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PROPERTIES OF SECOND ORDER DIFFERENTIAL EQUATIONS WITH ADVANCED AND DELAY ARGUMENT

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Abstract. In the paper, we study the bounded and unbounded oscillation of the second-order differential equations with deviating argument of the form

$$y''(t) = p(t)y(\tau(t)).$$

We introduce new monotonicities for the nonoscillatory solutions and apply them to offer new criteria for elimination of certain types of solutions. The presented results will be supported by set of examples to confirm our achieved progress in the oscillation theory.

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1. Introduction

In this paper, we shall study the asymptotic and oscillation behavior of the solutions for half-linear second order differential equations with deviating argument

$$y''(t) = p(t)y(\tau(t)). \tag{E}$$

We shall assume that

(H_1) $p(t) \in C^1([t_0,\infty))$, $p(t) > 0$,
(H_2) $\tau(t) \in C^1([t_0,\infty))$, $\tau'(t) > 0$, $\lim\limits_{t\to\infty} \tau(t) = \infty$.

By a solution of Eq. (E) we mean a function $y(t) \in C^2([T_y,\infty))$, $T_y \ge t_0$ which satisfies Eq. (E) on $[T_y,\infty)$. We consider only those solutions $y(t)$ of (E) for which $\sup\{|y(t)| : t \ge T\} > 0$ for all $T \ge T_y$. We assume that (E) possesses such a solution. A solution of (E) is called oscillatory if it has arbitrarily large zeros on $[T_y,\infty)$ and otherwise it is called to be nonoscillatory. An equation itself is said to be oscillatory if all its solutions are oscillatory.

Due to linearity of (E) it is sufficient to deal only with positive solutions of (E). The problem of establishing oscillatory criteria for various types of differential equations has been a very active research area over the past decades (see [1–12]).

It is known that the equation

$$y''(t) = p(t)y(t)$$

always possesses both positive decreasing and positive increasing solution. The situation for equation with deviating argument (E) may be different. If we denote by N the set of all positive solutions of (E), then it has the following decomposition

$$N = N_0 \cup N_2,$$

where the class N_0 involves positive decreasing solutions while N_2 includes positive increasing ones.

Koplatadze and Chanturia [10] have shown that for $\tau(t) \le t$ the condition

$$\limsup_{t\to\infty} \int_{\tau(t)}^{t} (s - \tau(t))p(s)\mathrm{d}s > 1 \tag{1.1}$$

does not allow the presence of positive decreasing solutions. i.e. $N_0 = \varnothing$.

On the other hand, (E) does not possess positive increasing solutions if $\tau(t) \ge t$ and

$$\limsup_{t\to\infty} \int_{t}^{\tau(t)} (\tau(t) - s)p(s)\mathrm{d}s > 1, \tag{1.2}$$

that is $N_2 = \varnothing$.

Both criteria (1.1) and (1.2) are based on the standard monotonicities of possible positive solutions. So, if we intend to refine on above mentioned criteria, we have to improve these monotonicities. And this is the first aim of this paper. The progress will be demonstrated via the following couple of differential equations

$$y''(t) = p_*^2 y(t \pm 2\tau_*)$$

and

$$y''(t) = \frac{p_*}{t^2} y(\lambda t).$$

Some attempts on this direction have been made by several authors (see e.g. [6–8]) included the present one [3–5]. But there is no one general monotonicity formula that would fully cover classes N_0 and N_2 of (E), respectively, There are only partial formulas separately distinguish between $p(t) = p_*$ and $p(t) = p_*/t^2$.

The second aim of this paper is to provide one general monotonicity formula for all possible functions $p(t)$. To achieve this, we introduce new technique for investigation of properties of nonoscillatory solutions. In generally, when establishing desired monotonicity, the authors always try to decrease the order of derivative, but we proceed in opposite direction and we increase order of derivative of nonoscillatory solution. The third goal of the paper is to imply new monotonicities to essentially improve criteria (1.1) and (1.2).

2. Main Results

For positive increasing solutions of (E) we can improve the monotonicity as follows.

Lemma 1. *Let*

$$\int_{t_0}^{\infty} sp(s)\,\mathrm{d}s = \infty. \tag{2.1}$$

Assume that $y(t) \in N_2$. Then

$$\frac{y(t)}{t} \quad \textit{is increasing} \quad \left(\textit{i.e.,} \quad \frac{y(t)}{t}\uparrow\right). \tag{2.2}$$

Proof. Assume that $y(t)$ is positive increasing solution of (E). It is easy to see that

$$\left(t^2\left(\frac{y(t)}{t}\right)'\right)' = tp(t)y(\tau(t)). \tag{2.3}$$

An integration of (2.3) from t_0 to t in view of (2.1) yields

$$\begin{aligned} t^2\left(\frac{y(t)}{t}\right)' &= k + \int_{t_0}^{t} sp(s)y(\tau(s))\,\mathrm{d}s \\ &\geq k + y(\tau(t_0))\int_{t_0}^{t} sp(s)\,\mathrm{d}s \to \infty \quad \text{as} \quad t \to \infty. \end{aligned}$$

Hence

$$t^2\left(\frac{y(t)}{t}\right)' > 0$$

for all t large enough. □

To simplify our notation we introduce the following triple of functions

$$\alpha_2(t) = \frac{p'(t)}{p(t)} + \frac{\tau'(t)}{\tau(t)}\int_t^{\tau(t)} p(s)\tau(s)\,\mathrm{d}s,$$

$$\beta_2'(t) = \alpha_2(t), \qquad \gamma_2(t) = p(t)\mathrm{e}^{-\beta_2(t)}.$$

Lemma 2. *Let (2.1) hold and $\tau(t) \geq t$. Assume that $y(t) \in N_2$. Then*

$$\gamma_2(t)y(\tau(t))\uparrow.$$

Proof. Assume that (E) possesses a positive increasing solution $y(t)$, i.e $y(t) \in N_2$. Thanks to assumptions $p(t), \tau(t) \in C^1([t_0,\infty])$, we can differentiate ($E$), which leads to

$$y'''(t) = p'(t)y(\tau(t)) + p(t)\tau'(t)y'(\tau(t)). \tag{2.4}$$

On the other hand, integrating (E) from t to $\tau(t)$ we are lead to

$$\begin{aligned} y'(\tau(t)) &= y'(t) + \int_t^{\tau(t)} p(s)\tau(s)\frac{y(\tau(s))}{\tau(s)}\,\mathrm{d}s \\ &\geq \frac{y(\tau(t))}{\tau(t)}\int_t^{\tau(t)} p(s)\tau(s)\,\mathrm{d}s, \end{aligned} \tag{2.5}$$

where we used (2.2). By combining inequalities (2.4) and (2.5), we conclude that

$$y'''(t) \geq y(\tau(t))\left(p'(t) + p(t)\frac{\tau'(t)}{\tau(t)}\int_t^{\tau(t)} p(s)\tau(s)\,\mathrm{d}s\right),$$

which in view of (E) means that

$$y'''(t) \geq \alpha_2(t)y''(t).$$

Consequently,

$$\left(\mathrm{e}^{-\beta_2(t)}y''(t)\right)' \geq 0$$

and we conclude that $\mathrm{e}^{-\beta_2(t)}y''(t)$ is increasing, which is in view of (E) equivalent to $\gamma_2(t)y(\tau(t))$ is increasing. □

Remark 1. The new monotonicity presented in Lemma 2 is in closed form for all possible functions $p(t)$ and as compared with previous results it does not require to distinguish whether $p(t) = p_*$ or $p(t) = p_*/t^2$ or something else. It is useful to notice that technique used in the proof of Lemma 2 is new and unique.

Having established new monotonicity we are prepared to improve criterion (1.2).

Theorem 1. *Let $\tau(t) \geq t$ and (2.1) hold. If*

$$\limsup_{t\to\infty} \gamma_2(t)\int_t^{\tau(t)} \frac{p(s)(\tau(t)-s)}{\gamma_2(s)}\,\mathrm{d}s > 1, \tag{2.6}$$

then $N_2 = \varnothing$ for (E).

Proof. We argue by contradiction. Assume that (E) possesses an eventually positive increasing solution $y(t)$. Integrating (E) from t to u and using the monotonicity of $\gamma_2(t)y(\tau(t))$, we obtain

$$y'(u) = y'(t) + \int_t^u p(s)y(\tau(s))\,\mathrm{d}s \geq \gamma_2(t)y(\tau(t))\int_t^u \frac{p(s)}{\gamma_2(s)}\,\mathrm{d}s.$$

Integrating once more from u to t, we get

$$y(u) \geq \gamma_2(t)y(\tau(t))\int_t^u\int_t^x \frac{p(s)}{\gamma_2(s)}\,\mathrm{d}s\mathrm{d}x = \gamma_2(t)y(\tau(t))\int_t^u \frac{p(s)(u-s)}{\gamma_2(s)}\,\mathrm{d}s.$$

Setting $u = \tau(t)$, we have

$$y(\tau(t)) \geq y(\tau(t))\gamma_2(t)\int_t^{\tau(t)} \frac{p(s)(\tau(t)-s)}{\gamma_2(s)}\,\mathrm{d}s,$$

which contradicts to condition (2.6) and we conclude, that class N_2 is empty. □

The above theorem can be reformulated in term of unbounded oscillation.

Theorem 2. *Let $\tau(t) \geq t$ and (2.1), (2.6) hold. Then every unbounded solution of (E) is oscillatory.*

If $1/\gamma_2(t)$ is increasing, then it is easy to see that (1.2) implies (2.6). In the following couple of illustrative examples we will demonstrate the real progress.

Example 1. We consider advanced differential equation with constant coefficients

$$y''(t) = p_*^2 y(t + 2\tau_*), \quad p_* > 0, \quad \tau_* > 0. \tag{E_{x1}}$$

It is easy to verify that criterion (1.2) guarantees $N_2 = \varnothing$ provided that

$$p_*\tau_* > \frac{1}{\sqrt{2}} \approx 0.707106.$$

On the other hand, simple calculation yields

$$\alpha_2(t) = 2p_*^2\tau_* + 2p_*^2\tau_*^2\frac{1}{t + 2\tau_*}.$$

Thus

$$\beta_2(t) = 2p_*^2\tau_* t + 2p_*^2\tau_*^2 \ln(t + 2\tau_*)$$

and

$$\gamma_2(t) = p_*^2 \frac{e^{-2p_*^2\tau_* t}}{(t + 2\tau_*)^{2p_*^2\tau_*^2}}$$

and we conclude that $\gamma_2(t)y(t + 2\tau_0)$ is increasing. But then also

$$\widetilde{\gamma}_2(t)y(t + 2\tau_*) \uparrow \qquad \text{with } \widetilde{\gamma}_2(t) = e^{-2p_*^2\tau_* t}.$$

The reason for introducing function $\widetilde{\gamma}_2(t)$ is the fact that it is more acceptable than $\gamma_2(t)$ because the term $(t + 2\tau_0)^{-2p_*^2\tau_*^2}$ does not bring any progress for monotonicity of $y(t) \in N_2$.

Condition (2.6) with $\gamma_2(t) = \widetilde{\gamma}_2(t)$ reduces to

$$e^{4p_*^2\tau_*^2} > 8p_*^2\tau_*^2 + 1. \tag{2.7}$$

By standard numerical methods we can verify that the inequality

$$e^x > 2x + 1, \quad x > 0$$

is satisfied for all $x > 1.256431$ and consequently (2.7) holds true provided that

$$p_*\tau_* > \sqrt{\frac{1.256431}{4}} \approx 0.560453.$$

Consequently, our progress is significant.

Example 2. We consider the advanced Euler differential equation

$$y''(t) = \frac{p_*}{t^2}\, y(\lambda t), \tag{E_{x2}}$$

with $p_* > 0$ and $\lambda > 1$. According to Koplatadze and Chanturia's criterion (1.2), the class $N_2 = \varnothing$ for Eq. (E_{x2}) provided that

$$p_*(\lambda - 1 - \ln\lambda) > 1. \tag{2.8}$$

On the other hand, simple calculation yields

$$\alpha_2(t) = \frac{-2 + p_*\lambda\ln\lambda}{t}.$$

Therefore,

$$\beta_2(t) = (-2 + p_*\lambda\ln\lambda)\ln t \quad \text{and} \quad \gamma_2(t) = t^{-p_*\lambda\ln\lambda}.$$

Now, some necessary calculation yields that (2.6) takes the form

$$p_*\left(\frac{\lambda^{p_*\lambda\ln\lambda} - \lambda}{p_*\lambda\ln\lambda - 1} - \frac{\lambda^{p_*\lambda\ln\lambda} - 1}{p_*\lambda\ln\lambda}\right) > 1. \tag{2.9}$$

To see our progress let us set e.g. $\lambda = \mathrm{e}$. Then (2.8) takes the form

$$p_* > \frac{1}{\mathrm{e} - 2} \approx 1.392211,$$

while (2.9) reduces to

$$\frac{(\mathrm{e}^{p_*\mathrm{e}} - e)\, p_*\mathrm{e} - (\mathrm{e}^{p_*\mathrm{e}} - 1)\,(p_*\mathrm{e} - 1)}{p_*\mathrm{e} - 1} > \mathrm{e}. \tag{2.10}$$

It easy to verify that the inequality

$$\frac{(\mathrm{e}^x - e)\, x - (\mathrm{e}^x - 1)\,(x - 1)}{x - 1} > \mathrm{e}, \quad x > 1$$

holds for all $x > 1.911063$. Thus (2.10) is satisfied for

$$p_* > \frac{1.911063}{\mathrm{e}} \approx 0.703041.$$

Our progress is outstanding.

The next considerations are intended to improve criterion (1.1). We define triple of functions as follows.

$$\begin{aligned} \alpha_0(t) &= -\frac{p'(t)}{p(t)} + \tau'(t)\int_{\tau(t)}^{t} p(s)\,\mathrm{d}s, \\ \beta_0'(t) &= \alpha_0(t), \qquad \gamma_0(t) = p(t)\mathrm{e}^{\beta_0(t)}. \end{aligned} \tag{2.11}$$

Lemma 3. *Let $\tau(t) \le t$. Assume that $y(t) \in N_0$. Then*

$$\gamma_0(t)y(\tau(t)) \downarrow.$$

Proof. Assume that (E) possesses a positive decreasing solution $y(t)$, i.e $y(t) \in N_0$. Proceeding exactly as in the proof of Theorem 2 we are led to (2.4). On the other hand, an integration (E) from $\tau(t)$ to t yields

$$-y'(\tau(t)) = -y'(t) + \int_{\tau(t)}^{t} p(s)y(\tau(s))\,\mathrm{d}s \geq y(\tau(t))\int_{\tau(t)}^{t} p(s)\,\mathrm{d}s. \tag{2.12}$$

We conclude

$$y'''(t) \leq y(\tau(t))\left(p'(t) - p(t)\tau'(t)\int_{\tau(t)}^{t} p(s)\,\mathrm{d}s\right),$$

by combining inequalities (2.4) and (2.12). Taking (E) into account, we obtain

$$y'''(t) + \alpha_0(t)y''(t) \leq 0.$$

Consequently,

$$\left(\mathrm{e}^{\beta_0(t)}y''(t)\right)' \leq 0$$

and one can see that $\mathrm{e}^{\beta_0(t)}y''(t)$ is decreasing, which is in view of (E) equivalent to $\gamma_0(t)y(\tau(t))$ is decreasing. □

Theorem 3. *Let $\tau(t) \leq t$. If*

$$\limsup_{t\to\infty} \gamma_0(t)\int_{\tau(t)}^{t} \frac{p(s)(s-\tau(t))}{\gamma_0(s)}\,\mathrm{d}s > 1, \tag{2.13}$$

then $N_0 = \varnothing$ for (E).

Proof. We admit that $N_0 \neq \varnothing$ which means that (E) possesses an eventually positive decreasing solution $y(t)$. Integrating (E) from u to t and using the monotonicity of $\gamma_0(t)y(\tau(t))$, we obtain

$$-y'(u) = -y'(t) + \int_{u}^{t} p(s)y(\tau(s))\,\mathrm{d}s \geq \gamma_0(t)y(\tau(t))\int_{u}^{t} \frac{p(s)}{\gamma_0(s)}\,\mathrm{d}s.$$

Integrating once more from u to t, we have

$$y(u) \geq \gamma_0(t)y(\tau(t))\int_{u}^{t}\int_{x}^{t} \frac{p(s)}{\gamma_0(s)}\,\mathrm{d}s\mathrm{d}x = \gamma_0(t)y(\tau(t))\int_{u}^{t} \frac{p(s)(s-u)}{\gamma_0(s)}\,\mathrm{d}s.$$

Setting $u = \tau(t)$, we have

$$y(\tau(t)) \geq y(\tau(t))\gamma_0(t)\int_{\tau(t)}^{t} \frac{p(s)(s-\tau(t))}{\gamma_0(s)}\,\mathrm{d}s,$$

which contradicts to condition (2.13) and we deduce that class N_0 is empty. □

We provide an alternative formulation of the previous theorem.

Theorem 4. *Let $\tau(t) \leq t$ and (2.13) hold. Then every bounded solution of (E) is oscillatory.*

We again support our results by couple of illustrative examples.

Example 3. We consider retarded differential equation with constant coefficients

$$y''(t) = p_*^2 y(t - 2\tau_*), \quad p_* > 0, \quad \tau_* > 0. \tag{E_{x3}}$$

By (1.1) the class $N_0 = \varnothing$ provided that

$$p_* \tau_* > \frac{1}{\sqrt{2}} \approx 0.707106.$$

On the other hand,

$$\alpha_0(t) = 2p_*^2 \tau_*, \quad \beta_0(t) = 2p_*^2 \tau_* t, \quad \gamma_0(t) = p_*^2 e^{2p_*^2 \tau_* t}.$$

Consequently condition (2.13) reduces to

$$e^{4p_*^2 \tau_*^2} > 8p_*^2 \tau_*^2 + 1. \tag{2.14}$$

Proceeding exactly as in Example 1 we see that (2.14) holds if

$$p_* \tau_* > \sqrt{\frac{1.256431}{4}} \approx 0.560453.$$

Example 4. Consider delay Euler differential equation

$$y''(t) = \frac{p_*}{t^2} y(\lambda t), \tag{E_{x4}}$$

with $p_* > 0$ and $\lambda \in (0, 1)$. By (1.1), the class $N_0 = \varnothing$ for Eq. (E_{x4}) provided that

$$p_*(\lambda - 1 - \ln \lambda) > 1$$

which for $\lambda = 1/e$ takes the form

$$p_* > e = 2.718281.$$

On the other hand,

$$\alpha_0(t) = \frac{2 + p_*(1 - \lambda)}{t}, \quad \beta_0(t) = (2 + p_*(1 - \lambda)) \ln t, \quad \gamma_0(t) = p_* t^{p_*(1-\lambda)}.$$

Now, (2.13) takes the form

$$p_* \left(\frac{\lambda^{-p_*(1-\lambda)} - 1}{p_*(1 - \lambda)} - \frac{\lambda^{-p_*(1-\lambda)} - \lambda}{1 + p_*(1 - \lambda)} \right) > 1$$

which is equivalent to

$$\lambda^{-p_*(1-\lambda)} > 2 + 2p_*(1 - \lambda) - 2\lambda p_*(1 - \lambda) - \lambda. \tag{2.15}$$

We denote $x = p_*(1 - \lambda)$ and set $\lambda = 1/e$.Then (2.15) becomes algebraic inequality

$$e^x > 2 + 2x - 2e^{-1} x - e^{-1}, \qquad x > 0$$

which holds true for all $x > 1.110604$ and so we deduce that

$$p_* > \frac{1.110604}{1 - e^{-1}} \approx 1.756950.$$

Our progress is relevant.

3. EXTENSION

From our previous results it is natural to expect that there will be no nonoscillatory solution, or equivalently all solutions will be oscillatory for certain differential equations involving both advanced and delayed arguments. The purpose of this section is to show that this is indeed for

$$y''(t) = p(t)y(\tau(t)) + q(t)y(\sigma(t)), \tag{E^*}$$

where $p(t), q(t), \tau(t), \sigma(t)$ obey the corresponding conditions presented in (H_1) and (H_2) and $\gamma_0(t)$ is defined by (2.11) with $p(t)$ and $\tau(t)$ replaced by $q(t)$ and $\sigma(t)$, respectively.

Theorem 5. *Let $\tau(t) \geq t$ and (2.1), (2.6) hold. Moreover, assume that $\sigma(t) \leq t$ and*

$$\limsup_{t\to\infty} \gamma_0(t) \int_{\sigma(t)}^{t} \frac{q(s)(s-\sigma(t))}{\gamma_0(s)}\,\mathrm{d}s > 1,$$

then (E^*) is oscillatory.

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