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ON QUASI-WEIBULL DISTRIBUTION

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Abstract. Exponential distribution together with a variety of its transformations is permanently used both in probability theory and related fields. The most popular one is the power transformation yielding the Weibull distribution.

In this paper, the power distribution of exponential random variable is supplemented by a logarithmic factor leading to a new distribution called quasi-Weibull. This is a three-parameter distribution, where one parameter is inherited from the underlying exponential distribution, and the others originate from the transformation. The properties of the quasi-Weibull distribution are studied. Specifically, the impact of the parameters on the analyticity of characteristic function, the existence of the moment generating function, the moment-determinacy/indeterminacy and the behaviour of the hazard function are investigated.

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1. INTRODUCTION

Exponential distribution and its various transformations, generalizations and analogues are of great interest not only in probability theory but also in its numerous applications. See, for example, [3, 5, 10] and references therein. The mostly used transformation of an exponential random variable is the power transformation, which possesses the Weibull distribution. Apart from its importance within probability theory and relations to other classical distributions, such as gamma and Rayleigh, the Weibull distribution is a central one used extensively in reliability theory, actuarial science, engineering and finance.

In this work, we study a perturbation of the Weibull distribution, generated by the power-logarithmic transformation of the exponential distribution, as defined below.

Definition 1. Given $X \sim \mathrm{Exp}(\lambda)$ and $\varphi(x) = x^a \ln^b(1+x)$, $x > 0$, where $a \in \mathbb{R}$, $b \geqslant 0$, the distribution of $Y = Y_{a,b} = \varphi(X)$ is said to be *quasi-Weibull* with parameters a, b and λ.

Notice that when $b = 0$, one recovers the Weibull distribution. For this reason, in the present work, we deal solely with the case $b > 0$. The positive value of the parameter b supplies an additional flexibility for the distribution of Y when describing real data sets as it provides more subtle matching. It has to be emphasized that, power-logarithmic functions of random variables occur in various disciplines. For example, in the nuclear physics, the energy loss of fast particles by ionization is expressed by

$$F = \frac{2\pi Ne^4}{E}\ln\left(\frac{4E}{I}\right),$$

where E is a random variable averaging energy of particles, I is a characteristic energy of atom ionization, and N is the density of atoms in the medium. See, e.g. [2].

Let us recall some basic notions of probability theory used in the sequel. Given a random variable X with a distribution function $F_X(x)$, the *reliability function* of X is $R_X(x) := 1 - F_X(x)$. Notice that in the actuarial mathematics, R_X is called a survival function. For $n \in \mathbb{N}_0$, the n-th order moment of X is defined as $M_n := \int_{\mathbb{R}} x^n dF_X(x)$, provided the integral is absolutely convergent. The distribution of X is said to be *moment-determinate* if X has moments of all orders and F_X is the only distribution possessing this moment sequence $\{M_n\}_{n=0}^{\infty}$. Otherwise, the distribution is *moment-indeterminate*. For more information, the reader is referred to, e.g. [9, Ch. 11].

The moment-(in)determinacy of probability distributions is not only one of the classical areas of probability theory owning deep interrelations with other branches of mathematics (see, e.g. [1]), but also it is a significant issue for applications. The respective direction of research, called the moment analysis of distributions, is getting increasingly popular in the fields such as econometrics, quantum physics, actuarial sciences, and mathematical finance. For example, Penson et al. [7] have shown that Stieltjes classes of moment indeterminate distributions emerge when describing coherent states in the quantum physics. Also, Lasserre et al. [4] proposed a novel approach involving the moments of appropriate distributions. For these type of problems, the moment-(in)determinacy is a crucial property. Additional relevant information on the subject can be found in [8] and references therein.

The focus of this work is on the impact of parameters a and b on the properties of the quasi-Weibull distribution. The paper is organized as follows. In section 2, the analyticity of the characteristic function is studied. Section 3 presents the results on the existence of moments for the quasi-Weibull distribution together with the conditions for moment-(in)determinacy in the case when all moments exist. Section 4 provides the conditions on parameters a and b for the quasi-Weibull distribution to be increasing/decreasing failure rate. The comparison with the respective properties of the classical Weibull distribution is performed. The obtained results are illustrated graphically.

2. Characteristic function

First, let us obtain a density of the quasi-Weibull distribution $Y_{a,b}$. To do this, the following cases have to be considered:

(i) If $a \leqslant -b$ or $a \geqslant 0$, then $\varphi(x)$, $x > 0$, is invertible with $g = \varphi^{-1}$. In this case, the density of Y is given by

$$f_Y(x) = \lambda e^{-\lambda g(x)} |g'(x)|, \quad x > 0, \tag{2.1}$$

and the reliability function by

$$R_Y(x) = \begin{cases} e^{-\lambda g(x)}, & \text{if } a \geqslant 0,\, x > 0, \\ 1 - e^{-\lambda g(x)}, & \text{if } a \leqslant -b,\, x > 0. \end{cases} \tag{2.2}$$

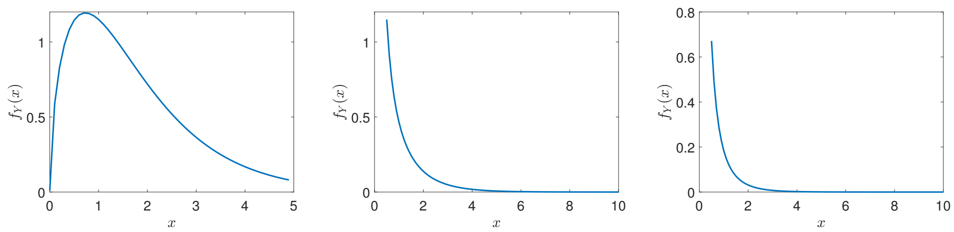
(ii) If $-b < a < 0$, then φ is not invertible, and we set $\varphi_1(x) = \varphi(x)$, $x \in (0, x_0)$, $\varphi_2(x) = \varphi(x)$, $x \in (x_0, \infty)$, where x_0 satisfies $\varphi'(x_0) = 0$. Then, one may consider $g_1 = \varphi_1^{-1}$ and $g_2 = \varphi_2^{-1}$. In these notations, the density of Y is given by

$$f_Y(x) = \lambda e^{-\lambda g_1(x)} g_1'(x) - \lambda e^{-\lambda g_2(x)} g_2'(x), \quad 0 < x < \varphi(x_0),$$

Correspondingly, the reliability function equals

$$R_Y(x) = \begin{cases} e^{-\lambda g_1(x)} - e^{-\lambda g_2(x)}, & 0 < x < \varphi(x_0), \\ 0, & x \geqslant \varphi(x_0), \end{cases}$$

The graphs of the density functions corresponding to $\lambda = 1$ and different values of a and b are demonstrated in Figure 1.



(a) $a = 0.2$, $b = 0.2$ (b) $a = 0.2$, $b = 2$ (c) $a = 2$, $b = 0.2$

FIGURE 1. Density functions for the quasi-Weibull distribution with $\lambda = 1$.

Our first result is concerned with some analytic properties of the characteristic function $\phi_{a,b}(t)$ of $Y_{a,b}$.

Theorem 1. *If $-b \leqslant a < 1$, then $\phi_{a,b}(t)$ is entire. Otherwise, it is not analytic in any neighbourhood of 0. Further, if $0 < a < 1$, then $\phi_{a,b}(t)$ is entire of order $\rho = 1/(1-a)$, and type $\sigma = \infty$, while if $-b \leqslant a < 0$, then $\phi_{a,b}(t)$ is entire of order 1 and type $\sigma = \varphi(x_0)$.*

Proof. By [6, formula (2.2.3)], since $Y_{a,b} \geqslant 0$, the characteristic function of its probability distribution is analytic in the disc $\mathcal{D}_{\mathcal{R}} = \{z : |z| < \mathcal{R}\}$ if and only if its reliability function satisfies the asymptotic estimate

$$R_Y(x) = O(e^{-rx}), \quad x \to +\infty \quad \text{for all } 0 < r < \mathcal{R}.$$

When $a \geqslant 0$, from (2.2), one has, for all $r > 0$,

$$\lim_{x\to+\infty} e^{rx} R_Y(x) = \lim_{t\to+\infty} e^{r\varphi(t)-\lambda t} = \begin{cases} 0, & \text{if } a < 1, \\ +\infty, & \text{if } a \geqslant 1. \end{cases} \tag{2.3}$$

This implies that $\phi_{a,b}(t)$ is entire when $0 \leqslant a < 1$ and not analytic in any neighbourhood of 0 when $a \geqslant 1$. To find the order of $\phi_{a,b}(t)$, we use Theorem 2.4.4 of [6]. First, we evaluate

$$\kappa = \lim_{x\to+\infty} \frac{\ln\ln(1/R_Y(x))}{\ln x} = \lim_{x\to+\infty} \frac{\ln(\lambda g(x))}{\ln x} = \lim_{t\to+\infty} \frac{\ln\lambda + \ln t}{a\ln t + b\ln\ln(1+t)} = \frac{1}{a},$$

whence

$$\frac{1}{\rho} = 1 - \frac{1}{\kappa} = 1 - a$$

and the order is $\rho = 1/(1-a)$. As for the type,

$$\tau = \lim_{x\to\infty} x^{-\kappa} \ln(1/R_Y(x)) = \lim_{x\to+\infty} \frac{\lambda g(x)}{x^{1/a}} = \lim_{t\to+\infty} \frac{\lambda t}{t\ln^{b/a}(1+t)} = 0,$$

yielding that $\phi_{a,b}(t)$ has the maximal type.

Next, let $-b \leqslant a < 0$. Then, the distribution of $Y_{a,b}$ has bounded support $(0, \varphi(x_0))$, and according to [6], $\phi_{a,b}(t)$ is entire of order 1 and type $\sigma = \varphi(x_0)$. Notice that when $a = -b$, we set $x_0 = 0$.

Finally, let $a < -b$. Using (2.2), one obtains

$$\lim_{x\to+\infty} e^{rx} R_Y(x) = \lim_{t\to 0^+} e^{r\varphi(t)}(1 - e^{-\lambda t}) = \lim_{t\to 0^+} \lambda t e^{r\varphi(t)} = +\infty$$

for all $r > 0$ since $\varphi(t) \sim t^{a+b}$ as $t \to 0^+$. Combining this with (2.3) completes the proof. □

Corollary 1. *The moment generating function of $Y_{a,b}$ exists if and only if $-b \leqslant a < 1$. This implies that for other values of the parameters, the quasi-Weibull distribution is heavy-tailed.*

Corollary 2. *For $-b \leqslant a < 1$, the distribution of $Y_{a,b}$ is moment-determinate by virtue of Cramer's condition.*

3. MOMENT-(IN)DETERMINACY OF QUASI-WEIBULL DISTRIBUTION

Prior to studying the moment-(in)determinacy of the distribution, it is necessary to establish the conditions under which the moments exist.

Theorem 2. *Random variable $Y_{a,b}$ has finite moments of all orders if and only if $a \geqslant -b$.*

Proof. The moment of order n equals

$$M_n = \lambda \int_0^\infty x^{an} \ln^{bn}(1+x) e^{-\lambda x} \mathrm{d}x.$$

When $a \geqslant 0$, one has $M_n < \infty$ for all $n \in \mathbb{N}_0$. When $a < 0$, since $x^{an} \ln^{bn}(1+x) \sim x^{(a+b)n}$ as $x \to 0^+$, the integral converges for all $n \in \mathbb{N}_0$ if and only if $a+b \geqslant 0$. $\square$

In view of the last theorem, the problem of moment-(in)determinacy for $Y_{a,b}$ is well-defined only when $a \geqslant -b$. In addition, Corollary 2 asserts that for $-b \leqslant a < 1$, the distribution of $Y_{a,b}$ is moment-determinate. Bearing this in mind, it suffices to examine only the case $a \geqslant 1$.

Theorem 3. *Let $a \geqslant 1$. The distribution of $Y_{a,b}$ is moment-determinate if and only if $-b \leqslant a < 2$ or $a = 2$, $b \leqslant 2$.*

Proof. By (2.1), the density is

$$f_{a,b}(x) := f_Y(x) = \lambda e^{-\lambda g(x)} g'(x), \quad x > 0,$$

where $g = \varphi^{-1}$.

First, we apply Krein's condition for moment-indeterminacy of an absolutely continuous distribution with support on $[0, +\infty)$, see, e.g. [9, Ch. 11]:

Let $f(x), x > 0$, be a probability density of a random variable X. If, for some $c > 0$,

$$I[f] = \int_c^\infty \frac{-\ln f(x^2)}{1+x^2} \mathrm{d}x < \infty,$$

then the distribution of X is moment-indeterminate. We have, therefore, for f_Y :

$$I[f_{a,b}] = \int_c^\infty \frac{-\ln\lambda + \lambda g(x^2) - \ln g'(x^2)}{1+x^2} \mathrm{d}x \sim \frac{1}{2} \int_{g(c^2)}^\infty \frac{[\lambda u + \ln\varphi'(u)]\varphi'(u)}{(1+\varphi(u))\sqrt{\varphi(u)}} \mathrm{d}u. \quad (3.1)$$

Since $\varphi'(u) \sim C u^{a-1} \ln^b(1+u)$, $u \to +\infty$, the numerator of (3.1) satisfies

$$[\lambda u + \ln\varphi'(u)]\varphi'(u) \sim \lambda u^a \ln^b(1+u), \quad u \to +\infty.$$

Likewise, for the denominator, we obtain

$$(1+\varphi(u))\sqrt{\varphi(u)} \sim u^{3a/2} \ln^{3b/2}(1+u), \quad u \to +\infty.$$

As a result, in terms of convergence,

$$I[f_{a,b}] \sim \int_{c_1}^\infty \frac{\mathrm{d}u}{u^{a/2} \ln^{b/2}(u)}, \quad c_1 > 0.$$

The latter integral converges when $a > 2$ and diverges when $a < 2$ whatever $b > 0$ is. As for $a = 2$, the integral converges only for $b > 2$.

This shows that, for $a > 2$ or $a = 2$, $b > 2$, the distribution of $Y_{a,b}$ is moment-indeterminate. Since Krein's condition is inconclusive in the case when $I[f]$ is divergent, we apply Carleman's condition for the remaining values of parameters.

Recall that Carleman's condition deals with the moments and states that if X is a non-negative random variable with moments M_n such that

$$\sum_{n=1}^{\infty} M_n^{-1/(2n)} = \infty,$$

then the distribution of X is moment-determinate.

To deal with the untouched values of the parameters, we split the rest of the proof into two cases.

Case 1: Let $a < 2$ and $0 < \delta < b$ satisfy $a + \delta < 2$. In this case, $\ln^b(1+x) \leqslant Cx^{\delta}$ for $x \geqslant 0$, and hence,

$$M_n \leqslant \lambda C^n \int_0^{\infty} x^{(a+\delta)n} e^{-\lambda x} \mathrm{d}x = \frac{C^n}{\lambda^{(a+\delta)n}} \Gamma((a+\delta)n+1) \leqslant (C_1)^n (2n)!, \quad n \geqslant n_0.$$

Therefore, by Stirling's formula, $M_n^{1/(2n)} \sim C_2 n$, $n \to \infty$, and, thus, $\sum_{n=1}^{\infty} M_n^{-1/(2n)}$ diverges, whence the distribution of $Y_{a,b}$ is moment-determinate.

Case 2: Let $a = 2$ and $b \leqslant 2$. Then,

$$\begin{aligned} M_n &= \lambda \int_0^{\infty} x^{2n} \ln^{bn}(1+x) e^{-\lambda x} \mathrm{d}x = \lambda^{-2n} \int_0^{\infty} x^{2n} \ln^{bn}\left(1+\frac{x}{\lambda}\right) e^{-x} \mathrm{d}x \\ &= \lambda^{-2n} \int_0^{10n^2} x^{2n} \ln^{bn}\left(1+\frac{x}{\lambda}\right) e^{-x} \mathrm{d}x + \lambda^{-2n} \int_{10n^2}^{\infty} x^{2n} \ln^{bn}\left(1+\frac{x}{\lambda}\right) e^{-x} \mathrm{d}x \\ &=: I_1 + I_2. \end{aligned}$$

Clearly, $I_1 \leqslant \lambda^{-2n} \ln^{bn}(1+10n^2/\lambda)(2n)!$. To estimate I_2, we write

$$I_2 \leqslant \lambda^{-(2+b)n} \int_{10n^2}^{\infty} \exp\{-x + (2+b)n \ln x\} \, \mathrm{d}x.$$

It can be readily seen that $v(x) := -x/2 + (2+b)n \ln x$ is strictly decreasing when $x > 2(2+b)n$ and also tends to $-\infty$. Since $2(2+b)n < 8n < 10n^2$ for $n \geqslant 1$ and $v(10n^2) < 0$ for $n \geqslant 5$, it follows that

$$(2+b)n \ln x \leqslant x/2 \quad \text{for all } x \geqslant 10n^2, \ n \geqslant 5.$$

Consequently, for all $n \geqslant 5$, one has:

$$I_2 \leqslant \lambda^{-(2+b)n} \int_{10n^2}^{\infty} e^{-x/2} \mathrm{d}x = 2\lambda^{-(2+b)n} e^{-5n^2} \to 0 \quad \text{as } n \to \infty.$$

As a result, for n large enough,

$$M_n = I_1 + I_2 \leqslant 2I_1 \leqslant 2\lambda^{-2n} \ln^{bn}(1 + 10n^2/\lambda)(2n)!.$$

The obtained estimate yields, with the help of Stirling's formula, that, when n is large enough,

$$M_n^{1/(2n)} \leqslant Cn \ln^{b/2} n.$$

By Carleman's condition, the distribution of $Y_{a,b}$ is moment-determinate.

□

4. THE HAZARD FUNCTION

In this section, we deal with the hazard function of a quasi-Weibull distribution. As the terminology for this function is rather diverse, we clarify the needed terms here.

Definition 2. For a random variable X possessing an absolutely continuous distribution with a density f_X, the *hazard function* is defined as

$$h_X(x) = \frac{f_X(x)}{R_X(x)},$$

where R_X stands for the reliability function of X.

A hazard function is one of the key characteristics pertinent to lifetime distributions occurring in the reliability theory, queuing, computer science, actuarial and financial mathematics. In different branches, it is also known as hazard rate, force of mortality, intensity rate, instantaneous failure rate, and others. See, e.g. [10, Sec. 3.3]. Notice that the exponential distribution is characterized by a constant hazard function, that is, if $X \sim \mathrm{Exp}(\lambda)$, then $h_X(x) = \lambda$. Its power transformation X^a has the Weibull distribution, whose hazard function is increasing when $a > 1$ and decreasing when $0 < a < 1$.

If the hazard function h_X is increasing, then the distribution of X is said to be *increasing failure rate* (IFR), while if h_X is decreasing, the distribution is said to be *decreasing failure rate* (DFR). Generally speaking, an increasing hazard function corresponds to the situations when conditional survival probability decreases with the ageing of a system whose lifetime is X; and a decreasing hazard function shows that ageing is beneficial for a system. In practical applications, hazard functions show different monotonicity behaviour on different intervals. The graph of a hazard function can also be called the *mortality curve* or *life characteristic.*

The next results reveal the behaviour of the hazard function of a quasi-Weibull distribution for various values of positive parameters a and b.

Theorem 4. *If $a \geqslant 1$, then the distribution of $Y_{a,b}$ is DFR for all $x > 0$.*

Proof. Notice that when $a \geqslant 1$, the functions $\varphi(t)$ and $g(x)$ are increasing on $(0,\infty)$. By virtue of (2.1), the hazard function is

$$h(x) = g'(x) = \frac{1}{\varphi'(t)}, \quad \text{where } x = \varphi(t).$$

Consequently, $h(x)$ decreases if and only if $\varphi'(t)$ increases. As

$$\varphi'(t) = at^{a-1}\ln^b(1+t) + bt^a\frac{\ln^{b-1}(1+t)}{1+t}, \tag{4.1}$$

one has

$$\varphi''(t) = a(a-1)t^{a-2}\ln^b(1+t) + 2abt^{a-1}\frac{\ln^{b-1}(1+t)}{1+t} \\ + bt^a\frac{\ln^{b-2}(1+t)}{(1+t)^2}[b-1-\ln(1+t)]. \tag{4.2}$$

Obviously, if $a \geqslant 1$, then

$$\varphi''(t) \geqslant bt^{a-1}\frac{\ln^{b-2}(1+t)}{(1+t)^2}\cdot\xi(t),$$

where

$$\begin{aligned}\xi(t) &= 2a(1+t)\ln(1+t) + (b-1)t - t\ln(1+t)\\ &\geqslant a(2+t)\ln(1+t) + (b-1)t > 0 \quad \text{for all } t > 0.\end{aligned}$$

Thus, $\varphi'(t)$ is increasing for all $t > 0$. □

Theorem 5. *If $a+b \leqslant 1$, then the distribution of $Y_{a,b}$ is IFR for all $x > 0$.*

Proof. Using (4.2), we write $\varphi''(t) = \xi_1(t) + \xi_2(t)$, where

$$\xi_1(t) = \frac{at^{a-2}}{1+t}\ln^{b-1}(1+t)[(a-1)(1+t)\ln(1+t) + bt]$$

and

$$\xi_2(t) = \frac{bt^{a-1}\ln^{b-2}(1+t)}{(1+t)^2}[a(1+t)\ln(1+t) + (b-1)t - t\ln(1+t)].$$

Observe that, in both cases, the functions in brackets are decreasing in t and vanish at 0. Thence, both $\xi_1(t) < 0$ and $\xi_2(t) < 0$ for all $t > 0$. The statement is proved. □

Theorem 6. *If $a < 1$ and $a+b > 1$, then the distribution of $Y_{a,b}$ is eventually IFR without being IFR for all $x > 0$.*

Proof. It follows from (4.1) that

$$\lim_{t\to 0^+}\varphi'(t) = \lim_{t\to+\infty}\varphi'(t) = 0,$$

whence $\varphi'(t)$ is not monotone for all $t \in (0,+\infty)$. Meanwhile,

$$\varphi''(t) = t^{a-2}\ln^b(1+t)\xi(t),$$

where

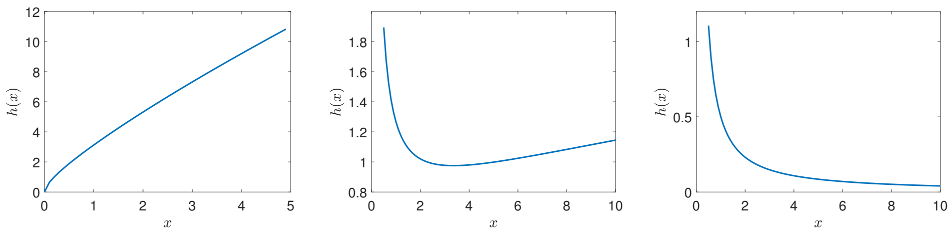
$$\xi(t) = a(a-1) + \frac{2abt}{(1+t)\ln(1+t)} + \frac{bt^2[t-1-\ln(1+t)]}{(1+t)^2\ln^2(1+t)}.$$

As

$$\lim_{t\to+\infty}\xi(t) = a(a-1) < 0,$$

it follows that $\varphi''(t) < 0$ for t large enough. □

The graphs of the hazard functions for the quasi-Weibull distribution possessing densities displayed in Figure 1 can be viewed in Figure 2.



(a) $a = 0.2$, $b = 0.2$ (b) $a = 0.2$, $b = 2$ (c) $a = 2$, $b = 0.2$

FIGURE 2. Hazard functions for the quasi-Weibull distribution with $\lambda = 1$.

On the whole, although there exist certain similarities between Weibull and quasi-Weibull distributions, in some aspects, their features are quite different. For the latter distribution, first of all, the density function of $Y_{a,b}$ does not appear in a closed form. Next, the moment generating function, for $a = 1$, does not exist in distinction to the Weibull distribution. Both distributions are moment-determinate when $a < 2$ and moment-indeterminate when $a > 2$. When it comes to the case $a = 2$, the Weibull distribution is moment-determinate, whereas the moment-(in)determinacy of $Y_{2,b}$ depends on b with the distribution being moment-determinate only when $b \leqslant 2$. The list of similarities and distinctions extends to the behaviour of the hazard function. The properties of $Y_{a,b}$ with respect to the behaviour of its hazard function are illustrated in Figure 3.

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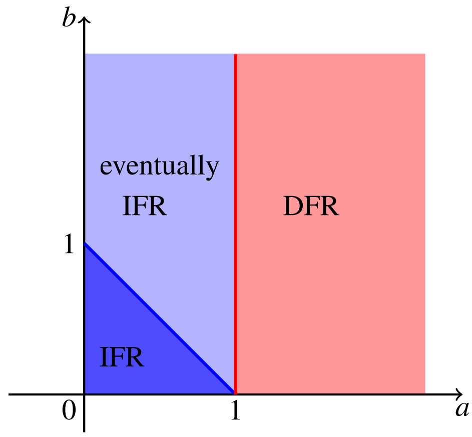

FIGURE 3. Regions of DFR/IFR for the quasi-Weibull distribution

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