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GENERAL $((k,p),\psi)$-HILFER FRACTIONAL INTEGRALS

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Abstract. The main motivation of this study is to establish a general version of the Riemann-Liouville fractional integrals with two exponential parameters k and p called $((k,p),\psi)$-Hilfer fractional integrals which is determined over the k-gamma function. We first prove that these operators are well-defined, continuous and have semi-group property. Then, particularly, we present the harmonic, geometric and arithmetic $(k,p),\psi$-Hilfer fractional integrals. Moreover, some special cases relating to general $((k,p),\psi)$-Riemann-Liouville fraction integrals are given.

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1. INTRODUCTION

The fractional calculus theory has been used as a mathematical tool in a variety of pure and practical fields. This approach has been used in different scientific fields. In applied mathematics, various fractional operators have been used to show a set of integral inequalities and their generalizations. One among the vital applications of fractional integrals is the k-Riemann-Liouville fractional integral operator which is an important tool and a source of many research works in field science such as the theory of inequalities, differential equations, integral inequalities. see for example [2], [9], [10], [11], [12]. The right and left-sided k-Riemann-Liouville fractional integrals of order $\alpha > 0$, for a continuous function f on $[a,b]$ are defined as

$$\mathrm{J}^{\alpha}_{a^+,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad a < x \le b, \tag{1.1}$$

$$\mathrm{J}^{\alpha}_{b^-,k}f(x) = \frac{1}{k\Gamma_k(\alpha)}\int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad a \le x < b, \tag{1.2}$$

where the k-gamma function[3] verified

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \tag{1.3}$$

$$\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha)$$

for all $\alpha, k > 0$.

Some basic equations satisfied by the k-gamma function are given in [3].

Property 1. *For all $\alpha, k > 0$ and $n \in \mathbb{N}$, the fundamental formulae satisfied by k-gamma function are*

$$\Gamma_k(\alpha+nk) = k^n \left(\frac{\alpha}{k}\right)\left(\frac{\alpha}{k}+1\right)\ldots\left(\frac{\alpha}{k}+(n-1)\right)\Gamma_k(\alpha), \tag{1.4}$$

$$\Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha), \tag{1.5}$$

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma(\frac{\alpha}{k}). \tag{1.6}$$

Remark 1. The above definition and properties reduce to the gamma function and its properties when $k \to 1$.

The k-beta function satisfies the following identities.

$$\beta_k(x,y) = \frac{1}{k}\int_0^1 t^{\frac{x}{k}-1}(1-t)^{\frac{y}{k}-1}\,dt. \tag{1.7}$$

$$\beta_k(x,y) = \frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}, \quad x > 0, \quad y > 0. \tag{1.8}$$

In 2017, Kuldeep [6] defined the two-parameter gamma function called (k,p) gamma function which is a generalization of k-gamma.

Definition 1. Given $\alpha \in \mathbb{C}/k\mathbb{Z}^-; k,p \in \mathbb{R}^+ - \{0\}$ and $Re(\alpha) > 0$, then the integral representation of (k,p)-Gamma Function is given by

$$\Gamma_{(k,p)}(\alpha) = \int_0^\infty t^{\alpha-1}e^{-\frac{t^k}{p}}\,dt. \tag{1.9}$$

We present certain base formulas related to the (k,p)-gamma function are mentioned in [5], [6].

Property 2. *For all $\alpha, k, p > 0$ and $n \in \mathbb{N}$, the fundamental formulae satisfied by (k,p)-gamma function are ,*

$$\Gamma_{(k,p)}(\alpha+nk) = p^n \left(\frac{\alpha}{k}\right)\left(\frac{\alpha}{k}+1\right)\cdots\left(\frac{\alpha}{k}+(n-1)\right)\Gamma_{(k,p)}(\alpha), \tag{1.10}$$

$$\Gamma_{(k,p)}(\alpha+k) = \frac{p\alpha}{k}\Gamma_{(k,p)}(\alpha), \tag{1.11}$$

$$\Gamma_{(k,p)}(\alpha) = \left(\frac{p}{k}\right)^{\frac{\alpha}{k}}\Gamma_k(\alpha) = \left(\frac{p^{\frac{\alpha}{k}}}{k}\right)\Gamma\left(\frac{\alpha}{k}\right). \tag{1.12}$$

We deduce that

$$\Gamma_{(k,p)}(1) = \left(\frac{p^{\frac{1}{k}}}{k}\right)\Gamma\left(\frac{1}{k}\right), \quad \Gamma_{(k,p)}(k) = \left(\frac{p}{k}\right), \tag{1.13}$$

$$\Gamma_{(k,p)}(p)=\left(\frac{p^{\frac{p}{k}}}{k}\right)\Gamma\left(\frac{p}{k}\right).$$

Remark 2. The above definition and properties reduce to the k-gamma function and its properties when $p=k$.

By using the formula (1.12), we get

$$\beta_k(x,y)=\frac{\Gamma_{(k,p)}(x)\Gamma_{(k,p)}(y)}{\Gamma_{(k,p)}(x+y)}=\frac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)},\quad x>0,\quad y>0. \tag{1.14}$$

2. $((k,p),\psi)$-HILFER FRACTIONAL INTEGRALS

In this section, we present the general $((k,p),\psi)$-Hilfer fractional integrals of order α with two exponential parameters k and p which generalize the k-Riemann-Liouville fractional integrals.

Let $\varphi:(0,+\infty)\times(0,+\infty)\longrightarrow(0,+\infty)$ be a map satisfying the condition $\varphi(k,k)=k$. For example

(1) the arithmetic mean $\varphi_1(k,p)=\frac{k+p}{2}$,

(2) the geometric mean $\varphi_2(k,p)=\sqrt{kp}$,

(3) $\varphi_3(k,p)=\frac{k^2}{p}$, called the H-case.

Now we will give definitions of generalized fractional integrals.

Definition 2. (See [1], [7], [13]) Let $\psi(x)$ be an increasing positive monotone function on $[a,b]$ such that $\psi'(x)$ is continuous on (a,b). The space $X^r_\psi(a,b)$ $(1\leq r<+\infty)$ is defined as the set of those real-valued Lebesgue measurable functions f on $[a,b]$ for which

$$\|f\|_{X^r_\psi}=\left(\int_a^b|f(x)|^r\psi'(x)dx\right)^{\frac{1}{r}}<\infty. \tag{2.1}$$

In particular, when $\psi(x)=x$ the space $X^r_\psi(a,b)$ coincides with the $L_r[a,b]$.

Definition 3. Let $[a,b]\subseteq[0,+\infty]$, $f\in X^1_\psi(a,b)$ and $k,p>0$ and for an strictly increasing function ψ, where $\psi\in\mathcal{C}^1[a,b]$ such that $\psi'(t)\neq0$, for all $t\in[a,b]$. The left and right-sided general (k,p)-Hilfer fractional integrals of a function f with respect to the function ψ on $[a,b]$ are defined, respectively, as follows

$${}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)=\frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_a^x\psi'(t)(\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1}f(t)dt,\quad a<x\leq b, \tag{2.2}$$

$$ {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)=\frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)}\int_x^b\psi'(t)(\psi(t)-\psi(x))^{\frac{\alpha}{\varphi(k,p)}-1}f(t)dt,\quad a\le x<b. \tag{2.3} $$

Here, $\Gamma_{(k,p)}$ is the (k,p)-Gamma Function defined by as in (1.9).

Remark 3. Set $p=k$ in the above definition, we obtain the (k,ψ)-Hilfer fractional integrals [8].

In the following theorem, we show that the $((k,p),\psi)$-Hilfer fractional integrals are well-defined.

Theorem 1. *The fractional integrals (2.2), (2.3) are defined for functions $f\in L_1[a,b]$, existing almost everywhere and*

$$ {}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x),\ {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)\in X^1_\psi(a,b). \tag{2.4} $$

Moreover

$$ \left\|\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}f(x)\right\|_{X^1_\psi(a,b)}\le C\left\|f(t)\right\|_{X^1_\psi(a,b)}, \tag{2.5} $$

where

$$ C=\max\left(\frac{(\psi(b)-\psi(a))^{\frac{\alpha_1}{\varphi(k,p)}}}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)},\frac{(\psi(b)-\psi(a))^{\frac{\alpha_2}{\varphi(k,p)}}}{\Gamma_{(k,p)}\left(k\left(\frac{\alpha_2}{\varphi(k,p)}+1\right)\right)}\right),\quad 0<\alpha_2<\varphi(k,p)<\alpha_1. $$

Proof. Let $f(x)\in L_1[a,b]$.

- Let $\frac{\alpha}{\varphi(k,p)}=1$, is clearly.
- Let $\frac{\alpha}{\varphi(k,p)}>1$. Let $\Omega=[a,b]\times[a,b]$, we pose for all $(x,t)\in\Omega$, posing

$$ G_1(x,t)=\begin{cases}(\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1}, & a\le t\le x,\\ 0, & x\le t\le b,\end{cases} $$

and

$$ G_2(x,t)=\begin{cases}0, & a\le t\le x,\\ (\psi(t)-\psi(x))^{\frac{\alpha}{\varphi(k,p)}-1}, & x\le t\le b.\end{cases} $$

Since ψ is continuous, strictly increasing function, for $i=1,2$ we have

$$ \int_a^b G_i(x,t)\psi'(x)dx\le\int_a^b\psi'(x)(\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}-1}dx=(\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}}. $$

By applying Fubini's Theorem, we get

$$ \begin{aligned}\int_a^b\left(\int_a^b G_i(x,t)\left|f(t)\right|\psi'(t)dt\right)\psi'(x)dx&=\int_a^b\left(\int_a^b G_i(x,t)\psi'(x)dx\right)\left|f(t)\right|\psi'(t)dt\\ &\le(\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}}\int_a^b\left|f(t)\right|\psi'(t)dtdt\\ &=(\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}}\left\|f(t)\right\|_{X^1_\psi(a,b)}<\infty.\end{aligned} $$

Therefore

$$\int_a^b G_i(x,t)\,|f(t)|\,\psi'(t)dt \in X^1_\psi(a,b).$$

This gives us (2.4) for $i=1,2$.

- Let $0<\frac{\alpha}{\varphi(k,p)}<1$, put $\eta = 1-\frac{\alpha}{\varphi(k,p)}$ hence $0<\eta<1$. By using Fubini's Theorem, we get

$$\begin{aligned}
&\int_a^b \left|{}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x)\right| \psi'(x)dx \\
&= \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^b \left|\int_a^x (\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1} f(t)\psi'(t)dt\right| \psi'(x)dx \\
&\leq \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^b \int_a^x \frac{|f(t)|}{(\psi(x)-\psi(t))^{\eta}} \psi'(t)\psi'(x)dtdx \\
&= \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^b |f(t)| \left(\int_t^b \frac{\psi'(x)}{(\psi(x)-\psi(t))^{\eta}} dx\right) \psi'(t)dt \\
&= \frac{1}{(1-\eta)\,k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^b |f(t)|\,(\psi(b)-\psi(t))^{1-\eta}\psi'(t)dt \\
&\leq \frac{(\psi(b)-\psi(a))^{1-\eta}}{(1-\eta)\,k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^b |f(t)|\,\psi'(t)dt.
\end{aligned}$$

Hence, we obtain

$$\int_a^b \left|{}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x)\right| \psi'(x)dx \leq \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}}}{\Gamma_{(k,p)}\left(k(\frac{\alpha}{\varphi(k,p)}+1)\right)} \|f(t)\|_{X^1_\psi(a,b)} < +\infty.$$

By the similar process, one can obtain

$$\int_a^b \left|{}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x)\right| \psi'(x)dx \leq \frac{(\psi(b)-\psi(a))^{\frac{\alpha}{\varphi(k,p)}}}{\Gamma_{(k,p)}\left(k(\frac{\alpha}{\varphi(k,p)}+1)\right)} \|f(t)\|_{X^1_\psi(a,b)} < +\infty.$$

This gives us (2.4).

□

Theorem 2. *Let* $\frac{\alpha}{\varphi(k,p)} > 1$ *and* $f \in L_1[a,b]$*, then the fractional integrals* (2.2)*,* (2.3) *are*

$$ {}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x),\ {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x) \in C[a,b]. \tag{2.6}$$

Proof. Let $x,y \in [a,b]$, $y \leq x$ and $y \to x$. Since ψ is strictly increasing function and $\psi \in \mathcal{C}^1[a,b]$, we get

$$\begin{aligned}
&k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)\left| {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x) - {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(y) \right| \\
&\quad = \left| \left(\int_x^b \left[(\psi(t)-\psi(x))^{\frac{\alpha}{\varphi(k,p)}-1} - (\psi(t)-\psi(y))^{\frac{\alpha}{\varphi(k,p)}-1} \right] f(t)\psi'(t)dt \right) \right. \\
&\qquad \left. - \int_y^x (\psi(t)-\psi(y))^{\frac{\alpha}{\varphi(k,p)}-1} f(t)\psi'(t)dt \right| \\
&\quad \leq \int_x^b \left| (\psi(t)-\psi(x))^{\frac{\alpha}{\varphi(k,p)}-1} - (\psi(t)-\psi(y))^{\frac{\alpha}{\varphi(k,p)}-1} \right| |f(t)|\,\psi'(t)dt \\
&\qquad + (\psi(x)-\psi(y))^{\frac{\alpha}{\varphi(k,p)}-1} \|f(t)\|_{X^1_\psi(a,b)} .
\end{aligned}$$

Thus

$$\left| {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x) - {}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(y) \right| \to 0 \quad \text{as } \ y \to x.$$

By similar process, one can prove

$$\left| {}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x) - {}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(y) \right| \to 0 \quad \text{as } \ y \to x,$$

which completes the proof. □

Now, we give the commutativity and the semigroup properties of the $((k,p),\psi)$-Hilfer fractional integrals.

Theorem 3. *Let $\alpha,\beta > 0$. Then we have the following equalities for $((k,p),\psi)$-Hilfer fractional integrals.*

$${}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}\left({}_{a^+}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)} f(x)\right) = {}_{a^+}\mathrm{I}^{\alpha+\beta,\psi}_{\varphi(k,p)} f(x) = {}_{a^+}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)}\left({}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x)\right). \tag{2.7}$$

$${}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}\left({}_{b^-}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)} f(x)\right) = {}_{b^-}\mathrm{I}^{\alpha+\beta,\psi}_{\varphi(k,p)} f(x) = {}_{b^-}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)}\left({}_{b^-}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)} f(x)\right). \tag{2.8}$$

Equalities (2.7) and (2.8) are satisfied in any point for $f(t) \in \mathcal{C}([a,b])$ and in almost every point for $f(t) \in L_1[a,b]$.

Proof. Using Fubini's Theorem, we get

$$\begin{aligned}
&\left[k^2\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)\Gamma_k\left(\frac{k\beta}{\varphi(k,p)}\right)\right] {}_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}\left({}_{a^+}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)} f(x)\right) \\
&\quad = \int_a^x (\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1} \left(\int_a^t (\psi(t)-\psi(s))^{\frac{\beta}{\varphi(k,p)}-1} f(s)\psi'(s)ds \right) \psi'(t)dt \\
&\quad = \int_a^x f(s) \left(\int_s^x (\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1} (\psi(t)-\psi(s))^{\frac{\beta}{\varphi(k,p)}-1} \psi'(t)dt \right) \psi'(s)ds.
\end{aligned} \tag{2.9}$$

If we use the change of variable $y = \frac{\psi(t)-\psi(s)}{\psi(x)-\psi(s)}$ in the inner integral in (2.9), we obtain

$$\begin{aligned}\int_s^x (\psi(x)-\psi(t))^{\frac{\alpha}{\varphi(k,p)}-1}(\psi(t)-\psi(s))^{\frac{\beta}{\varphi(k,p)}-1}dt \\ = (\psi(x)-\psi(s))^{\frac{\alpha+\beta}{\varphi(k,p)}-1}\int_0^1 (1-y)^{\frac{\alpha}{\varphi(k,p)}-1}(y)^{\frac{\beta}{\varphi(k,p)}-1}dy \\ = k\,(\psi(x)-\psi(s))^{\frac{\alpha+\beta}{\varphi(k,p)}-1}\beta_k\left(\frac{k\alpha}{\varphi(k,p)},\frac{k\beta}{\varphi(k,p)}\right).\end{aligned} \tag{2.10}$$

By using k-beta propriety (1.14) and (2.9) in (2.10), we deduce that

$$_{a^+}\mathrm{I}^{\alpha,\psi}_{\varphi(k,p)}\left(_{a^+}\mathrm{I}^{\beta,\psi}_{\varphi(k,p)}f(x)\right) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k(\alpha+\beta)}{\varphi(k,p)}\right)}\int_a^x(\psi(x)-\psi(s))^{\frac{\alpha+\beta}{\varphi(k,p)}-1}f(s)\psi'(s)ds.$$

This gives the equality (2.7). By similar way, the equality (2.8) can be proved easily. □

3. Special cases of general $((k,p),\psi)$-Hilfer fractional integrals

In this section, we present three interesting cases of the general integrals $(k,p),\psi)$-Hilfer depending on the choice of the function ψ.

3.1. *(k,p)-Riemann-Liouville fractional operators*

By setting $\psi(t)=t$, general fractional operators $(k,p),\psi)$-Hilfer reduce to fractional operators (k,p)-Riemann-Liouville as defined below.

Definition 4. Let $[a,b]\subseteq[0,+\infty]$, where $a<b$, $f\in X_t^1(a,b)$, for all $t\in[a,b]$ and $k,p>0$. The left and right-sided (k,p)- Riemann-Liouville fractional integrals of a function f are defined, respectively, as follows

$$_{a^+}\mathrm{I}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}(\frac{k\alpha}{\varphi(k,p)})}\int_a^x (x-t)^{\frac{\alpha}{\varphi(k,p)}-1}f(t)dt, \quad a<x\le b, \tag{3.1}$$

$$_{b^-}\mathrm{I}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}(\frac{k\alpha}{\varphi(k,p)})}\int_x^b (t-x)^{\frac{\alpha}{\varphi(k,p)}-1}f(t)dt, \quad a\le x<b, \tag{3.2}$$

where $\Gamma_{(k,p)}$ is the (k,p)-Gamma Function defined by (1.9).

We present some specially cases to (k,p)-Riemann-Liouville fractional operators

3.1.1. *H-(k,p)-Riemann-Liouville fractional integrals*

By choosing $\varphi(k,p)=\frac{k^2}{p}$ in Definition 4, we have:

Definition 5. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$ and $k,p>0$. The right and the left-sided harmonic (k,p)-Riemann-Liouville fractional integrals of order $\alpha>0$ are defined as

$$ {}_{a^+}\mathrm{H}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_a^x (x-t)^{\frac{p\alpha}{k^2}-1} f(t)dt, \quad a<x\leq b. \tag{3.3} $$

$$ {}_{b^-}\mathrm{H}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_x^b (t-x)^{\frac{p\alpha}{k^2}-1} f(t)dt, \quad a\leq x< b. \tag{3.4} $$

3.1.2. *Geometric (k,p)-Riemann-Liouville fractional integrals*

By taking $\varphi(k,p)=\sqrt{kp}$ in Definition 4, we have:

Definition 6. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$ and $k,p>0$. The right and the left-sided geometric (k,p)-Riemann-Liouville fractional integrals of order $\alpha>0$ are defined as

$$ {}_{a^+}\mathrm{G}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_a^x (x-t)^{\frac{\alpha}{\sqrt{kp}}-1} f(t)dt, \quad a<x\leq b. \tag{3.5} $$

$$ {}_{b^-}\mathrm{G}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_x^b (t-x)^{\frac{\alpha}{\sqrt{kp}}-1} f(t)dt, \quad a\leq x< b. \tag{3.6} $$

3.1.3. *Arithmetic (k,p)-Riemann-Liouville fractional integrals*

By putting $\varphi(k,p)=\frac{k+p}{2}$ in Definition 4, we have:

Definition 7. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$ and $k,p>0$. The right and the left-sided arithmetic (k,p)-Riemann-Liouville fractional integrals of order $\alpha>0$ are defined as

$$ {}_{a^+}\mathrm{A}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_a^x (x-t)^{\frac{2\alpha}{k+p}-1} f(t)dt, \quad a<x\leq b. \tag{3.7} $$

$$ {}_{b^-}\mathrm{A}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_x^b (t-x)^{\frac{2\alpha}{k+p}-1} f(t)dt, \quad a\leq x< b. \tag{3.8} $$

Remark 4. Choosing $p=k$ in Definitions 5, 6 and 7 give us the k-Riemann-Liouville fraction operators given in [9].

3.2. *(k,p)-Hadamard fractional operators*

Taking $\psi(t)=\ln(t)$ on the definition of general fractional operators $((k,p),\psi)$-Hilfer, we get (k,p)-Hadamard fractional operators defined out below.

Definition 8. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$, for all $t\in[a,b]$ and $k,p>0$. The left and right-sided (k,p)- Hadamard fractional integrals are defined, respectively, as follows

$$ {}_{a^+}\mathrm{I}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{\varphi(k,p)}-1} \frac{f(t)}{t}dt, \quad a < x \leq b, \tag{3.9} $$

$$ {}_{b^-}\mathrm{I}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_x^b \left(\ln\frac{t}{x}\right)^{\frac{\alpha}{\varphi(k,p)}-1} \frac{f(t)}{t}dt, \quad a \leq x < b, \tag{3.10} $$

where $\Gamma_{(k,p)}$ is the (k,p)-Gamma Function defined by (1.9).

Now, we give some particular cases to (k,p)-Hadamard fractional operators.

3.2.1. *H-(k,p)-Hadamard fractional integrals*

By choosing $\varphi(k,p) = \frac{k^2}{p}$ in Definition 8, we get:

Definition 9. Let $[a,b] \subseteq [0,+\infty]$, where $a < b$, $f \in X_t^1(a,b)$ and $k,p > 0$. The right and the left-sided harmonic (k,p) -Hadamard fractional integrals of order $\alpha > 0$ are defined as

$$ {}_{a^+}\mathrm{H}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_a^x \left(\ln\frac{x}{t}\right)^{\frac{p\alpha}{k^2}-1} \frac{f(t)}{t}dt, \quad a < x \leq b. \tag{3.11} $$

$$ {}_{b^-}\mathrm{H}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_x^b \left(\ln\frac{t}{x}\right)^{\frac{p\alpha}{k^2}-1} \frac{f(t)}{t}dt, \quad a \leq x < b. \tag{3.12} $$

3.2.2. *Geometric (k,p)-Hadamard fractional integrals*

By taking $\varphi(k,p) = \sqrt{kp}$ in Definition 8, we get:

Definition 10. Let $[a,b] \subseteq [0,+\infty]$, where $a < b$, $f \in X_t^1(a,b)$ and $k,p > 0$. The right and the left-sided geometric (k,p) -Hadamard fractional integrals of order $\alpha > 0$ are defined as

$$ {}_{a^+}\mathrm{G}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_a^x \left(\ln\frac{x}{t}\right)^{\frac{\alpha}{\sqrt{kp}}-1} \frac{f(t)}{t}dt, \quad a < x \leq b. \tag{3.13} $$

$$ {}_{b^-}\mathrm{G}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_x^b \left(\ln\frac{t}{x}\right)^{\frac{\alpha}{\sqrt{kp}}-1} \frac{f(t)}{t}dt, \quad a \leq x < b. \tag{3.14} $$

3.2.3. *Arithmetic (k,p)-Hadamard fractional integrals*

By putting $\varphi(k,p) = \frac{k+p}{2}$ in Definition 8, we get:

Definition 11. Let $[a,b] \subseteq [0,+\infty]$, where $a < b$, $f \in X_t^1(a,b)$ and $k,p > 0$. The right and the left-sided arithmetic (k,p) -Hadamard fractional integrals of order $\alpha > 0$ are defined as

$$ {}_{a^+}\mathrm{A}^{\alpha}_{\varphi(k,p)}f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_a^x \left(\ln\frac{x}{t}\right)^{\frac{2\alpha}{k+p}-1} \frac{f(t)}{t}dt, \quad a < x \leq b. \tag{3.15} $$

$$ {}_{b^-}\mathrm{A}^{\alpha}_{\varphi(k,p)} f(x) = \frac{1}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_x^b \left(\ln \frac{t}{x}\right)^{\frac{2\alpha}{k+p}-1} \frac{f(t)}{t} dt, \quad a \le x < b. \tag{3.16} $$

Remark 5. Choosing $p = k$ in Definitions 9, 10 and 11 give us the k-Hadamard fractional operators defined in [4].

3.3. *(k,p)-Katugompola fractional operators*

Putting $\psi(t) = \frac{t^{\rho+1}}{\rho+1}$ where $\rho > -1$, then the general $((k,p),\psi)$-Hilfer fractional operators reduce to the (k,p)-Katugompola operators define as follows.

Definition 12. Let $[a,b] \subseteq [0,+\infty]$, where $a < b$, $f \in X_t^1(a,b)$, for all $t \in [a,b]$, $\rho > -1$ and $k, p > 0$. The left and right-sided (k,p)- Katugompola fractional integrals of a function f are defined, respectively, as follows

$$ {}_{a^+}\mathrm{I}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\varphi(k,p)}}}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\frac{\alpha}{\varphi(k,p)}-1} t^{\rho} f(t) dt, \quad a < x \le b, \tag{3.17} $$

$$ {}_{b^-}\mathrm{I}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\varphi(k,p)}}}{k\Gamma_{(k,p)}\left(\frac{k\alpha}{\varphi(k,p)}\right)} \int_x^b (t^{\rho+1} - x^{\rho+1})^{\frac{\alpha}{\varphi(k,p)}-1} t^{\rho} f(t) dt, \quad a \le x < b, \tag{3.18} $$

where $\Gamma_{(k,p)}$ is the (k,p)-Gamma Function defined by (1.9).

The specially cases to (k,p)-Katugompola fractional operators are as follows.

3.3.1. *H-(k,p)-Katugompola fractional integrals*

By choosing $\varphi(k,p) = \frac{k^2}{p}$ in Definition 12, we have:

Definition 13. Let $[a,b] \subseteq [0,+\infty]$, where $a < b$, $f \in X_t^1(a,b)$, $\rho > -1$ and $k, p > 0$. The right and the left-sided harmonic (k,p)-Katugompola fractional integrals of order $\alpha > 0$ are defined as

$$ {}_{a^+}\mathrm{H}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{\frac{p\alpha}{k^2}-1}}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\frac{p\alpha}{k^2}-1} t^{\rho} f(t) dt, \quad a < x \le b. \tag{3.19} $$

$$ {}_{b^-}\mathrm{H}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{\frac{p\alpha}{k^2}-1}}{k\Gamma_{(k,p)}\left(\frac{p}{k}\alpha\right)} \int_x^b (t^{\rho+1} - x^{\rho+1})^{\frac{p\alpha}{k^2}-1} t^{\rho} f(t) dt, \quad a \le x < b. \tag{3.20} $$

3.3.2. *Geometric (k,p)-Katugompola fractional integrals*

By taking $\varphi(k,p) = \sqrt{kp}$ in Definition 12, we have:

Definition 14. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$, $\rho > -1$ and $k,p > 0$. The right and the left-sided geometric (k,p)-Katugompola fractional integrals of order $\alpha > 0$ are defined as

$$ {}_{a^+}\mathrm{G}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\sqrt{kp}}}}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\frac{\alpha}{\sqrt{kp}}-1} t^{\rho} f(t)dt, \quad a<x\leq b. \tag{3.21} $$

$$ {}_{b^-}\mathrm{G}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{1-\frac{\alpha}{\sqrt{kp}}}}{k\Gamma_{(k,p)}\left(\sqrt{\frac{k}{p}}\alpha\right)} \int_x^b (t^{\rho+1} - x^{\rho+1})^{\frac{\alpha}{\sqrt{kp}}-1} t^{\rho} f(t)dt, \quad a\leq x< b. \tag{3.22} $$

3.3.3. *Arithmetic (k,p)-Katugompola fractional integrals*

By putting $\varphi(k,p) = \frac{k+p}{2}$ in Definition 12, we have:

Definition 15. Let $[a,b] \subseteq [0,+\infty]$, where $a<b$, $f \in X_t^1(a,b)$, $\rho > -1$ and $k,p>0$. The right and the left-sided arithmetic (k,p)-Katugompola fractional integrals of order $\alpha > 0$ are defined as

$$ {}_{a^+}\mathrm{A}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{1-\frac{2\alpha}{k+p}}}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_a^x (x^{\rho+1} - t^{\rho+1})^{\frac{2\alpha}{k+p}-1} t^{\rho} f(t)dt, \quad a<x\leq b. \tag{3.23} $$

$$ {}_{b^-}\mathrm{A}^{\alpha,\rho}_{\varphi(k,p)} f(x) = \frac{(\rho+1)^{1-\frac{2\alpha}{k+p}}}{k\Gamma_{(k,p)}\left(\frac{2k\alpha}{k+p}\right)} \int_x^b (t^{\rho+1} - x^{\rho+1})^{\frac{2\alpha}{k+p}-1} t^{\rho} f(t)dt, \quad a\leq x< b. \tag{3.24} $$

Remark 6. Putting $p=k$ in Definitions 13, 14 and 15 give us the k-Katugompola fractional operators.

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