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STUDY OF BI-F-HARMONIC CURVE ALONG IMMERSIONS

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Abstract. In this paper, we characterize the bi-f-harmonic curve on surfaces and then we study the submanifold of a Riemannian manifold using the bi-f-harmonic curve. The conditions for curvature and torsion of bi-f-harmonic curve on surface, ruled surface and 3-dimensional space are derived. In addition, the geometry of the submanifold is studied by taking a bi-f-harmonic curve with immersion from the submanifold to the ambient space. Moreover, the conditions are given for isotropic submanifolds, totally geodesic submanifolds and umbilical submanifolds, so that the immersed curve is a bi-f-harmonic curve in ambient space. Finally, we investigate some important results for a particular case $f = 1$ of the bi-f-harmonic curve.

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1. Introduction

Harmonic maps are the generalization of geodesics, minimal surfaces and harmonic functions. These mappings have important applications and relationships in different fields of mathematics and physics with nonlinear partial differential equations and the concept of the stochastic procedure. Let (M_1, g_1) and (M_2, g_2) are Riemannian manifolds, then a harmonic map $\varphi\colon (M_2, g_2) \to (M_1, g_1)$ is a critical point of the energy functional,

$$E(\varphi) = \frac{1}{2}\int_{\Gamma_2} |\mathrm{d}\varphi|^2 v_{g_2},$$

where Γ_2 is some compact domain of M_2 and $\tau(\varphi) = \mathrm{Trace}_{g_2}(\nabla \mathrm{d}\varphi)$ is tension field of φ. The harmonic map equation is an Euler-Lagrange equation of the functional [6] $\tau(\varphi) \equiv \mathrm{Trace}_{g_2}(\nabla \mathrm{d}\varphi) = 0$, where $\tau(\varphi) = \mathrm{Trace}_{g_2}(\nabla \mathrm{d}\varphi)$ is a tension field of φ. In

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1964, J. Eells and J. H. Sampson [6], generalized the concept of harmonic maps to bi-harmonic maps. A bi-harmonic map $\varphi\colon (M_2,g_2)\to(M_1,g_1)$ is a critical point of the bi-energy functional, $E_2(\varphi)=\frac{1}{2}\int_{\Gamma_2}|\tau(\varphi)|^2 v_{g_2}$, where Γ_2 is a compact domain of M_2. The bi-harmonic map equation is an Euler-Lagrange equation of the functional [7],

$$\tau_2(\varphi)\equiv \mathrm{Trace}_{g_2}\left(\nabla^{\varphi}\nabla^{\varphi}-\nabla^{\varphi}_{\nabla^{M_2}}\right)\tau(\varphi)-\mathrm{Trace}_{g_2}\mathrm{R}^{M_1}\left(\mathrm{d}\varphi,\tau(\varphi)\right)\mathrm{d}\varphi=0,$$

where $\mathrm{R}^{M_1}=\left[\nabla^{M_1}_X,\nabla^{M_1}_Y\right]Z-\nabla^{M_1}_{[X,Y]}Z$, is a curvature operator of (M_2,g_2). Bi-harmonic submanifold of the Euclidean space was introduced by B. Y. Chen in 1991 [3]. If the inclusion map is bi-harmonic isometric immersion, then a submanifold is a bi-harmonic submanifold. In [10], the author introduced the f-bi-harmonic maps by combining the bi-harmonic maps and f-harmonic maps. A f-bi-harmonic map is a critical point of the bi-f-energy functional, $\frac{1}{2}\int_{\Gamma_2}|\tau_f(\varphi)|^2 v_{g_2}$. The bi-f-harmonic equation for curves in Euclidean space, hyperbolic space, sphere and hypersurfaces of manifolds were considered in [18]. A f-harmonic map $\varphi\colon (M_2,g_2)\to(M_1,g_1)$ is a critical point of the f-energy functional, $E_f(\varphi)=\frac{1}{2}\int_{\Gamma_2} f|\mathrm{d}\varphi|^2 v_{g_2}$, where Γ_2 is a compact domain of M. The f-harmonic map equation is an Euler-Lagrange equation of the functional [5, 16],

$$\tau_f(\varphi)\equiv f\tau(\varphi)+\mathrm{d}\varphi(\mathrm{grad}(f))=0,$$

where $\tau(\varphi)=\mathrm{Trace}_{g_2}(\nabla\mathrm{d}\varphi)$, is the tension field of φ. An f-bi-harmonic map from a compact Riemannian manifold to a curved manifold with constant f-bienergy density is a harmonic map [15].

A smooth map $f\colon (M_1^{m_1},g_1)\to(M_2^{m_2},g_2)$ is said to be an immersion if the differential map $df_{p_1}\colon T_{p_1}M_1\to T_{f(p_1)}M_2$ is one-one for all $p_1\in M_1$. The theory of isometric immersions is one of the active research areas in differential geometry [1, 2, 4]. In 1974, the authors showed that if a circle is mapped by immersion from a submanifold to the ambient manifold, then the submanifold is totally umbilical with a parallel mean curvature vector field [9]. In [8], the author characterizes the helix by a differential equation and studies the effect on the submanifold by moving a helix from the submanifold to the ambient manifold by immersion. By using isotropic immersion, the circles in complex projective space and complex hyperbolic space were studied in [11, 12, 17]. Recently authors studied the characterization of submanifold by taking the hyperelastic curves along an immersion [19].

We organize our paper as follows: In section 2, we give some basic and useful relations. In section 3, we study bi-f-harmonic curves on surfaces. We prove that if $\gamma(s)$ is a bi-f-harmonic curve on ruled surface $S\subset\mathbb{R}^3$ with $f=constant\neq 0$, then $\gamma(s)$ must be a straight line. Section 4, is divided into two parts; the first part deals with the characterization of submanifolds by moving the bi-f-harmonic curve from the submanifold to the ambient space by immersion. Then we show that an isometric immersion is a bi-f-harmonic immersion in the case of a totally geodesic submanifold. The second part of the section, discusses the characterization of a submanifold by taking a bi-harmonic map by immersion from the submanifold to the ambient space.

2. PRELIMINARIES

A bi-f-harmonic map $\varphi\colon (M_2,g_2)\to(M_1,g_1)$ from Riemannian manifold M_2 to a Riemannian manifold M_1 is a critical point of the bi-f-energy functional,

$$E_f^2(\varphi)=\frac{1}{2}\int_{\Gamma_2}|\tau_f(\varphi)|^2 v_{g_2},$$

where Γ_2 is a compact domain of M_2. The bi-f-harmonic map equation is an Euler-Lagrange equation of the functional,

$$\tau_f^2(\varphi)\equiv fJ^{\varphi}(\tau_f(\varphi))-\nabla^{\varphi}_{\mathrm{grad}(\varphi)}\tau_f(\varphi)=0,$$

where $\tau_f(\varphi)$ is the f-tension field of φ and J^{φ} is the Jacobi operator of the map defined by $J^{\varphi}(X)=-\left[Tr_{g_2}\nabla^{\varphi}\nabla^{\varphi}X-\nabla^{\varphi}_{\nabla^{M_1}}X-\mathrm{R}^{M_2}(\mathrm{d}\varphi,X)\mathrm{d}\varphi\right]$ [15, 16].

Let $\phi\colon I\to M_1$ be a curve in (M_1,g_1), then ϕ is a bi-f-harmonic curve on M_1 if and only if the following condition holds [18]

$$\begin{aligned}\left(ff'''+f'f''\right)T_1+\left(3ff''+2f'^2\right)\nabla^1_{T_1}T_1+4ff'\nabla^{1,2}_{T_1}T_1&\\+f^2\nabla^{1,3}_{T_1}T_1+f^2\mathrm{R}^1\left(\nabla^1_{T_1}T_1,T_1\right)T_1=0,&\quad(2.1)\end{aligned}$$

where $f\colon I\to(0,\infty)$ is a smooth function, $\nabla^{1,2}_{T_1}T_1=\nabla^1_{T_1}\nabla^1_{T_1}T_1$, $\nabla^{1,3}_{T_1}T_1=\nabla^1_{T_1}\nabla^1_{T_1}\nabla^1_{T_1}T_1$. and R^1 is a Riemann curvature tensor of M_1.

Let M be a Riemannian manifold of dimension m and M_1 is its submanifold of dimension m_1 with Riemannian connection ∇ on M and induced Riemannian connection ∇^1 on M_1. The set of all vector fields on M and M_1, are represented as $\Gamma(TM)$ and $\Gamma(TM_1)$, respectively. TM_1, $\Gamma(TM_1)^{\perp}$ and $TM_1^{\perp}$ will express the set of tangent vector bundle of M_1, all normal vector fields of M_1 and normal vector bundle of M_1, respectively. The Gauss and Weingarten formulae for M and M_1, are [20]

$$\nabla_{X_1}Y_1=\nabla^1_{X_1}Y_1+h^1(X_1,Y_1),\quad\forall\, X_1Y_1\in\Gamma(TM_1),\tag{2.2}$$

$$\nabla_{X_1}W_1=-A_{W_1}X_1+D_{X_1}W_1,\quad\forall\, W_1\in\Gamma(TM_1)^{\perp},\tag{2.3}$$

where D is a connection in normal bundle, $h^1(X_1,Y_1)$ is second fundamental form of M_1 and A is the shape operator of M_1. The tangential component and normal component in Gauss formula are $\nabla^1_{X_1}Y_1$ and $h^1(X_1,Y_1)$, respectively. Whereas in Weingarten formula the tangential component and normal component are $A_{W_1}X_1$ and $D_{X_1}W_1$, respectively. The second fundamental form is bilinear and symmetric in X_1 and Y_1. The relation between shape operator and second fundamental form is given by

$$\langle A_{W_1}X_1, Y_1\rangle = \langle h^1(X_1,Y_1), W_1\rangle. \tag{2.4}$$

If R and R^1 are the Riemannian curvature tensor fields of M and M^1, then from (2.2) and (2.3) , we have

$$\begin{aligned} \mathrm{R}(X_1,Y_1)Z_1 = \mathrm{R}^1(X_1,Y_1)Z_1 - A_{h^1(Y_1,Z_1)}X_1 + A_{h^1(Y_1,Z_1)}Y_1 \\ + \left(\nabla_{X_1}h^1\right)(Y_1,Z_1) - \left(\nabla_{Y_1}h^1\right)(X_1,Z_1), \end{aligned}$$

for all $X_1, Y_1, Z_1 \in \Gamma TM_1$. The covariant derivative of second fundamental form and shape operator are, respectively

$$\left(\nabla_{X_1}h^1\right)(Y_1,Z_1) = D_{X_1}h^1(Y_1,Z_1) - h^1\left(\nabla^1_{X_1}Y_1, Z_1\right) - h^1\left(Y_1, \nabla^1_{X_1}Z_1\right), \tag{2.5}$$

and

$$(\nabla_{X_1}A)_{W_1}Y_1 = \nabla^1_{X_1}(A_{W_1}Y_1) - A_{D_{X_1}W_1}Y_1 - A_{W_1}\left(\nabla^1_{X_1}Y_1\right). \tag{2.6}$$

The covariant derivative of h and A, satisfies the relation

$$\left\langle\left(\nabla_{X_1}h^1\right)(Y_1,Z_1), W_1\right\rangle = \left\langle(\nabla_{X_1}A)_{W_1}Y_1, Z_1\right\rangle. \tag{2.7}$$

The submanifold M_1 is totally geodesic in M if its second fundamental form is identically zero i.e. $h^1 = 0$ $(A = 0)$ and M_1 is totally umbilical if h^1 satisfies

$$h^1(X_1,Y_1) = \langle X_1,Y_1\rangle H. \tag{2.8}$$

If, $\langle h^1(X_{1p},X_{1p}), h^1(X_{1p},X_{1p})\rangle = constant$, $\forall\ X_{1p} \in T_pM_1$, then M_1 is said to be isotropic at p. The necessary and sufficient condition for M_1 to be isotropic is that

$$\langle h^1(X_{1p},X_{1p}), h^1(X_{1p},Y_{1p})\rangle = 0, \quad \forall\, X_{1p}, Y_{1p} \in T_pM_1.$$

Also, if $h^1(X_{1p},Y_{1p}) = 0$, $\forall\, X_{1p}, Y_{1p} \in T_pM_1$, then M_1 is totally umbilical [14].

3. Characterization of bi-f- harmonic curves

Let $\gamma(s)$ be a smooth unit speed curve immersed on a surface S, with $T = \gamma'(s)$ is the unit tangent vector and $N = JT$ is the unit normal vector along $\gamma(s)$ [13]. Where J is the anti-clockwise rotation by an angle $\frac{\pi}{2}$ defined in the tangent bundle of S. Let ∇ be the Levi-Civita connection on S, then the Frenet-Serret equation is

$$\nabla_T T = \kappa_g N, \tag{3.1}$$

where κ_g is a geodesic curvature.

Theorem 1. *A curve $\gamma(s)$ on a surface S is a bi-f-harmonic curve if and only if its geodesic curvature satisfies the equations*

$$ff''' + f'f'' - 4\kappa_g^2 - 3\kappa_g\kappa_g' f^2 = 0, \tag{3.2}$$

$$3\kappa_g ff'' + 2\kappa_g f'^2 + 4\kappa_g' ff' + \left(\kappa_g'' - \kappa_g^3\right)f^2 + \kappa_g f^2 K_S = 0, \tag{3.3}$$

where $K_S = \langle \mathrm{R}(T,N)N,T\rangle$ is the Gaussian curvature of surface S along γ and $()'$ represents the derivative with respect to parameter s.

Proof. Let $\gamma(s)$ be a curve on a surface S then from (3.1), we have

$$\begin{cases} \nabla_T^2 T = -\kappa_g^2 T + \kappa_g' N, \\ \nabla_T^3 T = -3\kappa_g\kappa_g' T + \left(\kappa_g'' - \kappa_g^3\right) N, \end{cases} \tag{3.4}$$

where T and N are tangent and normal along $\gamma(s)$. Substituting equation (3.4) in (2.1), we get

$$\left(ff''' + f'f'' - 4\kappa_g^2 ff' - 3\kappa_g\kappa_g' f^2\right) T + f^2\kappa_g \mathrm{R}(N,T)T \\ + \left(3\kappa_g ff'' + 2\kappa_g f'^2 + 4\kappa_g' ff' + f^2\kappa_g'' - f^2\kappa_g^3\right) N = 0. \tag{3.5}$$

Then the inner product of equation (3.5) with tangent and normal provide the required conditions. ☐

Let S be a ruled surface in $\mathbb{R}^3$ with parametrization $x(s,t) = \gamma(s) + tN(s)$, where $N(s)$ is a normal of curve in $\mathbb{R}^3$. Then the Gaussian curvature along $\gamma(s)$ satisfies $K_S(\gamma(s)) = -\tau^2(s)$. Also, the geodesic curvature of $\gamma(s)$, as a curve on S, is $\kappa_g = \pm\kappa(s)$, where $\kappa(s)$ is a curvature of $\gamma(s)$ in $\mathbb{R}^3$.

Theorem 2. *A curve $\gamma(s)$ is a bi-f-harmonic curve on ruled surface S if and only if curvature $(\kappa(s))$ and torsion $(\tau(s))$ of the curve satisfies the equations*

$$\kappa(s) = f^{-\frac{4}{3}}\left(\frac{2}{3}\int f^{\frac{2}{3}}(ff''' + f'f'')ds + C\right)^{\frac{1}{2}}, \tag{3.6}$$

$$\tau(s) = \frac{1}{3\kappa^2 f^2}\Big[27\kappa^4 f^3 f'' + 18\kappa^4 f^3 f'^2 + 3\kappa^2\left(f^2\{ff''' + f'f'' - 4\kappa^2 ff'\}\right)' \\ - \left(ff''' + f'f'' - 4\kappa^2 ff'\right)^2\Big]^{\frac{1}{2}}, \tag{3.7}$$

where C is some constant.

Proof. Let $\gamma(s)$ be a bi-f-harmonic curve on a ruled surface S. Taking $\kappa_g = \kappa$ and $K_S = \langle \mathrm{R}(T,N)N,T\rangle = -\tau^2$ in equation (3.2) and (3.3), we get

$$ff''' + f'f'' - 4\kappa^2 ff' - 3\kappa\kappa' f^2 = 0, \tag{3.8}$$

$$3\kappa ff'' + 2\kappa f'^2 + 4\kappa' ff' + \left(\kappa'' - \kappa^3\right) f^2 - \kappa f^2\tau^2 = 0. \tag{3.9}$$

Solution of the equation (3.8), provides the value of the curvature $\kappa(s)$ (using mapple). Also from equation (3.8), we have

$$\kappa' = \frac{1}{3f^2\kappa}(ff''' + f'f'' - 4\kappa^2 ff'). \tag{3.10}$$

The derivative of κ' from (3.10),

$$\kappa'' = -\frac{\kappa'^2}{\kappa} + \frac{1}{3\kappa}\left(\frac{1}{f^2}\left(ff''' + f'f'' - 4\kappa^2 ff'\right)\right)'.$$

Substituting the values of κ' and κ'' from above equations in (3.9), we get the required value of torsion $\tau(s)$.

Conversely, assume that $\gamma(s)$ be a curve on a ruled surface S, with curvature and torsion given by (3.6) and (3.7), respectively. Then it is easy to see that κ and τ satisfies the equation (3.8) and (3.9), hence $\gamma(s)$ is a bi-f-harmonic curve on ruled surface S. $\square$

Corollary 1. *If $\gamma(s)$ is a bi-f-harmonic curve with non zero curvature and $f = constant$ lying on a ruled surface S in $\mathbb{R}^3$, then f must be a zero function $(i.e., f = 0)$.*

Proof. Let $\gamma(s)$ is a bi-f-harmonic curve on ruled surface S with $f(s) = constant \neq 0$, then from equation (3.8), we have $\kappa = C$, where C is constant.
Since $f(s) = constant \neq 0$ and $\kappa = C$, therefore from equation (3.9), we have $\tau = iC$, which is not possible. $\square$

Hence from the above corollary, we can conclude that

Remark 1. If $\gamma(s)$ is a bi-f-harmonic curve lying on a ruled surface S in $\mathbb{R}^3$ with $f = constant \neq 0$, then $\gamma(s)$ is a straight line.

Corollary 2. *Let $\gamma(s)$ be a bi-f-harmonic curve lying on a ruled surface S, with $f(s) = \kappa(s) \neq 0$, then $\gamma(s)$ is a Frenet helix.*

Proof. Since $\gamma(s)$ is a bi-f-harmonic curve on a ruled surface S with $f(s) = \kappa(s) \neq 0$, therefore from (3.8), we get

$$\kappa\kappa''' + \kappa'\kappa'' - 4\kappa^3\kappa' - 3\kappa^3\kappa' = 0. \tag{3.11}$$

On integrating (3.11), we obtain

$$\kappa\kappa'' - \frac{7}{4}\kappa^4 = C_1,$$

where C_1 is some constant. Multiplying above equation with $2\frac{\kappa'}{\kappa}$ and then integrating,

$$\kappa'^2 = \frac{7}{8}\kappa^4 + 2C_1 \log\kappa + C_2,$$

where C_2 is another arbitrary constant, for simplicity taking $C_1 = C_2 = 0$, we have

$$\kappa = \sqrt{\frac{8}{7}}\frac{1}{s}.$$

Substituting the value of $f = \kappa = \sqrt{\frac{8}{7}}\frac{1}{s}$ in (3.9), we get

$$\tau = 3\sqrt{\frac{10}{7}}\frac{1}{s}.$$

Since the ratio of κ and τ is constant, hence γ is a Frenet helix. $\square$

Theorem 3. *Let γ be a bi-f-harmonic curve on a surface S with Gaussian curvature $K_S = -\frac{1}{9f^2}\left(70f'^2 + 39C\right)$ and $ff'' = constant$, then γ is a geodesic on S.*

Proof. Since γ is a bi-f-harmonic curve and $ff'' = constant(C)$, therefore from (3.2), $\kappa_g' = \frac{4}{3}\left(\frac{\kappa_g f'}{f}\right)$. Further the derivative of κ_g',

$$\kappa_g'' = \frac{4}{3}\left(\frac{\kappa_g ff'' + \kappa_g' ff' - \kappa_g f'^2}{f^2}\right).$$

Substituting κ_g' and κ_g'' in (3.3), we have

$$\kappa_g = \pm\frac{\sqrt{39ff'' + 70f'^2 + 9f^2 K_S}}{3f}, \tag{3.12}$$

where K_S is a Gaussian curvature. Thus, from (3.12) and $K_S = -\frac{1}{9f^2}\left(70f'^2 + 39C\right)$, we have $\kappa_g = 0$. Hence γ is a geodesics on surface S. $\square$

Let $\gamma(s)$ be a bi-f-harmonic curve immersed on a Riemannian manifold M^3 and $\{T(s), N(s), B(s)\}$ is a Frenet frame along $\gamma(s)$, where $\gamma'(s) = T(s)$ is a unit tangent vector field. Then the Frenet equations are

$$\begin{cases} \nabla_T T(s) = \kappa N, \\ \nabla_T N(s) = -\kappa T + \tau B, \\ \nabla_T B(s) = -\tau N, \end{cases} \tag{3.13}$$

where $\kappa = \kappa(s)$ and $\tau = \tau(s)$ are the curvature and the torsion of γ, respectively. Now, substituting equation (3.13) in equation (2.1) and taking corresponding tangent, normal and binormal components of the equation (2.1), we obtain

$$\begin{cases} 3\kappa ff'' + 2\kappa f'^2 + 4\kappa' ff' + \kappa'' f^2 + \kappa f^2 \langle \mathrm{R}(N,T)T, N\rangle = \kappa^3 f^2 + \kappa\tau^2 f^2, \\ ff''' + f'f'' - 4\kappa^2 ff' - 3\kappa\kappa' f^2 = 0, \\ 4\kappa\tau ff' + 2\kappa'\tau f^2 + \kappa\tau' f^2 + \kappa f^2 \langle \mathrm{R}(N,T)T, B\rangle = 0. \end{cases} \tag{3.14}$$

Theorem 4. *If a curve $\gamma(s)$ in three dimensional Riemannian manifold M^3 is a bi-f-harmonic curve, then the torsion ($\tau(s)$) and curvature ($\kappa(s)$) of $\gamma(s)$, are*

$$\begin{aligned} \tau(s) = \frac{1}{3\kappa^2 f^2}\Big(& -9\kappa^6 f^4 + \left(27f^3 f'' + 36f^2 f'^2 + 9f^4 K_S\right)\kappa^4 \\ & + 3\kappa^2\left(f^2\{ff''' + f'f'' - 4\kappa^2 ff'\}\right)' - (ff''' + f'f'' - 4\kappa^2 ff')^2\Big)^{\frac{1}{2}}, \end{aligned} \tag{3.15}$$

$$\kappa(s) = f^{-\frac{4}{3}}\left(\frac{2}{3}\int f^{\frac{2}{3}}\left(ff''' + f'f''\right)ds + \frac{3}{2}C\right)^{\frac{1}{2}}, \tag{3.16}$$

where $\kappa = \kappa(s)$, $\tau = \tau(s)$, and c is some constant.

Proof. Let γ be a bi-f-harmonic curve on a Riemannian manifold M^3, then the solution of equation (3.14)), gives us (3.16) (obtain by using mapple). On the other hand, the first part of equation (3.14), provide us

$$\kappa' = \frac{1}{3f^2\kappa}\left(ff''' + f'f'' - 4\kappa^2 ff'\right). \tag{3.17}$$

Taking the derivative of (3.17), we get

$$\kappa'' = -\frac{\kappa'^2}{\kappa} + \frac{1}{3\kappa}\left(\frac{1}{f^2}\left(ff''' + f'f'' - 4\kappa^2 ff'\right)\right)'. \tag{3.18}$$

Substituting the values of κ' and κ'' from equations (3.17) and (3.18) in the second part of (3.14), we get the equation (3.15). □

4. Characterization of the submanifold of a Riemannian manifold by bi-f-harmonic curves

Let $\gamma_1 : I \to M_1$ be a curve in an m_1-dimensional Riemannian manifold M_1. Let $\left\{N_1^0, N_1^1, N_1^2, \ldots, N_1^{m_1-1}\right\}$ be an orthonormal frame in ΓTM_1, where $N_1^0 = T_1$, $N_1^1 = N_1$ and $N_1^2 = B_1$ are the unit tangent vector, the unit normal vector and the unit binormal vector of γ_1, respectively. Then the corresponding Frenet equations are

$$\nabla^1_{T_1} N_1^i = -\kappa_i N_1^{i-1} + \kappa_{i+1} N_1^{i+1}, \qquad 0 \le i \le m_1 - 1,$$

where $\kappa_0 = \kappa_n = 0$, $\kappa_1 = ||\nabla^1_{T_1} N_1^0||$ is curvature and $\tau_1 = \kappa_2 = -\langle \nabla^1_{T_1} N_1, N_1^2\rangle$ is torsion of γ_1 in M_1, respectively. Next, we introduce the concept of bi-f-harmonic immersion

Definition 1. Let $i\colon M_1 \to M$ be an isometric immersion from Riemannian manifold M_1 to a Riemannian manifold M such that γ_1 is a bi-f-harmonic curve on M_1. If the curve $\gamma = i \circ \gamma_1$ is also a bi-f-harmonic curve on M, then the immersion i is a bi-f-harmonic immersion.

Lemma 1. *Let $i\colon M_1 \to M$ be an isometric immersion from Riemannian manifold M_1 to a Riemannian manifold M. Then for a curve γ_1 on M_1 and the curve $\gamma(s) = i \circ \gamma_1(s)$ with curvature κ on M, satisfies the following equations*

$$\begin{aligned}
\nabla^3_{T_1} T_1 = {} & h^1\left(T_1, \nabla^{1,2}_{T_1} T_1\right) - A_{h^1\left(T_1, \nabla^1_{T_1} T_1\right)} T_1 + D_{T_1} h^1\left(T_1, \nabla^1_{T_1} T_1\right) \\
& - A_{D_{T_1} h^1(T_1,T_1)} T_1 - \nabla^1_{T_1}\left(A_{h^1(T_1,T_1)} T_1\right) \\
& - h^1\left(T_1, A_{h^1(T_1,T_1)} T_1\right) + D^2_{T_1} h^1(T_1,T_1) + \nabla^{1,3}_{T_1} T_1,
\end{aligned} \tag{4.1}$$

$$\begin{aligned}
\mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 = {} & \mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1 - A_{h^1(T_1,T_1)} \nabla^1_{T_1} T_1 + A_{h^1\left(\nabla^1_{T_1} T_1, T_1\right)} T_1 \\
& + \left(\nabla_{\nabla^1_{T_1} T_1} h\right)(T_1, T_1) - \left(\nabla_{T_1} h\right)\left(\nabla^1_{T_1} T_1, T_1\right),
\end{aligned} \tag{4.2}$$

where $\nabla^{1,2}_{T_1} = \nabla^1_{T_1}\nabla^1_{T_1}$ *and* $\nabla^{1,3}_{T_1} = \nabla^1_{T_1}\nabla^1_{T_1}\nabla^1_{T_1}$.

Proof. From Gauss and Weingarten formula, we have

$$\begin{aligned}\nabla^2_{T_1} T_1 &= \nabla_{T_1}\left(\nabla^1_{T_1} T_1, h^1(T_1,T_1)\right)\\ &= \nabla^{1,2}_{T_1} T_1 + h^1(T_1,\nabla^1_{T_1} T_1) - A_{h^1(T_1,T_1)} T_1 + D_{T_1} h^1(T_1,T_1).\end{aligned}$$

Further taking the covariant derivative of above equation with respect to T and using (2.2) and (2.3), we get the equation (4.1) . Whereas equation (4.2) is directly followed from equations (2.2) and (2.6). □

Let γ be a bi-f-harmonic curve on M, then from (2.1), we have

$$\begin{aligned}\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right)\nabla_{T_1} T_1 + 4ff'\nabla^2_{T_1} T_1 + f^2\nabla^3_{T_1} T_1&\\ + f^2 \mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 = 0.&\quad (4.3)\end{aligned}$$

Using Lemma (1) in the above equation, we get

$$\begin{aligned}&\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right)\left(\nabla^1_{T_1} T_1\right) + \left(3ff'' + 2f'^2\right) h^1(T_1,T_1)\\ &+ 4ff'\nabla^{1,2}_{T_1} T_1 + 4ff' h^1\left(T_1,\nabla^1_{T_1} T_1\right) - 4ff' A_{h^1(T_1,T_1)} T_1 + 4ff' D_{T_1} h^1(T_1,T_1)\\ &+ f^2\nabla^{1,3}_{T_1} T_1 + f^2 h^1\left(T_1,\nabla^{1,2}_{T_1} T_1\right) - f^2 A_{h^1(T_1,\nabla^1_{T_1} T_1)} T_1 + f^2 D_{T_1} h^1\left(T_1,\nabla^1_{T_1} T_1\right)\\ &- f^2\nabla^1_{T_1}\left(A_{h^1(T_1,T_1)} T_1\right) - f^2 h^1\left(T_1, A_{h^1(T_1,T_1)} T_1\right) - ff^2 A_{D_{T_1} h^1(T_1,T_1)} T_1\\ &+ f^2 D^2_{T_1} h^1(T_1,T_1) + f^2\mathrm{R}\left(\nabla^1_{T_1} T_1, T_1\right) T_1 - f^2 A_{h^1(T_1,T_1)}\nabla^1_{T_1} T_1 + f^2 A_{h^1\left(\nabla^1_{T_1} T_1, T_1\right)} T_1\\ &+ f^2\left(\nabla_{\nabla^1_{T_1} T_1} h\right)(T_1,T_1) - f^2\left(\nabla_{T_1} h^1\right)\left(\nabla^1_{T_1} T_1, T_1\right) = 0. \qquad (4.4)\end{aligned}$$

The tangent part of above equation, is

$$\begin{aligned}&\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right)\nabla^1_{T_1} T_1 + 4ff'\nabla^{1,2}_{T_1} T_1 - 4ff' A_{h^1(T_1,T_1)} T_1\\ &+ f^2\nabla^{1,3}_{T_1} T_1 - f^2\nabla^1_{T_1}\left(A_{h^1(T_1,T_1)}\right) - f^2 A_{D_{T_1} h^1(T_1,T_1)} T_1 + f^2\mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1\\ &- f^2 A_{h^1(T_1,T_1)}\nabla^1_{T_1} T_1 = 0. \qquad (4.5)\end{aligned}$$

Substituting the equations (2.6) and (2.5) in (4.5), we have

$$\begin{aligned}&\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right)\nabla^1_{T_1} T_1 + 4ff'\nabla^{1,2}_{T_1} T_1 - 4ff' A_{h^1(T_1,T_1)} T_1\\ &+ f^2\nabla^{1,3}_{T_1} T_1 - f^2\left(\nabla^1_{T_1} A\right)_{h^1(T_1,T_1)} T_1 - 2f^2 A_{(\nabla_{T_1} h^1)(T_1,T_1)} T_1 - 4f^2 A_{h^1\left(\nabla^1_{T_1} T_1, T_1\right)} T_1\\ &- 2f^2 A_{h^1(T_1,T_1)}\nabla^1_{T_1} T_1 + f^2\mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1 = 0. \qquad (4.6)\end{aligned}$$

By using Frenet Serret equations of γ_1, (4.6) reduces to

$$\begin{aligned}&\left(ff''' + f'f'' - 4\kappa_1^2 ff' - 3\kappa_1\kappa_1' f^2\right) T_1 + \left(4\kappa_1\tau_1 ff' + 2\kappa_1'\tau_1 f^2 + \kappa_1\tau_1' f^2\right) B_1\\ &+ \left(3\kappa_1 ff'' + 2\kappa_1 f'^2 + 4\kappa_1' ff' - \kappa_1^3 f^2 + \kappa_1'' f^2 - \kappa_1\tau_1^2 f^2\right) N_1 - 4ff' A_{h^1(T_1,T_1)} T_1\end{aligned}$$

$$-f^2\left(\nabla^1_{T_1}A\right)_{h^1(T_1,T_1)}T_1-2f^2A_{\left(\nabla_{T_1}h^1\right)(T_1,T_1)}T_1-4\kappa_1f^2A_{h^1(N_1,T_1)}T_1$$
$$-2\kappa_1f^2A_{h^1(T_1,T_1)}N_1+\kappa_1f^2\mathrm{R}^1\left(N_1,T_1\right)T_1=0.$$

Taking inner product of above equation with T_1 and using equation (2.5), we have

$$ff'''+f'f''-3\kappa_1\kappa_1'f^2-4\kappa_1^2ff'-4ff'\left\langle A_{h^1(T_1,T_1)}T_1,T_1\right\rangle$$
$$-f^2\left\langle\left(\nabla^1_{T_1}A\right)_{h^1(T_1,T_1)}T_1,T_1\right\rangle-2f^2\left\langle A_{D_{T_1}h^1(T_1,T_1)}T_1,T_1\right\rangle+4f^2\left\langle A_{h^1\left(\nabla_{T_1}T_1,T_1\right)}T_1,T_1\right\rangle$$
$$-4\kappa_1f^2\left\langle A_{h^1(N_1,T_1)}T_1,T_1\right\rangle-2\kappa_1f^2\left\langle A_{h^1(T_1,T_1)}N_1,T_1\right\rangle=0. \tag{4.7}$$

From equations (2.4) and (4.7), we obtain

$$ff'''+f'f''-4\kappa_1^2ff'-3\kappa_1\kappa_1'f^2-4ff'\left\langle h^1\left(T_1,T_1\right),h^1\left(T_1,T_1\right)\right\rangle$$
$$-2f^2\left\langle h^1\left(T_1,T_1\right),D_{T_1}h^1\left(T_1,T_1\right)\right\rangle-2\kappa_1f^2\left\langle h^1\left(N_1,T_1\right),h^1\left(T_1,T_1\right)\right\rangle$$
$$-f^2\left\langle\left(\nabla^1_{T_1}A\right)_{h^1(T_1,T_1)}T_1,T_1\right\rangle=0. \tag{4.8}$$

Now, taking $\left\langle h^1\left(T_1,T_1\right),D_{T_1}h^1\left(T_1,T_1\right)\right\rangle=\frac{1}{2}D_{T_1}\left\langle h^1\left(T_1,T_1\right),h^1\left(T_1,T_1\right)\right\rangle$ and equation (2.7) in equation (4.8), we get

$$ff'''+f'f''-3\kappa_1\kappa_1'f^2-4\kappa_1^2ff'-4ff'\left\langle h^1\left(T_1,T_1\right),h^1(T_1,T_1)\right\rangle$$
$$-f^2D_{T_1}\left\langle h^1\left(T_1,T_1\right),h^1\left(T_1,T_1\right)\right\rangle-2\kappa_1f^2\langle h^1(N_1,T_1),h^1(T_1,T_1)\rangle$$
$$-f^2\left\langle\left(\nabla_{T_1}h^1\right)\left(T_1,T_1\right),h^1\left(T_1,T_1\right)\right\rangle=0. \tag{4.9}$$

Using the equation (2.5) in equation (4.9), we have

$$ff'''+f'f''-3\kappa_1\kappa_1'f^2-4\kappa_1^2ff'-4ff'||h^1\left(T_1,T_1\right)||^2$$
$$-f^2D_{T_1}||h^1\left(T_1,T_1\right)||^2-2\kappa_1f^2\left\langle h^1(N_1,T_1),h^1\left(T_1,T_1\right)\right\rangle$$
$$-f^2\left\langle D_{T_1}h^1(T_1,T_1),h^1(T_1,T_1)\right\rangle+2f^2\left\langle h^1(\nabla^1_{T_1}T_1,T_1),h^1(T_1,T_1)\right\rangle=0. \tag{4.10}$$

Since,

$$-2f^2\left\langle h^1\left(\nabla^1_{T_1}T_1,T_1\right),h^1\left(T_1,T_1\right)\right\rangle=-2f^2\kappa_1\left\langle h^1(N_1,T_1),h^1(T_1,T_1)\right\rangle,$$

and,

$$f^2\left\langle D_{T_1}h^1\left(T_1,T_1\right),h^1\left(T_1,T_1\right)\right\rangle=\frac{1}{2}f^2D_{T_1}||h^1\left(T_1,T_1\right)||^2,$$

then equation (4.10), reduces to

$$ff'''+f'f''-3\kappa_1\kappa_1'f^2-4\kappa_1^2ff'$$
$$=4ff'||h^1\left(T_1,T_1\right)||^2+\frac{3}{2}f^2D_{T_1}||h^1\left(T_1,T_1\right)||^2. \tag{4.11}$$

Again the normal part of equation (4.4),

$$\left(3ff''+2f'^2\right)h^1\left(T_1,T_1\right)+4ff'h^1\left(T_1,\nabla^1_{T_1}T_1\right)+4ff'D_{T_1}h^1\left(T_1,T_1\right)$$

$$+f^2h^1\left(T_1,\nabla^{1,2}_{T_1}T_1\right)+7f^2D_{T_1}h^1\left(T_1,\nabla^1_{T_1}T_1\right)-f^2h^1\left(T_1,A_{h^1(T_1,T_1)}T_1\right)$$
$$+f^2D^2_{T_1}h^1\left(T_1,T_1\right)+f^2\left(\nabla_{\nabla^1_{T_1}T_1}h^1\right)\left(T_1,T_1\right)-f^2\left(\nabla_{T_1}h^1\right)\left(\nabla^1_{T_1}T_1,T_1\right)=0. \quad (4.12)$$

Thus equation (2.5) can be written as

$$D^2_{T_1}h^1\left(T_1,T_1\right)+2h^1\left(\nabla^1_{T_1}T_1,\nabla^1_{T_1}T_1\right)+2h^1\left(T_1,\nabla^{1,2}_{T_1}T_1\right)$$
$$=\left(\nabla^2_{T_1}h^1\right)\left(T_1,T_1\right)+4D_{T_1}h^1\left(T_1,\nabla^1_{T_1}T_1\right). \quad (4.13)$$

So, from equations (4.13) and (4.12), we have

$$\left(3ff''+2f'^2\right)h^1\left(T_1,T_1\right)+4ff'h^1\left(T_1,\nabla^1_{T_1}T_1\right)+4ff'D_{T_1}h^1\left(T_1,T_1\right)$$
$$-f^2h^1\left(T_1,\nabla^{1,2}_{T_1}T_1\right)+5f^2D_{T_1}h^1\left(T_1,\nabla^1_{T_1}T_1\right)-f^2h^1\left(T_1,A_{h^1(T_1,T_1)}T_1\right)$$
$$+f^2\left(\nabla^2_{T_1}h^1\right)\left(T_1,T_1\right)-2f^2h^1\left(\nabla^1_{T_1}T_1,\nabla^1_{T_1}T_1\right)+f^2\left(\nabla_{\nabla^1_{T_1}T_1}h^1\right)\left(T_1,T_1\right)$$
$$-f^2\left(\nabla_{T_1}h^1\right)\left(\nabla^1_{T_1}T_1,T_1\right)=0. \quad (4.14)$$

Equation (2.5) and equation (4.14), gives us

$$\left(3ff''+2f'^2\right)h^1\left(T_1,T_1\right)+12ff'h^1\left(T_1,\nabla^1_{T_1}T_1\right)+4ff'\left(\nabla_{T_1}h^1\right)\left(T_1,T_1\right)$$
$$+4f^2h^1\left(T_1,\nabla^{1,2}_{T_1}T_1\right)+3f^2h^1\left(\nabla^1_{T_1}T_1,\nabla^1_{T_1}T_1\right)+5f^2\left(\nabla_{T_1}h^1\right)\left(T_1,\nabla^1_{T_1}T_1\right)$$
$$-f^2h^1\left(T_1,A_{h^1(T_1,T_1)}T_1\right)+f^2\left(\nabla^{1,2}_{T_1}h^1\right)\left(T_1,T_1\right)+f^2\left(\nabla_{\nabla^1_{T_1}T_1}h^1\right)\left(T_1,T_1\right)$$
$$-f^2\left(\nabla_{T_1}h^1\right)\left(\nabla^1_{T_1}T_1,T_1\right)=0. \quad (4.15)$$

Using Frenet Serret equation in (4.15), we have

$$\left(12ff'\kappa_1+4f^2\kappa_1'\right)h^1\left(T_1,N_1\right)+4f^2\kappa_1\tau_1h^1\left(T_1,B_1\right)+4f^2\kappa_1\left(\nabla_{T_1}h^1\right)\left(T_1,N_1\right)$$
$$+3f^2\kappa_1^2h^1\left(N_1,N_1\right)+4ff'\left(\nabla_{T_1}h^1\right)\left(T_1,T_1\right)+\kappa_1f^2\left(\nabla_{N_1}h^1\right)\left(T_1,T_1\right) \quad (4.16)$$
$$=\left(4f^2\kappa_1^2-3ff''-2f'^2\right)h^1\left(T_1,T_1\right)+f^2\left(h^1(T_1,A_{h^1(T_1,T_1)}T_1)-(\nabla^2_{T_1}h^1)(T_1,T_1)\right).$$

Replacing B_1 with - B_1 in above equation and subtracting from equation (4.16), we obtain

$$h^1\left(T_1,B_1\right)=0.$$

Now from all the above discussion, we get the characterization of the submanifold of a Riemannian manifold with the help of bi-f-harmonic curves.

Proposition 1. *Let M_1 be an isotropic submanifold of a Riemannian manifold M and $i\colon M_1\to M$ be an isometric immersion, such that the curve $\gamma_1(s)$ on M_1 and the curve $\gamma(s)=i\circ\gamma_1(s)$ is a bi-f-harmonic curve with curvature κ on M, then*

$$ff'''+f'f''-3\kappa_1\kappa_1'f^2-4ff'\left(\kappa_1^2+C\right)=0, \quad (4.17)$$

where $||h^1\left(T_1,T_1\right)||^2=C$ is some constant.

Proof. Using the properties of isotropic submanifold in (4.4), we obtain the result. □

Corollary 3. *Let $i\colon M_1 \to M$ be an isometric immersion between Riemannian manifold M_1 to a Riemannian manifold M, such that $\gamma(s) = i \circ \gamma_1(s)$ is a bi-f-harmonic curve with curvature κ on M corresponding to curve $\gamma_1(s)$ on M_1. Suppose that M_1 is an isotropic submanifold and f is constant then curvature κ_1 on M_1 is constant.*

Proof. From equation (4.17), we have

$$\kappa_1 \kappa_1' = 0. \tag{4.18}$$

Integrating equation (4.18) with respect to parameter s, we obtain

$$\kappa_1 = C_1,$$

where C_1 is a constant. □

Theorem 5. *Let M_1 be a totally geodesic submanifold of Riemannian manifold M and $i\colon M_1 \to M$ be an isometric immersion such that $\gamma(s) = i \circ \gamma_1(s)$ is a bi-f-harmonic curve with curvature κ on M corresponding to $\gamma_1(s)$ on M_1, then*

$$ff''' + f'f'' - 3\kappa_1\kappa_1' f^2 - 4\kappa_1^2 ff' = 0.$$

Proof. Since M_1 is a totally geodesic submanifold of M, so taking $h^1 = 0$ in the tangent part of equation (4.11), we get the required result. □

Theorem 6. *Let $i\colon M_1 \to M$ be a bi-f-harmonic immersion between Riemannian manifold M_1 to a Riemannian manifold M, then M_1 is a totally umbilical submanifold of Riemannian manifold M. Then mean curvature (H), holds the equation*

$$D_{T_1}^2 H = \left(||H||^2 + \kappa_1^2 - 2\frac{f'^2}{f^2} - 3\frac{f''}{f} \right) H - 4\frac{f'}{f} D_{T_1} H. \tag{4.19}$$

Conversely, if $i\colon M_1 \to M$ be an isometric immersion and M_1 is totally geodesic, then i is a bi-f-harmonic immersion.

Proof. Let i be a bi-f-harmonic immersion from Riemannian manifold M_1 to a Riemannian manifold M. Since M_1 is a totally umbilical submanifold, so equation (4.15), reduces to

$$\begin{aligned} &4f^2\kappa_1 \left(\nabla_{T_1} h^1\right)(T_1, N_1) + 3f^2\kappa_1^2 H + 4ff' D_{T_1} H + \kappa_1 f^2 \left(\nabla_{N_1} h^1\right)(T_1, T_1) \\ &\quad = \left(4f^2\kappa_1^2 - 3ff'' - 2f'^2\right) H + f^2 h^1 (T_1, A_H T_1) - f^2 \left(\nabla_{T_1}^2 h^1\right)(T_1, T_1). \end{aligned} \tag{4.20}$$

Replacing N_1 with $-N_1$ in (4.20) and using (4.20), we get

$$4\kappa_1 f^2 \left(\nabla_{T_1} h^1\right)(T_1, N_1) + \kappa_1 f^2 \left(\nabla_{N_1} h^1\right)(T_1, T_1) = 0. \tag{4.21}$$

Substituting equation (4.21) from equation (4.20), we obtain

$$\left(-\kappa_1^2 f^2 + 3ff'' + 2f'^2\right) H$$

$$= f^2 h^1\left(T_1, A_H T_1\right) - f^2\left(\nabla^2_{T_1} h^1\right)\left(T_1, T_1\right) - 4ff' D_{T_1} H. \quad (4.22)$$

Using the equations (2.4) and (4.13) in equation (4.22), we get

$$f^2 D^2_{T_1} H = \left(f^2 ||H||^2 + f^2 \kappa_1^2 - 3ff'' - 2f'^2\right) H - 4ff' D_{T_1} H.$$

Therefore

$$D^2_{T_1} H = \left(||H||^2 + \kappa_1^2 - 3\frac{f''}{f} - 2\frac{f'^2}{f^2}\right) H - 4\frac{f'}{f} D_{T_1} H.$$

Conversely, we suppose that the M_1 is totally geodesic and $i\colon M_1 \mapsto M$ is an isometric immersion. Let γ_1 be a bi-f-harmonic curve with curvature κ_1 on M_1, and $\gamma(s) = i \circ \gamma_1$ be a curve on M. Then from (4.3) and (4.4), we have

$$\begin{aligned}
&\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla_{T_1} T_1 + 4ff' \nabla^2_{T_1} T_1 + f^2 \nabla^3_{T_1} T_1 + f^2 \mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 \\
&= \left(ff''' + f'f''\right) T_1 + 4ff'\left(\nabla^{1,2}_{T_1} + h^1\left(T_1, \nabla^1_{T_1} T_1\right) - A_{h^1(T_1,T_1)} T_1 + D_{T_1} h^1(T_1, T_1)\right) \\
&\quad + \left(3ff'' + 2f'^2\right) \nabla^1_{T_1} T_1 + \left(3ff'' + 2f'^2\right) h^1\left(T_1, T_1\right) + f^2\Big(\nabla^{1,3}_{T_1} T_1 + h^1\left(T_1, \nabla^{1,2}_{T_1} T_1\right) \\
&\quad - A_{h^1(T_1, \nabla^1_{T_1} T_1)} T_1 + D_{T_1} h^1\left(T_1, \nabla^1_{T_1} T_1\right) - \left(\nabla^1_{T_1} A\right)_{h^1(T_1,T_1)} T_1 - h^1\left(T_1, A_{h^1(T_1,T_1)} T_1\right) \\
&\quad - A_{D_{T_1} h^1(T_1,T_1)} T_1 + D^2_{T_1} h^1\left(T_1, T_1\right) + \mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1 - A_{h^1(T_1,T_1)} \nabla^1_{T_1} T_1 \\
&\quad + A_{h^1(\nabla^1_{T_1} T_1, T_1)} T_1 + \left(\nabla_{\nabla^1_{T_1} T_1} h^1\right)\left(T_1, T_1\right) - \left(\nabla_{T_1} h^1\right)\left(\nabla^1_{T_1} T_1, T_1\right)\Big). \qquad (4.23)
\end{aligned}$$

Using Frenet Serret equation and taking second fundamental form $h^1 = A = 0$ in equation (4.23), we obtain

$$\begin{aligned}
&\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla_{T_1} T_1 + 4ff' \nabla^2_{T_1} T_1 + f^2 \mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 + f^2 \nabla^3_{T_1} T_1 \\
&= \left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla^1_{T_1} T_1 + 4ff' \nabla^{1,2}_{T_1} T_1 + f^2 \nabla^{1,3}_{T_1} T_1 \\
&\quad + f^2 \mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1 = 0.
\end{aligned}$$

Hence γ is a bi-f-harmonic curve on M, and i is a bi-f-harmonic immersion. □

Corollary 4. *Let M_1 be a totally umbilical submanifold of Riemannian manifold M and $i\colon M_1 \to M$ be an isometric immersion such that the mean curvature H satisfies the equations (4.19) and*

$$f^2 D_{\nabla^1_{T_1} T_1} H = 4ff' ||H||^2 T_1 + 2f^2 ||H||^2 \nabla^1_{T_1} T_1 + 2f^2 h^1\left(\nabla^1_{\nabla^1_{T_1} T_1} T_1, T_1\right). \qquad (4.24)$$

Then i is a bi-f-harmonic immersion.

Proof. Let $\gamma_1(s)$ be a bi-f harmonic curve on M_1 and $i\colon M_1 \to M$ is an isometric immersion on M such that M_1 is a totally umbilical submanifold, then for a curve $\gamma(s)$ on M, from equations (4.3) and (4.4), we have

$$\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla_{T_1} T_1 + 4ff' \nabla^2_{T_1} T_1 + f^2 \mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 + f^2 \nabla^3_{T_1} T_1$$

$$
\begin{aligned}
&= \left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla^1_{T_1} T_1 + f^2 \nabla^{1,3}_{T_1} T_1 + \left(3ff'' + 2f'^2\right) h^1\left(T_1, T_1\right) \\
&\quad + 4ff' \nabla^{1,2}_{T_1} T_1 + 4ff' h^1\left(T_1, \nabla^1_{T_1} T_1\right) - 4ff' A_{h^1(T_1,T_1)} T_1 + 4ff' D_{T_1} h^1\left(T_1, T_1\right) \\
&\quad + f^2 h^1\left(T_1, \nabla^{1,2}_{T_1} T_1\right) - f^2 A_{h^1\left(T_1, \nabla^1_{T_1} T_1\right)} T_1 + f^2 D_{T_1} h^1\left(T_1, \nabla^1_{T_1} T_1\right) \\
&\quad - f^2 \nabla^1_{T_1}\left(A_{h^1(T_1,T_1)} T_1\right) - f^2 h^1\left(T_1, A_{h^1(T_1,T_1)} T_1\right) - f^2 A_{D_{T_1} h^1(T_1,T_1)} T_1 \\
&\quad + f^2 D^2_{T_1} h^1\left(T_1, T_1\right) + f^2 \mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1 - f^2 A_{h^1(T_1,T_1)} \nabla^1_{T_1} T_1 + f^2 A_{h^1\left(\nabla^1_{T_1} T_1, T_1\right)} T_1 \\
&\quad + f^2 \left(\nabla_{\nabla_{T_1} T_1} h^1\right)\left(T_1, T_1\right) - 7f^2 \left(\nabla_{T_1} h^1\right)\left(\nabla^1_{T_1} T_1, T_1\right).
\end{aligned} \tag{4.25}
$$

Using the Frenet- Serret formulae and (2.8) in equation (4.25), we get

$$
\begin{aligned}
&\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla_{T_1} T_1 + 4ff' \nabla^2_{T_1} T_1 + f^2 \mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 + f^2 \nabla^3_{T_1} T_1 \\
&= \left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla^1_{T_1} T_1 + 4ff' \nabla^{1,2}_{T_1} T_1 + \left(3ff'' + 2f'^2\right) H \\
&\quad - 4ff' A_H T_1 + 4ff' D_{T_1} H + f^2 D^2_{T_1} H + f^2 \nabla^{1,3}_{T_1} T_1 - \kappa_1^2 f^2 - f^2 \nabla^1_{T_1}\left(A_H T_1\right) \\
&\quad - f^2 h^1\left(T_1, A_H T_1\right) - f^2 A_{D_{T_1} H} T_1 + f^2 \mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1 - f^2 \kappa_1 A_H N_1 \\
&\quad + f^2 D_{\nabla^1_{T_1} T_1} H - 2f^2 h^1\left(\nabla^1_{\nabla^1_{T_1} T_1} T_1, T_1\right).
\end{aligned} \tag{4.26}
$$

Using equations (4.19) in (4.26), we obtain

$$
\begin{aligned}
&\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla_{T_1} T_1 + 4ff' \nabla^2_{T_1} T_1 + f^2 \mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 + f^2 \nabla^3_{T_1} T_1 \\
&= \left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'r\right) \nabla^1_{T_1} T_1 + 4ff' \nabla^{1,2}_{T_1} T_1 + f^2 \nabla^{1,3}_{T_1} T_1 \\
&\quad + f^2 \mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1 - 4ff' A_H T_1 - f^2 \nabla^1_{T_1}\left(A_H T_1\right) - f^2 A_{D_{T_1} H} T_1 \\
&\quad - \kappa_1 f^2 A_H N_1 + f^2 \left(\nabla_{\nabla^1_{T_1} T_1} H^1\right)\left(T_1, T_1\right) - f^2 \left(\nabla_{T_1} h^1\right)\left(\nabla^1_{T_1} T_1, T_1\right).
\end{aligned} \tag{4.27}
$$

Since $A_H T_1 = ||H||^2 T_1$, $A_H N_1 = ||H||^2 N_1$, $\nabla^1_{T_1}\left(A_H T_1\right) = ||H||^2 \nabla^1_{T_1} T_1$, $A_{D_{T_1} H} T_1 = 0$, $\left(\nabla_{\nabla^1_{T_1} T_1} h^1\right)\left(T_1, T_1\right) = D_{\nabla^1_{T_1} T_1} H - 2h^1\left(\nabla^1_{\nabla^1_{T_1} T_1} T_1, T_1\right)$ and $\left(\nabla_{T_1} h^1\right)\left(\nabla^1_{T_1} T_1, T_1\right) = 0$, therefore (4.27), gives us

$$
\begin{aligned}
&\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla_{T_1} T_1 + 4ff' \nabla^2_{T_1} T_1 + f^2 \mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 + f^2 \nabla^3_{T_1} T_1 \\
&= \left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla^1_{T_1} T_1 + 4ff' \nabla^{1,2}_{T_1} T_1 + f^2 \nabla^{1,3}_{T_1} T_1 \\
&\quad + f^2 \mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1 - 4ff' ||H||^2 T_1 - 2f^2 ||H||^2 \nabla^1_{T_1} T_1 + f^2 D_{\nabla^1_{T_1} T_1} H \\
&\quad - 2f^2 h^1(\nabla^1_{\nabla^1_{T_1} T_1} T_1, T_1).
\end{aligned} \tag{4.28}
$$

As γ_1 is a bi-f-harmonic curve on M_1, so equation (4.28), implies that

$$
\left(ff''' + f'f''\right) T_1 + \left(3ff'' + 2f'^2\right) \nabla_{T_1} T_1
$$

$$+4ff'\nabla^2_{T_1}T_1+f^2\nabla^3_{T_1}T_1+f^2\mathrm{R}\left(\nabla_{T_1}T_1,T_1\right)T_1$$
$$=-4ff'||H||^2T_1-2f^2||H||^2\nabla^1_{T_1}T_1+f^2D_{\nabla^1_{T_1}T_1}H-2f^2h^1\left(\nabla^1_{\nabla^1_{T_1}T_1}T_1,T_1\right). \quad (4.29)$$

The equations (4.24) and (4.29), together complete the proof. □

4.1. *Characterization of submanifold of a Riemannian manifold by bi-harmonic curves*

In this subsection, we study the characterization of bi-1-harmonic curve (bi-harmonic curve). A bi-harmonic curve (bi-1-harmonic curve) is a special case of bi-f-harmonic curve, where $f=1$. Taking $f=1$ in (2.1), we have

$$\nabla^{1,3}_{T_1}T_1+\mathrm{R}^1\left(\nabla^1_{T_1}T_1,T_1\right)T_1=0.$$

Let $i\colon M_1\to M$ is an isometric immersion from the Riemannian manifold M_1 to a Riemannian manifold M, and $\gamma(s)$ is a bi-harmonic curve on M, then

$$\nabla^3_{T_1}T_1+\mathrm{R}\left(\nabla^1_{T_1}T_1,T_1\right)T_1=0. \quad (4.30)$$

The tangential part of (4.30) and from equation (4.11), we get

$$-3\kappa_1\kappa_1'f^2=\frac{3}{2}f^2D_{T_1}||h^1(T_1,T_1)||^2. \quad (4.31)$$

Also from equation (4.16) and normal part of (4.30), we obtain

$$-4\kappa_1^2h^1\left(T_1,T_1\right)+4\kappa_1'h^1\left(T_1,N_1\right)+4\kappa_1\tau_1h^1\left(T_1,B_1\right)+3\kappa_1^2h^1\left(N_1,N_1\right)$$
$$+4\kappa_1\left(\nabla_{T_1}h^1\right)\left(T_1,N_1\right)-h^1\left(T_1,A_{h^1(T_1,T_1)}T_1\right)+\left(\nabla^2_{T_1}h^1\right)\left(T_1,T_1\right)$$
$$+\kappa_1\left(\nabla_{N_1}h^1\right)\left(T_1,T_1\right)=0. \quad (4.32)$$

Changing B_1 into - B_1 in equation (4.32) and then subtracting from equation (4.32), we get

$$h^1\left(T_1,B_1\right)=0.$$

Theorem 7. *Let $i\colon M_1\to M$ be an isometric immersion between Riemannian manifolds M_1 and M such that γ_1 is a curve with curvature κ_1 in M_1 and $\gamma(s)=i\circ\gamma_1(s)$ is a bi-f-harmonic curve with curvature κ in M. Then M_1 is an isotropic submanifold if and only if its curvature is constant.*

Proof. Let γ_1 is a curve with constant curvature in M_1, then from (4.31), we have

$$D_{T_1}||h^1\left(T_1,T_1\right)||^2=0.$$

Thus $||h^1\left(T_1,T_1\right)||$ is constant, hence M_1 is an isotropic submanifold.

Conversely, let M_1 be an isotropic submanifold, then from equation (4.31), we get $\kappa_1=constant$. □

Theorem 8. *Let i be a bi-harmonic immersion from Riemannian manifold M_1 to a Riemannian manifold M and let M_1 be a totally umbilical submanifold of M. Then the mean curvature vector field H satisfies*

$$D_{T_1}^2 H = \left(\kappa_1^2 + ||H||^2\right) H, \tag{4.33}$$

Conversely, Let $i\colon M_1 \mapsto M$ be an isometric immersion from Riemannian manifold M_1 to a Riemannian manifold M, then

(1) *If M_1 be a totally geodesic, then i is a bi-harmonic immersion.*
(2) *If M_1 be a totally umbilical, such that H satisfies the equation (4.33) and $D_{\nabla^1_{T_1} T_1} H = 2\nabla^1_{T_1}\left(||H||^2 T_1\right)$, then i is a bi-harmonic immersion.*

Proof. Let i be a bi-harmonic immersion between Riemannian manifold M_1 and M. Since M_1 is a totally umbilical submanifold, therefore equation (4.16), reduces to

$$\begin{aligned} 4\kappa_1 \left(\nabla_{T_1} h^1\right)(T_1, N_1) + 3\kappa_1^2 H + \kappa_1 \left(\nabla_{N_1} h^1\right)(T_1, T_1) \\ = 4\kappa_1^2 H + h^1(T_1, A_H T_1) - \left(\nabla_{T_1}^2 h^1\right)(T_1, T_1). \end{aligned} \tag{4.34}$$

Replacing N_1 into - N_1 in equation (4.34) and then substracting from equation (4.34), we get

$$4\kappa_1 \left(\nabla_{T_1} h^1\right)(T_1, N_1) + \kappa_1 \left(\nabla_{N_1} h^1\right)(T_1, T_1) = 0. \tag{4.35}$$

Substituting equation (4.35) in equation (4.34), we have

$$3\kappa_1^2 H = 4\kappa_1^2 H + \langle A_H T_1, T_1\rangle H - \left(\nabla_{T_1}^2 h^1\right)(T_1, T_1). \tag{4.36}$$

Using the equations (2.4) and (4.13) in equation (4.36), we have

$$D_{T_1}^2 H = \left(\kappa_1^2 + ||H||^2\right) H.$$

Conversely, (1) let $i\colon M_1 \mapsto M$ be an isometric immersion such that M_1 is a totally geodesic submanifold of M and γ_1 be a bi-harmonic curve with curvature κ_1 on M_1. Taking $f = 1$ in (4.23), we have

$$\begin{aligned} &\nabla_{T_1}^3 T_1 + \mathrm{R}(\nabla_{T_1} T_1, T_1) T_1 = \nabla_{T_1}^{1,3} T_1 + h^1(T_1, \nabla_{T_1}^{1,2} T_1) - A_{h^1(T_1, \nabla^1_{T_1} T_1)} T_1 \\ &+ D_{T_1} h^1(T_1, \nabla^1_{T_1} T_1) - \nabla^1_{T_1}\left(A_{h^1(T_1,T_1)} T_1\right) - h^1\left(T_1, A_{h^1(T_1,T_1)} T_1\right) - A_{D_{T_1} h^1(T_1,T_1)} T_1 \\ &+ D_{T_1}^2 h^1(T_1, T_1) + \mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1 - A_{h^1(T_1,T_1)} \nabla^1_{T_1} T_1 + A_{h^1\left(\nabla^1_{T_1} T_1, T_1\right)} T_1 \\ &+ \left(\nabla_{\nabla^1_{T_1} T_1} h^1\right)(T_1, T_1) - \left(\nabla_{T_1} h^1\right)\left(\nabla^1_{T_1} T_1, T_1\right). \end{aligned} \tag{4.37}$$

Using the fact that M_1 is totally geodesic and the Frenet Serret formulae in equation (4.37), we get

$$\nabla_{T_1}^3 T_1 + \mathrm{R}(\nabla_{T_1} T_1, T_1) T_1 = \nabla_{T_1}^{1,3} T_1 + \mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1. \tag{4.38}$$

Since γ_1 is a bi-harmonic curve on M_1, so from equation (4.38), we can say that $\gamma(s) = i \circ \gamma_1$ is bi-harmonic curve on M.

Now for (2), let M_1 be a totally umbilical submanifold and satisfies the equation (4.33), then

$$\nabla^3_{T_1} T_1 = \nabla^{1,3}_{T_1} T_1 - \nabla^1_{T_1}\left(||H||^2 T_1\right) - \left(\kappa_1^2 + ||H||^2\right) H + D^2_{T_1} H, \tag{4.39}$$

and

$$\mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 = \mathrm{R}^1\left(\nabla^1_{T_1} T_1, T_1\right) T_1 - \nabla^1_{T_1}\left(||H||^2 T_1\right) + D_{\nabla^1_{T_1} T_1} H. \tag{4.40}$$

Adding the equations (4.39) and (4.40), we obtain

$$\begin{aligned}\nabla^3_{T_1} T_1 + \mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 = \nabla^{1,3}_{T_1} T_1 - \nabla^1_{T_1}\left(2||H||^2 T_1\right) - \left(\kappa_1^2 + ||H||^2\right) H \\ + D^2_{T_1} H + \mathrm{R}^1\left(\nabla_{T_1} T_1, T_1\right) T_1 + D_{\nabla^1_{T_1} T_1} H.\end{aligned} \tag{4.41}$$

Using the fact that γ_1 is a bi-harmonic curve on M_1, equation (4.41) can be written as

$$\nabla^3_{T_1} T_1 + \mathrm{R}\left(\nabla_{T_1} T_1, T_1\right) T_1 = -\nabla^1_{T_1}\left(2||H||^2 T_1\right) + D_{\nabla^1_{T_1} T_1} H.$$

Thus from above equation, we have

$$\nabla^3_{T_1} T_1 + \left(\nabla_{T_1} T_1, T_1\right) T_1 = 0,$$

if and only if $D_{\nabla^1_{T_1} T_1} H = 2\nabla^1_{T_1}\left(||H||^2 T_1\right)$, hence i is a bi-harmonic immersion. $\square$

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