots \oplus e_kR)^{(\mathbb{N})}.$$

Note that $(e_1R \oplus e_2R \oplus \cdots \oplus e_kR)^{(\mathbb{N})}$ is isomorphic to a direct summand of $R_R^{(\mathbb{N})}$ and each e_iR is injective and indecomposable. We have that $R_R^{(\mathbb{N})}$ is ef-extending and obtain that $(e_1R \oplus e_2R \oplus \cdots \oplus e_kR)^{(\mathbb{N})}$ is ef-extending. Thus, by [4, Lemma 8.10], $E(R_R)^{(\mathbb{N})} \cong (e_1R \oplus e_2R \oplus \cdots \oplus e_kR)^{(\mathbb{N})}$ is self-injective, and so it is injective. This implies that R has ACC on right annihilators.

It is similar if R is left perfect. Note that $E(R_R)$ is projective by Corollary 4 and [15, Theorem 3.3]. □

Corollary 8 ([6, Theorem 3])**.** *Let R be a ring. Then the following statements are equivalent.*

(1) *R is right co-Harada.*
(2) *R is right perfect and $R_R^{(\mathbb{N})}$ is an extending module.*

Proposition 7. *Let R be a right nonsingular ring. Then the following statements are equivalent.*

(1) *R is right co-Harada.*
(2) *R is right (or left) perfect and $R \oplus R$ is ef-extending as a right R-module.*

Proof. By Lemma 6, Theorem 7 in [3]. □

Lemma 6. *The following statements are equivalent:*

(1) *The direct sum of any two uniform modules is ef-extending.*
(2) *Any uniform self-injective module has length at most 2.*
(3) *Any direct sum of uniform modules is extending.*

Proof. $(1) \Rightarrow (2)$. Consider any uniform injective module U. Suppose $x \in Rad(U)$ and T is a maximal nonzero submodule of xR. Then, U and xR/T have local endomorphism rings and $U \oplus xR/T$ is ef-extending by assumption. Hence the map $f\colon xR \to xR/T$ can be extended to $\bar{f}\colon U \to xR/T$ by Lemma 5. However $xR \leq Rad(U) \leq Ker\bar{f}$, which yields a contradiction. We conclude that $Rad(U)$ is semisimple and hence simple.

Assume that K_1 and K_2 are two distinct maximal submodules of U. Then any monomorphism $f\colon K_i \to K_j$ is onto for $i, j \in \{1,2\}$, since f extends to a monomorphism of U which has to be an automorphism. So the endomorphism rings of K_1 and K_2 are local. Now $K_i \oplus K_j$ is extending for $i, j \in \{1,2\}$ and hence K_1 is both K_2-injective and K_1-injective by Lemma 5. We have $K_1 + K_2 = U$ and obtain that K_1 is U-injective and hence it is a direct summand in U, a contradiction.

$(2) \Rightarrow (3)$. By [4, 13.1].

$(3) \Rightarrow (1)$. Obvious. □

Proposition 8. *Assume that R has property the direct sum of any two ef-extending right R-modules is ef-extending and $E(R_R) = \bigoplus_{i\in I} E_i$ where E_i is indecomposable for all $i \in I$. Then R is a right Artinian ring whose uniform right R-modules have length at most two.*

Proof. Assume that $E(R_R) = \bigoplus_{i\in I} E_i$ where E_i is indecomposable for all $i \in I$. Then, E_i is a uniform module for every $i \in I$. Let $\{V_i\}_{i\in I_1}$ be any nonempty family of injective hulls of simple modules. Call $V = \oplus_i V_i$. Then the module

$$M = E(R_R) \oplus V$$

is extending by Lemma 6, hence quasi-injective by [4, Lemma 8.10]. Therefore V is a quasi-injective module which is $E(R_R)$-injective. Thus, V is injective. This implies that R is right Noetherian.

By the above argument, every injective module is a direct sum of uniform modules. Since by Lemma 5, each uniform module has length at most two, every injective is a direct sum of injective hulls of simple modules. This proves that R is right Artinian. Hence the proof is complete. □

Corollary 9. *Assume that R has finite Goldie dimension and the direct sum of any two ef-extending right R-modules is ef-extending. Then, R is a right Artinian ring whose uniform right R-modules have length at most two.*

3. EF-EXTENDING MODULES AND EXCELLENT EXTENSIONS

When R is a subring of the ring S, and R and S have the same identity 1, the ring S is a *right excellent extension* of R if

(1) S_R and ${}_RS$ are free modules with a basis $\{1 = a_1, a_2, \ldots, a_n\}$ such that $a_iR = Ra_i$ for $i = 1, \ldots, n$.
(2) For any submodule A_S of a module M_S, if A_R is a direct summand of M_R, then A_S is a direct summand of M_S.

Excellent extensions were introduced by Passman ([11]). Let R be an associative ring with identity and let G be a finite multiplicative group. Then the crossed product $R*G$ is an associative ring constructed from R and G analogous. If $|G|^{-1} \in R$, then $R*G$ is a right excellent extension of R (in [7] and [12]).

Through this section, the ring S is a right excellent extension of R.

Lemma 7 ([8, Proposition 1.6], [7, Proposition 1.1])**.** *Let A_S be a submodule of an S-module M. Then*

(1) *A_S is closed in M_S if and only if A_R is closed in M_R.*
(2) *A_S is essential in M_S if and only if A_R is essential in M_R.*

Lemma 8 ([9, Lemma 2.4])**.** *Let A_R be a submodule of M_R. Then A_R is a closed submodule of M_R if and only if $(A \otimes_R S)_S$ is a closed submodule of $(M \otimes_R S)_S$.*

Theorem 2. *Let M be a right S-module. If M_R is an ef-extending module then M_S is an ef-extending module.*

Proof. Let M_R be an ef-extending module. Let A_S be a closed submodule of M_S which contains essentially a finitely generated submodule T_S. By Lemma 7, A_R is a closed submodule of M_R and T_R is essetial in A_R. Take $T = t_1S + t_2S + \cdots + t_kS$. One can check that T_R is generated by $\{t_ia_j | 1 \leq i \leq k, 1 \leq j \leq n\}$. We have that M_R is an ef-extending module and obtain that A_R is a direct summand of M_R. Since S is a right excellent extension of R, A_S is a direct summand of M_S. We deduce that M_S is an ef-extending module. □

Theorem 3. *Let M be a right R-module. If $(M \otimes_R S)_S$ is an ef-extending module then M_R is an ef-extending module.*

Proof. Let $(M \otimes_R S)_S$ be an ef-extending module. Let A_R be a closed submodule of M_R which contains essentially a finitely generated submodule T_R. By Lemma 8, $(A \otimes_R S)_S$ is a closed submodule of $(M \otimes_R S)_S$. We have that $(T \otimes_R S)_S$ is a finitely generated submodule of $(A \otimes_R S)_S$. On the other hand, we have

$$(T \otimes_R S)_R = (T \otimes a_1) \oplus (T \otimes a_2) \oplus \cdots \oplus (T \otimes a_n),$$
$$(A \otimes_R S)_R = (A \otimes a_1) \oplus (A \otimes a_2) \oplus \cdots \oplus (A \otimes a_n).$$

Note that $T \otimes a_i \cong T \otimes_R R \cong T$ and $A \otimes a_i \cong A \otimes_R R \cong A$ for all $i = 1, 2, \ldots, n$. One can check that $(T \otimes a_i)_R$ is essential in $(A \otimes a_i)_R$ for all $i = 1, 2, \ldots, n$. It follows that $(T \otimes_R S)_R$ is essential in $(A \otimes_R S)_R$, and so $(T \otimes_R S)_S$ is essential in $(A \otimes_R S)_S$ by Lemma 7. Since $(M \otimes_R S)_S$ is an ef-extending module, $(A \otimes_R S)_S$ is a direct summand of $(M \otimes_R S)_S$. Then, $(A \otimes_R S)_R$ is a direct summand of $(M \otimes_R S)_R$. We have that $(A \otimes 1)_R$ is a direct summand of $(M \otimes_R S)_R$ and $(A \otimes 1)_R \leq (M \otimes 1)_R$ and obtain that $(A \otimes 1)_R$ is a direct summand of $(M \otimes 1)_R$. We deduce that A is a direct summand of M. Thus, M_R is an ef-extending module. □

An R-module M is called *finitely* Σ*-ef-extending* if every finite direct sum of copies of M is ef-extending. The ring R is called *right finitely* Σ*-ef-extending* if R_R is finitely Σ-ef-extending ([17]).

Theorem 4. *Let S be a right excellent extension of R. Then R is right finitely Σ-ef-extending if and only if S is right finitely Σ-ef-extending.*

Proof. Suppose R_R is finitely Σ-ef-extending. Then for any $k > 0$, $(S^k)_R \cong (R^{nk})_R$ and so $(S^k)_S$ is ef-extending by Theorem 2. For the converse, suppose S_S is finitely Σ-ef-extending. Then for any $k > 0$, $(R^k \otimes_R S)_S \cong (S^k)_S$ is ef-extending. By Theorem 3, $(R^k)_R$ is ef-extending. □

It can easily be checked that the theorem still holds (with the same proof) if "finitely Σ-ef-extending" is replaced by "coutably Σ-ef-extending" or "Σ-ef-extending".

Corollary 10. *Let S be a right excellent extension of R.*

(1) *Every right R-module is ef-extending if and only if every right S-module is ef-extending.*
(2) *Every singular right R-module is ef-extending if and only if every singular right S-module is ef-extending.*
(3) *Every nonsingular right R-module is ef-extending if and only if every nonsingular right S-module is ef-extending.*

Proof. It follows immediately by Theorem 2 and 3. □

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