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SOME BOUNDS OF HERMITE–HADAMARD-TYPE INEQUALITIES BASED ON CONFORMABLE FRACTIONAL INTEGRALS

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Abstract. In this research, we establish the above and below bounds via the left and right sides of Hermite–Hadamard-type inequalities including conformable fractional integrals with the aid of the mappings whose second derivatives are bounded. Instead of using the convexity condition in these obtained inequalities, we used condition $f'(a+b-t)-f'(t)\geq 0$, $t\in\left[a,\frac{a+b}{2}\right]$. We have presented examples of the inequalities acquired. We have given the graph showing the correctness of the presented examples.

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1. Introduction and preliminaries

Convex function theory has many uses in the fields of mathematics, physics, and engineering. Let I be convex set on $\mathbb{R}$. The mapping $f\colon I\to\mathbb{R}$ is called convex function on I, if it satisfies the following inequality:

$$f(tx+(1-t)\,y)\leq tf(x)+(1-t)f(y)$$

for all $(x,y)\in I$ and $t\in[0,1]$. (see, e.g. [11], and the reference therein). There is a great connection between inequalities and the theory of convex functions. This bond emerged starting from the definition of the convex function and its acceleration is increased by the researches made afterwards. The Hermite–Hadamard-type inequalities, which are obtained by using convex functions and have been the subject of many studies, has been studied in the literature extensively. For more information on the Hermite-Hadamard type inequality obtained for different classes of convex, please refer to the references [5, 15–17]. The right-hand side of the Hermite–Hadamard-type inequality is called trapezoid-type inequality in the literature. Dragomir and Agarwal proved trapezoid-type inequalities based on convex mappings in [8]. The left side of the Hermite–Hadamard-type inequality is called midpoint-type inequality. Kirmacı acquired midpoint-type inequalities with the aid of the convex mappings in [20]. Researchers began to come up with ideas on how to obtain these inequalities when functions are not convex. With this motivation, many researchers have worked on this subject. Dragomir et al. presented trapezoid-type and midpoint-type extensions by means of the bounds of the twice-differentiable rather than the condition of convexity in [9] and [10], respectively. More precisely, Dragomir et al. obtained new bounds for classical midpoint-type and trapezoid-type inequalities.

Theorem 1 ([10, page 6], [9, page 8])**.** *Assume that $f\colon [a,b]\to\mathbb{R}$ is a twice-differentiable mapping such that there exists real constants m and M so that $m\leq f''\leq M$. Then, the following inequalities hold:*

$$m\frac{(b-a)^2}{24}\leq\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x-f\left(\frac{a+b}{2}\right)\leq M\frac{(b-a)^2}{24} \tag{1.1}$$

and

$$m\frac{(b-a)^2}{24}\leq\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\leq M\frac{(b-a)^2}{24}. \tag{1.2}$$

For different studies obtained with this motivation, the following references can be consulted [2, 6, 25].

Fractional analysis, ”Can fractional derivatives and fractional integrals be taken?” emerged from the question. Today, it is the subject of study by many researchers. Many types of fractional integrals are studied in the literature. Especially, there are many results obtained with the help of Riemann-Liouville fractional integrals, generalized fractional integrals, and conformable fractional integrals.

The Euler Gamma function and Euler Beta function are defined

$$\Gamma(x):=\int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$$

and

$$\mathcal{B}(x,y):=\int_0^1 t^{x-1}(1-t)^{y-1}\mathrm{d}t,$$

respectively for $x,y\in\mathbb{R}$. Let us give the definitions of Riemann-Liouville fractional integrals in the literature.

Definition 1 (see [19, page 71], [21])**.** Let us consider $f\in L_1[a,b]$. The Riemann-Liouville fractional integrals of order $\beta>0$ are described by

$$J_{a^+}^{\beta}f(x)=\frac{1}{\Gamma(\beta)}\int_a^x(x-t)^{\beta-1}f(t)\mathrm{d}t,\qquad x>a$$

and

$$J_{b-}^{\beta} f(x) = \frac{1}{\Gamma(\beta)} \int_x^b (t-x)^{\beta-1} f(t)\mathrm{d}t, \qquad x < b.$$

Here, Γ is the Euler Gamma function and $J_{a+}^{0} f(x) = J_{b-}^{0} f(x) = f(x)$.

Many studies contributing to the literature have been considered, especially using the convexity of the function. Using Riemann-Liouville fractional integrals for functions of one variable, Hermite–Hadamard-type inequalities for convex functions were obtained by Sarıkaya et al. in [23]. Sarikaya first [23] presented the Hermite–Hadamard-type inequalities involving Riemann-Liouville integrals as follows:

Theorem 2. *Let $f\colon [a,b] \to \mathbb{R}$ be a positive mapping with $f \in L_1[a,b]$. If f is a convex mapping on $[a,b]$, then the following inequalities hold:*

$$f\left(\frac{a+b}{2}\right) \le \frac{\Gamma(\beta+1)}{2(b-a)^{\beta}} \left[J_{a+}^{\beta} f(b) + J_{b-}^{\beta} f(a)\right] \le \frac{f(a)+f(b)}{2}$$

with $\beta > 0$.

Riemann-Liouville fractional integrals for convex functions of two variables were given by Sarıkaya [22] and Hermite–Hadamard-type inequalities for convex functions were proved in the coordinates obtained using these integrals. Chen obtained the midpoint-type and trapezoid-type inequality for Riemann-Liouville fractional integrals by using the condition $f'(a+b-x) \ge f'(x)$, $x \in \left[a, \frac{a+b}{2}\right]$ instead of the convexity of f in paper [7].

Theorem 3 (see [7, page 2,3])**.** *Consider $f\colon [a,b] \longrightarrow \mathbb{R}$ is a positive, twice-differentiable function and $f \in L_1[a,b]$. If f'' is bounded $[a,b]$, then we derive*

$$\begin{aligned} m\frac{(b-a)^2\left(\beta^2-\beta+2\right)}{8(\beta+1)(\beta+2)} &\le \frac{\Gamma(\beta+1)}{2(b-a)^{\beta}}\left[J_{a+}^{\beta} f(b) + J_{b-}^{\beta} f(a)\right] - f\left(\frac{a+b}{2}\right) \\ &\le \frac{(b-a)^2\left(\beta^2-\beta+2\right)}{8(\beta+1)(\beta+2)}M \end{aligned}$$

and

$$\begin{aligned} m\frac{(b-a)^2\beta}{2(\beta+1)(\beta+2)} &\le \frac{f(a)+f(b)}{2} - \frac{\Gamma(\beta+1)}{2(b-a)^{\beta}}\left[J_{a+}^{\beta} f(b) + J_{b-}^{\beta} f(a)\right] \\ &\le \frac{(b-a)^2\beta}{2(\beta+1)(\beta+2)}M \end{aligned}$$

for $\beta > 0$. Here, $m = \inf_{t\in[a,b]} f''(t)$ and $M = \sup_{t\in[a,b]} f''(t)$.

Budak et al. acquired the left and right-hand sides of fractional Hermite–Hadamard-type inequalities with the aid of the bounds of the second derivative in [3].

Theorem 4 (see [3, page 3,8]). *Let $f : [a,b] \longrightarrow \mathbb{R}$ denote a positive, twice-differentiable function and $f \in L_1[a,b]$. If f'' is bounded i.e. $m \leq f''(t) \leq M$, $t \in [a,b]$, $m, M \in \mathbb{R}$, then we have the inequalities*

$$m\frac{(b-a)^2}{4(\beta+1)(\beta+2)} \leq \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J^\beta_{\left(\frac{a+b}{2}\right)^+}f(b)+J^\beta_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)$$
$$\leq \frac{(b-a)^2}{4(\beta+1)(\beta+2)}M, \tag{1.3}$$

$$m\frac{(b-a)^2\beta(\beta+3)}{8(\beta+1)(\beta+2)} \leq \frac{f(a)+f(b)}{2}-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J^\beta_{\left(\frac{a+b}{2}\right)^+}f(b)+J^\beta_{\left(\frac{a+b}{2}\right)^-}f(a)\right]$$
$$\leq \frac{(b-a)^2\beta(\beta+3)}{8(\beta+1)(\beta+2)}M, \tag{1.4}$$

$$m\frac{\beta(b-a)^2}{8(\beta+2)} \leq \frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J^\beta_{a^+}f\left(\frac{a+b}{2}\right)+J^\beta_{b^-}f\left(\frac{a+b}{2}\right)\right]$$
$$-f\left(\frac{a+b}{2}\right) \leq \frac{\beta(b-a)^2}{8(\beta+2)}M$$

and

$$m\frac{(b-a)^2}{4(\beta+2)} \leq \frac{f(a)+f(b)}{2}-\frac{2^{\beta-1}\Gamma(\beta+1)}{(b-a)^\beta}\left[J^\beta_{a^+}f\left(\frac{a+b}{2}\right)+J^\beta_{b^-}f\left(\frac{a+b}{2}\right)\right]$$
$$\leq \frac{(b-a)^2}{4(\beta+2)}M$$

for $\beta > 0$.

Budak et al. obtained some trapezoid-type and midpoint-type inequalities via generalized fractional integrals with the help of the functions whose second derivatives are bounded [4].

In 2017, Jarad et al. [14] defined the conformable fractional integral operators. For studies on the conformable fractional approach and conformable fractional integrals, see references [1, 13, 18].

Definition 2 (see [14, page 5]). For $f \in L_1[a,b]$, the conformable fractional integral operators of order $\beta \in \mathbb{C}$, $Re(\beta) > 0$ and $\alpha \in (0,1]$ are given by

$$^\beta_{a^+}\Upsilon^\alpha f(x) = \frac{1}{\Gamma(\beta)}\int_a^x\left(\frac{(x-a)^\alpha-(t-a)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(t)}{(t-a)^{1-\alpha}}\mathrm{d}t,$$
$$^\beta\Upsilon^\alpha_{b^-} f(x) = \frac{1}{\Gamma(\beta)}\int_x^b\left(\frac{(b-x)^\alpha-(b-t)^\alpha}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}\mathrm{d}t.$$

Set et al. proved Hermite–Hadamard-type inequalities by means of conformable fractional integral operators in [24].

Theorem 5 (see [12, page 35]). *Note that f is a convex function on $[a,b]$. Then, the following double inequality is satisfied*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] \leq \frac{f(a)+f(b)}{2}. \tag{1.5}$$

Here, $\beta > 0$, $\alpha \in (0,1]$ and Γ is Euler Gamma fuction.

This study consists of three sections including the introduction and preliminaries. In section 2, we will acquire some new versions midpoint-type and trapezoid-type inequalities based on conformable fractional integrals. In some of these acquired inequalities, we will use the bounds of the second derivative of the function rather than the convexity of the function. We also acquired the new version of Hermite–Hadamard-type inequalities via conformable fractional integrals. In this obtained inequality, we used the condition $f'(a+b-t) - f'(t) \geq 0$, $t \in \left[a, \frac{a+b}{2}\right]$ rather than the convexity condition.

2. Main results

In this section, we obtain two new midpoint-type inequalities for the case of conformable fractional integrals. We also establish two new versions of trapezoid-type inequalities with the help of conformable fractional integrals. In these obtained four inequalities, the bounds of the second derivative of the function are used instead of the convexity of the function. We also obtained some new Hermite–Hadamard-type inequalities for the case of conformable fractional integrals. In this obtained inequality, we used the condition $f'(a+b-x) \geq f'(x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$ instead of the convexity of f.

Theorem 6. *Let us consider that $f\colon [a,b] \to \mathbb{R}$ is a twice-differentiable mapping. Let us also consider that there exist real constants m and M so that $m \leq f'' \leq M$ and $\beta > 0$, $\alpha \in (0,1]$. The following midpoint-type inequalities*

$$\begin{aligned} & m\frac{\beta(b-a)^2}{8}\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) \\ & \leq \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a)\right] - f\left(\frac{a+b}{2}\right) \\ & \leq M\frac{\beta(b-a)^2}{8}\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) \end{aligned} \tag{2.1}$$

are valid. Here, $\mathcal{B}$ is Euler Beta function.

Proof. By using the Definition 2, we establish

$$\begin{aligned}
&\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\\
&=\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[\frac{1}{\Gamma(\beta)}\int_{\frac{a+b}{2}}^{b}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(t-\frac{a+b}{2}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{\left(t-\frac{a+b}{2}\right)^{1-\alpha}}\mathrm{d}t\right.\\
&\qquad+\frac{1}{\Gamma(\beta)}\int_{a}^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\\
&\qquad\left.\times\frac{f(t)}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}}\mathrm{d}t\right].
\end{aligned}$$

Taking advantage of the change of variables, we acquire

$$\begin{aligned}
&\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\\
&=\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[\frac{1}{\Gamma(\beta)}\int_{a}^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(a+b-t)}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}}\mathrm{d}t\right.\\
&\qquad\left.+\frac{1}{\Gamma(\beta)}\int_{a}^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}}\mathrm{d}t\right]\\
&=\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_{a}^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\left(f(t)+f(a+b-t)\right)}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}}\mathrm{d}t.
\end{aligned}$$

With the help of the inequality (1.5), we obtain

$$\begin{aligned}
&\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right) \qquad (2.2)\\
&=\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_{a}^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\\
&\quad\times\frac{\left(f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)\right)}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}}\mathrm{d}t.
\end{aligned}$$

If we take advantage of the facts that

$$f(t)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{t}f'(s)\mathrm{d}s$$

and

$$f(a+b-t)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-t}f'(s)\mathrm{d}s,$$

then we acquire

$$\begin{aligned}f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)&=\int_{\frac{a+b}{2}}^{t}f'(s)\mathrm{d}s+\int_{\frac{a+b}{2}}^{a+b-t}f'(s)\mathrm{d}s\\&=\int_{\frac{a+b}{2}}^{a+b-t}f'(s)\mathrm{d}s-\int_{\frac{a+b}{2}}^{a+b-t}f'(a+b-s)\mathrm{d}s\\&=\int_{\frac{a+b}{2}}^{a+b-t}\left[f'(s)-f'(a+b-s)\right]\mathrm{d}s.\end{aligned}\tag{2.3}$$

We also acquire

$$f'(s)-f'(a+b-s)=\int_{a+b-s}^{s}f''(y)\mathrm{d}y.\tag{2.4}$$

With the help of the condition $m\le f''(y)\le M$ for all $y\in[a,b]$ and from the equality (2.4), we derive

$$\int_{a+b-s}^{s}m\mathrm{d}y\le\int_{a+b-s}^{s}f''(y)\mathrm{d}y\le\int_{a+b-s}^{s}M\mathrm{d}y,$$

which gives

$$m(2s-a-b)\le f'(s)-f'(a+b-s)\le M(2s-a-b).$$

From the equality (2.3), we obtain

$$m\int_{\frac{a+b}{2}}^{a+b-t}(2s-a-b)\le\int_{\frac{a+b}{2}}^{a+b-t}\left[f'(s)-f'(a+b-s)\right]\mathrm{d}s\le M\int_{\frac{a+b}{2}}^{a+b-t}(2s-a-b).$$

Hence, it yields

$$m\left(\frac{a+b}{2}-t\right)^2\le f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)\le M\left(\frac{a+b}{2}-t\right)^2.\tag{2.5}$$

Multiplying the inequality (2.5) by $\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{a+b}{2}-t\right)^{\alpha-1}$ and then integrating with respect to t on the interval $\left[a,\frac{a+b}{2}\right]$, we obtain

$$\begin{aligned}&m\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{a+b}{2}-t\right)^{\alpha+1}\mathrm{d}t\\&\le\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\end{aligned}$$

$$\times \frac{\left(f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)\right)}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}}dt$$

$$\leq M\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{a+b}{2}-t\right)^{\alpha+1}dt.$$

Finally, we obtain

$$m\frac{\beta(b-a)^2}{8}\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)$$
$$\leq \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)$$
$$\leq M\frac{\beta(b-a)^2}{8}\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right).$$

Finally, the proof of Theorem 6 is completed. □

Remark 1. If we choose $\alpha=1$ in Theorem 6, then the inequalities (2.1) become to (1.3).

Remark 2. If we assign $\alpha=1$ and $\beta=1$ in Theorem 6, then the inequalities (2.1) reduce to the inequalities (1.1).

Example 1. If the mapping $f\colon [a,b]=[0,1]\to\mathbb{R}$ is defined as $f(t)=t^3+t^2$ such that $2\leq f''(t)\leq 5$ for $t\in[0,1]$. Under these conditions, we obtain the mid-term of inequality (2.1) as follows

$$\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right]-f\left(\frac{a+b}{2}\right)$$
$$=2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}\left[{}^{\beta}_{\frac{1}{2}^+}\Upsilon^{\alpha}f(1)+{}^{\beta}\Upsilon^{\alpha}_{\frac{1}{2}^-}f(0)\right]-f\left(\frac{1}{2}\right)$$
$$=2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}\left[\frac{1}{\Gamma(\beta)}\int_{\frac{1}{2}}^{1}\left(\frac{\left(\frac{1}{2}\right)^{\alpha}-\left(t-\frac{1}{2}\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(t^3+t^2)}{\left(t-\frac{1}{2}\right)^{1-\alpha}}dt\right.$$
$$\left.+\frac{1}{\Gamma(\beta)}\int_0^{\frac{1}{2}}\left(\frac{\left(\frac{1}{2}\right)^{\alpha}-\left(\frac{1}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(t^3+t^2)}{\left(\frac{1}{2}-t\right)^{1-\alpha}}dt\right]$$
$$-\left(\frac{1}{2^3}+\frac{1}{2^2}\right).$$

By using the change of variables, we have

$$2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}\left[{}^{\beta}_{\frac{1}{2}^+}\Upsilon^{\alpha}f(1)+{}^{\beta}\Upsilon^{\alpha}_{\frac{1}{2}^-}f(0)\right]-f\left(\frac{1}{2}\right)$$

$$= 2^{\alpha\beta-1}\beta\alpha^{\beta}\int_0^{\frac{1}{2}}\left(\frac{\left(\frac{1}{2}\right)^{\alpha}-\left(\frac{1}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}$$
$$\times\frac{\left(t^3+t^2+(1-t)^3+(1-t)^2-2\left(\frac{1}{2^3}+\frac{1}{2^2}\right)\right)}{\left(\frac{1}{2}-t\right)^{1-\alpha}}\mathrm{d}t$$
$$= \frac{5\beta}{8}\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right).$$

As a result, the inequalities (2.1) can be found as follows

$$\frac{\beta}{4}\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\leq\frac{5\beta}{8}\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\leq\frac{5\beta}{8}\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right).$$

Theorem 7. *Note that $f\colon [a,b]\to\mathbb{R}$ is a twice-differentiable mapping such that there exists real constants m and M so that $m\leq f''\leq M$. Then, we derive*

$$m\frac{\beta(b-a)^2}{8}\left[\frac{1}{\beta}-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \tag{2.6}$$
$$\leq\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right]$$
$$\leq M\frac{\beta(b-a)^2}{8}\left[\frac{1}{\beta}-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right],$$

where $\mathcal{B}$ is Euler Beta function.

Proof. From the equality (1.5), we acquire

$$\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right] \tag{2.7}$$
$$=\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}$$
$$\times\frac{(f(t)+f(a+b-t))}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}}\mathrm{d}t$$
$$=\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}$$
$$\times\frac{(f(a)+f(b)-f(t)-f(a+b-t))}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}}\mathrm{d}t.$$

With the help of the equalities

$$f(a)-f(t)=-\int_a^t f'(s)\mathrm{d}s$$

and

$$f(b)-f(a+b-t)=\int_{a+b-t}^b f'(s)\mathrm{d}s,$$

we derive

$$\begin{aligned} f(a)+f(b)-(f(t)+f(a+b-t)) &= \int_{a+b-t}^b f'(s)\mathrm{d}s-\int_a^t f'(s)\mathrm{d}s \\ &= \int_a^t f'(a+b-s)\mathrm{d}s-\int_a^t f'(s)\mathrm{d}s \\ &= \int_a^t \left[f'(a+b-s)-f'(s)\right]\mathrm{d}s. \end{aligned} \tag{2.8}$$

We also provide

$$f'(a+b-s)-f'(s)=\int_s^{a+b-s} f''(y)\mathrm{d}y. \tag{2.9}$$

From the equality (2.9) and the condition $m\leq f''\leq M$, we get

$$m(a+b-2s)\leq f'(s)-f'(a+b-s)\leq M(a+b-2s). \tag{2.10}$$

With the help of the equality (2.8) and the inequality (2.10), we have

$$\int_a^t m(a+b-2s)\mathrm{d}s\leq\int_a^t\left[f'(s)-f'(a+b-s)\right]\mathrm{d}s\leq\int_a^t M(a+b-2s)\mathrm{d}s$$

such that

$$m(b-t)(t-a)\leq f(a)+f(b)-(f(t)+f(a+b-t))\leq M(b-t)(t-a). \tag{2.11}$$

Multiplying the inequality (2.11) by $\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{a+b}{2}-t\right)^{\alpha-1}$ and then integrating with respect to t on the interval $\left[a,\frac{a+b}{2}\right]$, we readily obtain

$$\begin{aligned} & m\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{a+b}{2}-t\right)^{\alpha-1}(b-t)(t-a)\mathrm{d}t \\ & \leq\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1} \\ & \quad\times\frac{(f(a)+f(b)-f(t)-f(a+b-t))}{\left(\frac{a+b}{2}-t\right)^{1-\alpha}}\mathrm{d}t \end{aligned}$$

$$\leq M\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{a+b}{2}-t\right)^{\alpha-1}(b-t)(t-a)\mathrm{d}t.$$

By using the above inequalities, we can easily get

$$\begin{aligned}
&m\frac{\beta(b-a)^2}{8}\left[\frac{1}{\beta}-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right]\\
&\leq\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right]\\
&\leq M\frac{\beta(b-a)^2}{8}\left[\frac{1}{\beta}-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].
\end{aligned}$$

Finally, the proof of Theorem 7 is accomplished. □

Remark 3. If we choose $\alpha=1$ in Theorem 7, then the inequalities (2.6) coincides with the inequalities (1.4).

Remark 4. If we take $\alpha=\beta=1$ in Theorem 7, then Theorem 7 equals to (1.2).

Example 2. Let us note that the function $f\colon [a,b]=[0,1]\to\mathbb{R}$ is defined as $f(t)=t^2-t^4$ such that $-10\leq f''(t)\leq 2$ for $t\in[0,1]$. By these conditions, the right-side of (2.6) becomes as follows

$$\begin{aligned}
&\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^+}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)\right]\\
&=\frac{f(0)+f(1)}{2}-2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}\left[{}^{\beta}_{\left(\frac{1}{2}\right)^+}\Upsilon^{\alpha}f(1)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{1}{2}\right)^-}f(0)\right]\\
&=-2^{\alpha\beta-1}\beta\alpha^{\beta}\int_0^{\frac{1}{2}}\left(\frac{\left(\frac{1}{2}\right)^{\alpha}-\left(\frac{1}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{(f(t)+f(1-t))}{\left(\frac{1}{2}-t\right)^{1-\alpha}}\mathrm{d}t\\
&=2^{\alpha\beta-1}\beta\alpha^{\beta}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{1}{2}\right)^{\alpha}-\left(\frac{1}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\left(-t^2+t^4-(1-t)^2+(1-t)^4\right)}{\left(\frac{1}{2}-t\right)^{1-\alpha}}\mathrm{d}t.\\
&=\frac{\beta}{16}\left[\left(\mathcal{B}\left(\frac{4}{\alpha}+1,\beta\right)+2\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\frac{3}{\beta}\right)\right].
\end{aligned}$$

Lastly, the inequality (2.6) becomes to

$$\begin{aligned}
\frac{5\beta}{4}\left[\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\frac{1}{\beta}\right]&\leq\frac{\beta}{16}\left[\left(\mathcal{B}\left(\frac{4}{\alpha}+1,\beta\right)+2\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)-\frac{3}{\beta}\right)\right]\\
&\leq\frac{\beta}{4}\left[\frac{1}{\beta}-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].
\end{aligned}$$

To illustrate the correctness of Example 2, one can refer to the Figure 1.

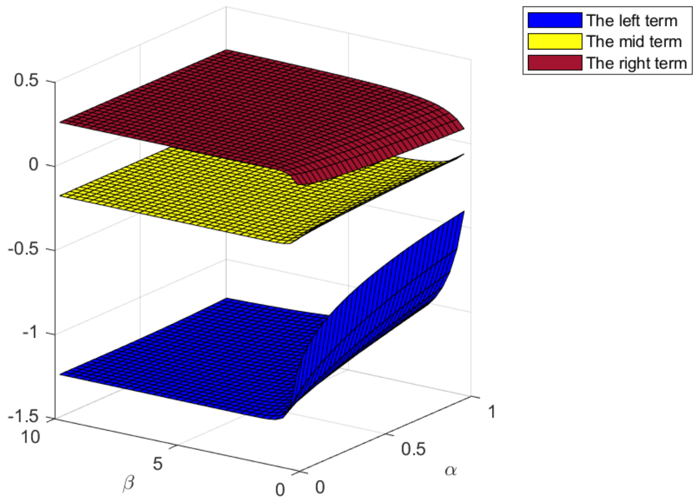

FIGURE 1. Graph for the result of Example 2 computed and plotted in MATLAB.

Theorem 8. *Let $f\colon [a,b]\to\mathbb{R}$ be a differentiable mapping and $f\in L_1[a,b]$. If $f'(a+b-x)\geq f'(x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$. Then, the following Hermite–Hadamard-type inequality for conformable fractional integrals*

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]\leq\frac{f(a)+f(b)}{2}\tag{2.12}$$

is valid.

Proof. From the equalities (2.2) and (2.3), we have

$$\begin{aligned}&\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}\Upsilon^{\alpha}f(b)+{}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}}f(a)\right]-f\left(\frac{a+b}{2}\right)\\&=\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{a+b}{2}-t\right)^{\alpha-1}\\&\quad\times\left(f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)\right)\mathrm{d}t\\&=\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-\left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha}\right)^{\beta-1}\left(\frac{a+b}{2}-t\right)^{\alpha-1}\end{aligned}$$

$$\times \left[\int_{\frac{a+b}{2}}^{a+b-t} \left[f'(s) - f'(a+b-s) \right] \mathrm{d}s \right] \mathrm{d}t$$

$$= \frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}} \int_{a}^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^{\alpha} - \left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha-1}$$

$$\times \left[\int_{t}^{\frac{a+b}{2}} \left[f'(a+b-u) - f'(u) \right] \mathrm{d}s \right] \mathrm{d}t \geq 0,$$

which presents the first inequality in (2.12).

Similar to foregoing process, by the equalities (2.7) and (2.8), we obtain

$$\frac{f(a)+f(b)}{2} - \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}} \left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right]$$

$$= \frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}} \int_{a}^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^{\alpha} - \left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha-1}$$

$$\times \left[f(a) + f(b) - (f(t) + f(a+b-t)) \right] \mathrm{d}t$$

$$= \frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}} \int_{a}^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^{\alpha} - \left(\frac{a+b}{2}-t\right)^{\alpha}}{\alpha} \right)^{\beta-1} \left(\frac{a+b}{2} - t \right)^{\alpha-1}$$

$$\times \left[\int_{a}^{t} \left[f'(a+b-s) - f'(s) \right] \mathrm{d}s \right] \mathrm{d}t \geq 0.$$

Thus, the proof of Theorem 8 is finished. □

Theorem 9. *Note that f is a convex mapping on $[a,b]$. Then, the following double inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}} \left[{}^{\beta}_{a^{+}}\Upsilon^{\alpha} f\left(\frac{a+b}{2}\right) + {}^{\beta}\Upsilon^{\alpha}_{b^{-}} f\left(\frac{a+b}{2}\right) \right]$$

$$\leq \frac{f(a)+f(b)}{2}. \tag{2.13}$$

Here, $\beta > 0$, $\alpha \in (0,1]$ and Γ is Euler Gamma fuction.

Theorem 10. *Assume that $f\colon [a,b] \to \mathbb{R}$ is a twice-differentiable mapping such that there exists real constants m and M so that $m \leq f'' \leq M$ and $\beta > 0$, $\alpha \in (0,1]$. The following double inequality*

$$m\frac{\beta(b-a)^2}{8} \left[\frac{1}{\beta} - 2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right) + \mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right) \right] \tag{2.14}$$

$$\leq \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}} \left[{}^{\beta}_{a^{+}}\Upsilon^{\alpha} f\left(\frac{a+b}{2}\right) + {}^{\beta}\Upsilon^{\alpha}_{b^{-}} f\left(\frac{a+b}{2}\right) \right] - f\left(\frac{a+b}{2}\right)$$

$$\leq M\frac{\beta\left(b-a\right)^{2}}{8}\left[\frac{1}{\beta}-2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right]$$

is valid. Here, $\mathcal{B}$ is Euler Beta function.

Proof. With the help of the Definition 2, we have

$$\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]$$
$$=\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[\frac{1}{\Gamma(\beta)}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(t-a)^{1-\alpha}}dt\right.$$
$$\left.+\frac{1}{\Gamma(\beta)}\int_{\frac{a+b}{2}}^{b}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(b-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(b-t)^{1-\alpha}}dt\right].$$

With the aid of the change of variables, we acquire

$$=\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[\frac{1}{\Gamma(\beta)}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(a+b-t)}{(t-a)^{1-\alpha}}dt\right.$$
$$\left.+\frac{1}{\Gamma(\beta)}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{f(t)}{(t-a)^{1-\alpha}}dt\right]$$
$$=\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{1}{(t-a)^{1-\alpha}}\left(f(t)+f(a+b-t)\right)dt.$$

From the inequality (2.13), we have

$$\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\tag{2.15}$$
$$=\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\left(f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)\right)}{(t-a)^{1-\alpha}}dt.$$

If we take advantage of the facts that

$$f(t)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{t}f'(s)ds$$

and

$$f(a+b-t)-f\left(\frac{a+b}{2}\right)=\int_{\frac{a+b}{2}}^{a+b-t}f'(s)ds,$$

then we have

$$f(t)+f(a+b-t)-2f(\frac{a+b}{2}) = \int_{\frac{a+b}{2}}^{t} f'(s)\mathrm{d}s + \int_{\frac{a+b}{2}}^{a+b-t} f'(s)\mathrm{d}s \tag{2.16}$$
$$= \int_{\frac{a+b}{2}}^{a+b-t} f'(s)\mathrm{d}s - \int_{\frac{a+b}{2}}^{a+b-t} f'(a+b-s)\mathrm{d}s$$
$$= \int_{\frac{a+b}{2}}^{a+b-t} \left[f'(s)-f'(a+b-s)\right]\mathrm{d}s.$$

We also have

$$f'(s)-f'(a+b-s) = \int_{a+b-s}^{s} f''(y)\mathrm{d}y. \tag{2.17}$$

By using the condition $m \le f''(y) \le M$ for all $y \in [a,b]$ and with the help of the equality (2.17), we have

$$\int_{a+b-s}^{s} m\mathrm{d}y \le \int_{a+b-s}^{s} f''(y)\mathrm{d}y \le \int_{a+b-s}^{s} M\mathrm{d}y,$$

which gives

$$m(2s-a-b) \le f'(s)-f'(a+b-s) \le M(2s-a-b).$$

Since the equality (2.16), we have

$$m\int_{\frac{a+b}{2}}^{a+b-t}(2s-a-b) \le \int_{\frac{a+b}{2}}^{a+b-t}\left[f'(s)-f'(a+b-s)\right]\mathrm{d}s \le M\int_{\frac{a+b}{2}}^{a+b-t}(2s-a-b).$$

This implies that

$$m\left(\frac{a+b}{2}-t\right)^2 \le f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right) \le M\left(\frac{a+b}{2}-t\right)^2. \tag{2.18}$$

Multiplying the inequality (2.18) by $\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}(t-a)^{\alpha-1}$ and then integrating with respect to t on the interval $\left[a,\frac{a+b}{2}\right]$, we obtain

$$m\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_{a}^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}(t-a)^{\alpha-1}\left(\frac{a+b}{2}-t\right)^2\mathrm{d}t$$
$$\le \frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_{a}^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{\left(f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)\right)}{(t-a)^{1-\alpha}}\mathrm{d}t$$
$$\le M\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_{a}^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}(t-a)^{\alpha-1}\left(\frac{a+b}{2}-t\right)^2\mathrm{d}t.$$

Then, we have

$$
\begin{aligned}
& m\frac{\beta(b-a)^2}{8}\left[\frac{1}{\beta}-2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \\
& \leq \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a^+}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right) \\
& \leq M\frac{\beta(b-a)^2}{8}\left[\frac{1}{\beta}-2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].
\end{aligned}
$$

Finally, the proof of Theorem 10 is completed. □

Example 3. If the mapping $f\colon\ [a,b]=[-1,1]\to\mathbb{R}$ is defined as $f(t)=t^3+2t^2+1$ such that $-2\leq f''(t)\leq 10$ for $t\in[-1,1]$. Under these assumptions, the mid-term of inequality (2.14) reduces to

$$
\begin{aligned}
& \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a^+}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right) \\
& =\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(2)^{\alpha\beta}}\left[{}_{-1^+}^{\beta}\Upsilon^{\alpha}f(0)+{}^{\beta}\Upsilon_{-1^-}^{\alpha}f(0)\right]-f(0) \\
& =\frac{\Gamma(\beta+1)\alpha^{\beta}}{2}\left[\frac{1}{\Gamma(\beta)}\int_{-1}^{0}\left(\frac{1-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{t^3+2t^2+1}{(t+1)^{1-\alpha}}\mathrm{d}t\right. \\
& \qquad\qquad \left.+\frac{1}{\Gamma(\beta)}\int_{0}^{1}\left(\frac{1-(1-t)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{t^3+2t^2+1}{(1-t)^{1-\alpha}}\mathrm{d}t\right] \\
& =2\beta\alpha^{\beta}\int_{-1}^{0}\left(\frac{1-(t+1)^{\alpha}}{\alpha}\right)^{\beta-1}\frac{t^2}{(t+1)^{1-\alpha}}\mathrm{d}t \\
& =2\beta\left[\frac{1}{\beta}-2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].
\end{aligned}
$$

Consequently, the inequality (2.14) can be written as follows

$$
\begin{aligned}
& -\beta\left[\frac{1}{\beta}-2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \\
& \leq 2\beta\left[\frac{1}{\beta}-2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \\
& \leq 5\beta\left[\frac{1}{\beta}-2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)+\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].
\end{aligned}
$$

Theorem 11. *Let $f\colon [a,b] \to \mathbb{R}$ be a twice-differentiable mapping such that there exists real constants m and M so that $m \leq f'' \leq M$. Then, we establish*

$$\begin{aligned} & m\frac{\beta(b-a)^2}{8}\left[2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \\ & \leq \frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right] \\ & \leq M\frac{\beta(b-a)^2}{8}\left[2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right], \end{aligned} \tag{2.19}$$

where $\mathcal{B}$ is Euler Beta function.

Proof. With the help of the inequality (2.13), we obtain

$$\begin{aligned} & \frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right] \\ & = \frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\ & \quad \times \frac{(f(t)+f(a+b-t))}{(t-a)^{1-\alpha}}\mathrm{d}t \\ & = \frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\ & \quad \times \frac{(f(a)+f(b)-f(t)-f(a+b-t))}{(t-a)^{1-\alpha}}\mathrm{d}t. \end{aligned} \tag{2.20}$$

With the aid of the equalities

$$f(a)-f(t)=-\int_a^t f'(s)\mathrm{d}s$$

and

$$f(b)-f(a+b-t)=\int_{a+b-t}^b f'(s)\mathrm{d}s,$$

we have

$$\begin{aligned} f(a)+f(b)-(f(t)+f(a+b-t)) &= \int_{a+b-t}^b f'(s)\mathrm{d}s-\int_a^t f'(s)\mathrm{d}s \\ &= \int_a^t f'(a+b-s)\mathrm{d}s-\int_a^t f'(s)\mathrm{d}s \\ &= \int_a^t \left[f'(a+b-s)-f'(s)\right]\mathrm{d}s. \end{aligned} \tag{2.21}$$

We also obtain

$$f'(a+b-s)-f'(s)=\int_s^{a+b-s} f''(y)\mathrm{d}y. \tag{2.22}$$

By the equality (2.22) and the condition $m \leq f'' \leq M$, we get

$$m(a+b-2s) \leq f'(s)-f'(a+b-s) \leq M(a+b-2s). \tag{2.23}$$

From the equality (2.21) and the inequality (2.23), we acquire

$$\int_a^t m(a+b-2s)\mathrm{d}s \leq \int_a^t \left[f'(s)-f'(a+b-s)\right]\mathrm{d}s \leq \int_a^t M(a+b-2s)\mathrm{d}s$$

so that

$$m(b-t)(t-a) \leq f(a)+f(b)-(f(t)+f(a+b-t)) \leq M(b-t)(t-a). \tag{2.24}$$

Multiplying the inequality (2.24) by $\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}(t-a)^{\alpha-1}$ and then integrating with respect to t on the interval $\left[a,\frac{a+b}{2}\right]$, we get

$$\begin{aligned}
& m\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}(t-a)^{\alpha}(b-t)\mathrm{d}t \\
\leq & \frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1} \\
& \times\frac{(f(a)+f(b)-f(t)-f(a+b-t))\mathrm{d}t}{(t-a)^{1-\alpha}} \\
\leq & M\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}(t-a)^{\alpha}(b-t)\mathrm{d}t.
\end{aligned}$$

So that

$$\begin{aligned}
& m\frac{\beta(b-a)^2}{8}\left[2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right] \\
\leq & \frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}_{a^+}^{\beta}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon_{b^-}^{\alpha}f\left(\frac{a+b}{2}\right)\right] \\
\leq & M\frac{\beta(b-a)^2}{8}\left[2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].
\end{aligned}$$

Hence, the proof of Theorem 11 is completed. □

Example 4. Let us consider that the function $f\colon [a,b]=[-2,3]\to\mathbb{R}$ is described as $f(t)=2t^3-t^2+t-1$ such that $-24\le f''(t)\le 36$ for $t\in[-2,3]$. By these assumptions, the right-side of (2.19) becomes as follows

$$\begin{aligned}&\frac{f(a)+f(b)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]\\&=\frac{f(-2)+f(3)}{2}-\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(5)^{\alpha\beta}}\left[{}^{\beta}_{-2^+}\Upsilon^{\alpha}f\left(\frac{1}{2}\right)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f\left(\frac{1}{2}\right)\right]\\&=\frac{25\beta}{2}\left[2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].\end{aligned}$$

Finally, the inequality (2.19) can be written as follows

$$\begin{aligned}&-75\beta\left[2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right]\\&\le\frac{25\beta}{2}\left[2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right]\\&\le\frac{225\beta}{2}\left[2\mathcal{B}\left(\frac{1}{\alpha}+1,\beta\right)-\mathcal{B}\left(\frac{2}{\alpha}+1,\beta\right)\right].\end{aligned}$$

Theorem 12. *Let $f\colon [a,b]\to\mathbb{R}$ be a differentiable mapping and $f\in L_1[a,b]$. If $f'(a+b-x)\ge f'(x)$ for all $x\in\left[a,\frac{a+b}{2}\right]$. Then, the following Hemite–Hadamard-type inequality for conformable fractional integrals holds*

$$\begin{aligned}f\left(\frac{a+b}{2}\right)&\le\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]\\&\le\frac{f(a)+f(b)}{2}.\end{aligned}$$

Proof. Since the equalities (2.15) and (2.16), we can write

$$\begin{aligned}&\frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}}\left[{}^{\beta}_{a^+}\Upsilon^{\alpha}f\left(\frac{a+b}{2}\right)+{}^{\beta}\Upsilon^{\alpha}_{b^-}f\left(\frac{a+b}{2}\right)\right]-f\left(\frac{a+b}{2}\right)\\&=\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}(t-a)^{\alpha-1}\\&\quad\times\left(f(t)+f(a+b-t)-2f\left(\frac{a+b}{2}\right)\right)\mathrm{d}t\\&=\frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}}\int_a^{\frac{a+b}{2}}\left(\frac{\left(\frac{b-a}{2}\right)^{\alpha}-(t-a)^{\alpha}}{\alpha}\right)^{\beta-1}(t-a)^{\alpha-1}\end{aligned}$$

$$\times \left[\int_{\frac{a+b}{2}}^{a+b-t} \left[f'(s) - f'(a+b-s) \right] ds \right] \mathrm{d}t$$

$$= \frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}} \int_{a}^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1}$$

$$\times \left[\int_{t}^{\frac{a+b}{2}} \left[f'(a+b-u) - f'(u) \right] ds \right] \mathrm{d}t$$

$$\geq 0,$$

which presents the first inequality in (2.13).

Likewise, by the equalities (2.20) and (2.21), we establish

$$\frac{f(a)+f(b)}{2} - \frac{2^{\alpha\beta-1}\Gamma(\beta+1)\alpha^{\beta}}{(b-a)^{\alpha\beta}} \left[{}^{\beta}_{\left(\frac{a+b}{2}\right)^{+}}\Upsilon^{\alpha} f(b) + {}^{\beta}\Upsilon^{\alpha}_{\left(\frac{a+b}{2}\right)^{-}} f(a) \right]$$

$$= \frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}} \int_{a}^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1}$$

$$\times \left[f(a) + f(b) - (f(t) + f(a+b-t)) \right] \mathrm{d}t$$

$$= \frac{2^{\alpha\beta-1}\beta\alpha^{\beta}}{(b-a)^{\alpha\beta}} \int_{a}^{\frac{a+b}{2}} \left(\frac{\left(\frac{b-a}{2}\right)^{\alpha} - (t-a)^{\alpha}}{\alpha} \right)^{\beta-1} (t-a)^{\alpha-1}$$

$$\times \left[\int_{a}^{t} \left[f'(a+b-s) - f'(s) \right] ds \right] \mathrm{d}t \geq 0.$$

This ends the proof of Theorem 12. □

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