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ANALYSIS OF MIXED DIFFERENTIAL EQUATIONS INVOLVING THE CAPUTO AND RIEMANN-LIOUVILLE FRACTIONAL DERIVATIVES

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Abstract. In this paper, we investigate the existence of positive solution as well as the uniqueness results for a fractional order differential equation involving the Caputo fractional derivative and the Riemann-Liouville fractional derivative. First of all, we show the existence and uniqueness of the positive solution by means of the fixed point theory, namely, Banach's contraction Principle. Second of all, we convert the posed problem to a sum of two integral operators, then we apply Kronelskii's fixed point theorem to conclude the existence of nontrivial solutions. As applications, we present examples for the demonstration of our main results.

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1. INTRODUCTION

Fractional calculus is an extension of classical calculus and deals with the generalization of integration and differentiation to an arbitrary real order. Boundary value problems for Caputo fractional differential equations, Riemann-Liouville fractional differential equations and mixed fractional differential equations of great importance for the researches due to their applications, such as physics, chemistry, probability, many other branches of engineering. There has been a noticeable development in the study of fractional differential equations in recent years, see the books of Kilbas et al. [6], Podlubny [17], Samko et al. [18] and Miller et al. [15] are mostly cited for the theory and applications of fractional calculus.

The existence of positive solutions for fractional-order nonlinear boundary value problems has been studied by numerous authors by using the fixed-point theorem

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in cones. To identify a few, we refer the reader to [1, 4, 5, 9, 10, 12, 20, 21] and the references therein. Some authors investigated the existence of solutions for a class of mixed fractional differential equations by using different methods. In [2], Agraval presented the mixed differential equation involving both the Caputo and the Riemann-Liouville fractional derivatives. In [3], Blaszczyk presented the numerical solutions of the mixed boundary value problem. For some interesting results on mixed fractional boundary value problems we refer the literature [8, 11, 12, 14, 16].

In [19], Song and Cui concerned the existence of solutions of nonlinear mixed fractional differential equation with the integral boundary value problem under resonance

$$\begin{cases} {}^cD_{1-}^{\alpha}D_{0+}^{\beta}u(t) = f(t,u(t),D_{0+}^{\beta+1}u(t),D_{0+}^{\beta}u(t)),\ t\in(0,1),\\ u(0)=u'(0)=0,\ u(1)=\int_0^1 u(t)\mathrm{d}A(t),\end{cases}$$

where $1<\alpha\le 2$, $0<\beta\le 1$, $f\in C([0,1]\times\mathbb{R}^3,\mathbb{R})$.

Liu et al. [13] investigated the existence of the unique nontrivial solution for mixed fractional differential equation

$$\begin{cases} {}^cD_{1-}^{\alpha}D_{0+}^{\beta}x(t)+f(t,x(t))=b,\ t\in(0,1),\\ x(0)=0,\ x'(1)=D_{0+}^{\beta}x(1)=0,\end{cases}$$

where $0<\alpha\le 1$, $1<\beta\le 2$ and $\alpha+\beta>2$. Here $f\colon [0,1]\times(-\infty,\infty)\to(-\infty,\infty)$ is continuous, and $b>0$ is a constant real number.

Guezane et al.[8] investigated the following mixed fractional boundary value problem

$$\begin{cases} -{}^cD_{0^+}^{\alpha}D_{0^+}^{\beta}u(t)+f(t,u(t))=0,\ t\in(0,1),\\ u(0)=u'(0)=u(1)=0,\end{cases}$$

where $0<\alpha\le 1$, $1<\beta\le 2$. They used Krasnoselskii's fixed point theorem to prove the existence of nontrivial solution.

Motivated by the above papers, we consider the nonlinear mixed fractional differential equation

$$D_{0^+}^{\alpha}\,{}^cD_{0^+}^{\beta}u(t)+f(t,u(t))=0,\ t\in(0,1),$$

with the multi-point fractional boundary conditions:

$$\begin{cases} u(0)=u'(0)={}^cD_{0^+}^{\beta}u(0)=0,\\ D_{0^+}^{\gamma}u(1)=\displaystyle\sum_{i=1}^{m-2}\eta_i D_{0^+}^{\gamma}u(\tau_i),\end{cases}$$

where $1<\alpha,\beta\le 2$, $\gamma<\alpha+\beta-1$, $D_{0^+}^{\alpha}$ denotes the Riemann-Liouville derivative of order α, ${}^cD_{0^+}^{\beta}$ denotes the Caputo derivative of order β, $0<\tau_1<\tau_2<\ldots<\tau_{m-2}<1$, $0<\eta_i<1$ for $i=1,2,\ldots,m-2$, $f\in C([0,1]\times[0,\infty),[0,\infty))$.

The organization of this paper is as follows. In Section 2 and Section 3, we provide some definitions and preliminary lemmas which are key tools for our main result. In

Section 4, we give and prove our main results. We give examples to illustrate how the main results can be used in practice.

2. Preliminaries and lemmas

In order to assert our main results, we assemble some necessary definitions and lemmas from the fractional calculus, which can be found in [6, 17, 18].

Definition 1. The fractional integral of order $\alpha > 0$ of a function $y\colon (0,+\infty) \to \mathbb{R}$ is given by

$$I_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)\mathrm{d}s,$$

provided that the right side is pointwise defined on $(0,\infty)$, where

$$\Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} \mathrm{d}x.$$

Definition 2. For a continuous function $y\colon (0,+\infty) \to \mathbb{R}$, the Caputo derivative of fractional order $\alpha > 0$ is defined as

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} y(s)\mathrm{d}s,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0,+\infty)$.

Definition 3. For a continuous function $y\colon (0,+\infty) \to \mathbb{R}$, the Riemann-Liouville derivative of fractional order $\alpha > 0$ is defined as

$$D_{0^+}^{\alpha} y(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^n \int_0^t (t-s)^{n-\alpha-1} y(s)\mathrm{d}s,$$

where $n = [\alpha] + 1$, provided that the right side is pointwise defined on $(0,+\infty)$.

Lemma 1. *Let $\alpha > 0$, then*

$$I_{0^+}^{\alpha}\, {}^cD_{0^+}^{\alpha} u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \ldots + c_{n-1} t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, $n = [\alpha] + 1$.

Lemma 2. *Let $\alpha > 0$, then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_n t^{\alpha-n}$$

for some $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, $n = [\alpha] + 1$.

3. Existence of nontrivial positive solutions

We consider the fractional boundary value problem with mixed derivative given as follows.

$$\begin{cases} D_{0+}^{\alpha}\, {}^{c}D_{0+}^{\beta} u(t) + f(t,u(t)) = 0,\ t \in (0,1), \\ u(0) = u'(0) = {}^{c}D_{0+}^{\beta} u(0) = 0, \\ D_{0+}^{\gamma} u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\gamma} u(\tau_i), \end{cases} \tag{3.1}$$

where $1 < \alpha, \beta \le 2$, $\gamma < \alpha+\beta-1$, D_{0+}^{α} denotes the Riemann-Liouville derivative of order α, ${}^{c}D_{0+}^{\beta}$ denotes the Caputo derivative of order β, $0 < \tau_1 < \tau_2 < \ldots < \tau_{m-2} < 1$, $0 < \eta_i < 1$ for $i = 1,2,\ldots,m-2$, $f \in C([0,1]\times[0,\infty),[0,\infty))$.

Throughout this paper we assume that the following conditions hold:

(H1) $0 < \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1} < 1$,

(H2) $f\colon [0,1]\times[0,\infty) \to [0,\infty)$ is a continuous function.

Lemma 3. *If $h \in C[0,1]$, the fractional boundary value problem*

$$\begin{cases} D_{0+}^{\alpha}\, {}^{c}D_{0+}^{\beta} u(t) + h(t) = 0,\ t \in (0,1), \\ u(0) = u'(0) = {}^{c}D_{0+}^{\beta} u(0) = 0, \\ D_{0+}^{\gamma} u(1) = \sum_{i=1}^{m-2} \eta_i D_{0+}^{\gamma} u(\tau_i), \end{cases} \tag{3.2}$$

has an integral expression

$$u(t) = \int_0^1 G(t,s)h(s)\mathrm{d}s \tag{3.3}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s) \tag{3.4}$$

and

$$G_1(t,s) = \frac{1}{\Gamma(\alpha+\beta)} \begin{cases} t^{\alpha+\beta-1}(1-s)^{\alpha+\beta-\gamma-1} - (t-s)^{\alpha+\beta-1}, & 0 \le s \le t \le 1, \\ t^{\alpha+\beta-1}(1-s)^{\alpha+\beta-\gamma-1}, & 0 \le t \le s \le 1, \end{cases} \tag{3.5}$$

$$G_2(t,s) = \frac{\sum_{i=1}^{m-2} \eta_i t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\left[1 - \sum_{i=1}^{m-2} \eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]} \times \begin{cases} (\tau_i(1-s))^{\alpha+\beta-\gamma-1} - (\tau_i - s)^{\alpha+\beta-\gamma-1}, & 0 \le s \le \tau_i \le 1, \\ (\tau_i(1-s))^{\alpha+\beta-\gamma-1}, & 0 \le \tau_i \le s \le 1. \end{cases} \tag{3.6}$$

Proof. According to Lemma 2, we can obtain that

$${}^{c}D_{0+}^{\beta} u(t) = -I_{0+}^{\alpha} h(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

We assume that the boundary conditions are satisfied. Firstly, using ${}^{c}D^{\beta}_{0^+}u(0) = 0$, we have $c_2 = 0$. We may apply Lemma 1, and find that

$$u(t) = -I^{\alpha+\beta}_{0^+}h(t) + c_1 I^{\beta}_{0^+}t^{\alpha-1} + d_1 + d_2 t.$$

From $u(0) = u^{'}(0) = 0$ conditions, $d_1 = d_2 = 0$ is found. From condition

$$D^{\gamma}_{0^+}u(1) = \sum_{i=1}^{m-2}\eta_i D^{\gamma}_{0^+}u(\tau_i),$$

we find that

$$c_1 = \frac{\displaystyle\int_0^1 (1-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s - \sum_{i=1}^{m-2}\eta_i \int_0^{\tau_i}(\tau_i - s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s}{\Gamma(\alpha)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]},$$

substituting this value of c_1, we have

$$\begin{aligned} u(t) = &\frac{-1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}h(s)\mathrm{d}s + \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)} \cdot \frac{\int_0^1 (1-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s}{\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]} \\ &- \frac{\sum_{i=1}^{m-2}\eta_i t^{\alpha+\beta-1}[\int_0^{\tau_i}(\tau_i - s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s]}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}. \end{aligned}$$

Now let's create the Green function:

$$\begin{aligned} u(t) = &\frac{-1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}h(s)\mathrm{d}s + \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^1 (1-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s \\ &+ \frac{t^{\alpha+\beta-1}\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\int_0^1 (1-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s \\ &- \frac{\sum_{i=1}^{m-2}\eta_i t^{\alpha+\beta-1}\left[\int_0^{\tau_i}(\tau_i - s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s\right]}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]} \\ = &\frac{-1}{\Gamma(\alpha+\beta)}\int_0^t (t-s)^{\alpha+\beta-1}h(s)\mathrm{d}s + \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_0^t (1-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s \\ &+ \frac{t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\int_t^1 (1-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s \\ &+ \frac{t^{\alpha+\beta-1}\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\int_0^1 (1-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s \end{aligned}$$

$$
\begin{aligned}
&-\frac{\sum_{i=1}^{m-2}\eta_i t^{\alpha+\beta-1}[\int_0^{\tau_i}(\tau_i-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s]}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\\
&=\int_0^1 G_1(t,s)h(s)\mathrm{d}s+\frac{t^{\alpha+\beta-1}\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\\
&\quad\int_0^1(1-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s-\frac{\sum_{i=1}^{m-2}\eta_i t^{\alpha+\beta-1}[\int_0^{\tau_i}(\tau_i-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s]}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\\
&=\int_0^1 G_1(t,s)h(s)\mathrm{d}s+\frac{t^{\alpha+\beta-1}\sum_{i=1}^{m-2}\eta_i}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\\
&\quad\times\left[\int_0^1\tau_i^{\alpha+\beta-\gamma-1}(1-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s-\int_0^{\tau_i}(\tau_i-s)^{\alpha+\beta-\gamma-1}h(s)\mathrm{d}s\right]\\
&=\int_0^1 G_1(t,s)h(s)\mathrm{d}s+\int_0^1 G_2(t,s)h(s)\mathrm{d}s\\
&=\int_0^1 G(t,s)h(s)\mathrm{d}s.
\end{aligned}
$$

The functions $G(t,s)$, $G_1(t,s)$ and $G_2(t,s)$ are defined in (3.4), (3.5) and (3.6), respectively. □

Lemma 4. *The function $G_1(t,s)$ defined by (3.5) satisfies the following properties:*

(i) *$G_1(t,s)$ is a continuous function and $G_1(t,s)\geq 0$ for any $(t,s)\in[0,1]\times[0,1]$,*

(ii) *$G_1(t,s)\leq \dfrac{1}{\Gamma(\alpha+\beta)}$ for any $(t,s)\in[0,1]\times[0,1]$.*

Proof. **(i)** It is obvious.

(ii) Let $s\leq t$.

$$
G_1(t,s)=\frac{t^{\alpha+\beta-1}(1-s)^{\alpha+\beta-\gamma-1}-(t-s)^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)}\leq\frac{t^{\alpha+\beta-1}(1-s)}{\Gamma(\alpha+\beta)}\leq\frac{1}{\Gamma(\alpha+\beta)}.
$$

Let $t\leq s$.

$$
G_1(t,s)=\frac{t^{\alpha+\beta-1}(1-s)^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta)}\leq\frac{1}{\Gamma(\alpha+\beta)}.
$$

In both cases, we obtain $G_1(t,s)\leq\dfrac{1}{\Gamma(\alpha+\beta)}$. □

Lemma 5. *The function $G_2(t,s)$ defined by (3.6) satisfies the following properties:*

(i) *$G_2(t,s)$ is a continuous function and $G_2(t,s)\geq 0$ for any $(t,s)\in[0,1]\times[0,1]$,*

(ii) *$G_2(t,s)\leq\dfrac{\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}$ for any $(t,s)\in[0,1]\times[0,1]$.*

Proof. **(i)** It is obvious.
(ii) Let $s \leq \tau_i$.

$$G_2(t,s) = \frac{\sum_{i=1}^{m-2} \eta_i t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]} \left((\tau_i(1-s))^{\alpha+\beta-\gamma-1} - (\tau_i - s)^{\alpha+\beta-\gamma-1}\right)$$
$$\leq \frac{\sum_{i=1}^{m-2} \eta_i}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]} \tau_i^{\alpha+\beta-\gamma-1}.$$

Let $\tau_i \leq s$.

$$G_2(t,s) = \frac{\sum_{i=1}^{m-2} \eta_i t^{\alpha+\beta-1}}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]} \left((\tau_i(1-s))^{\alpha+\beta-\gamma-1}\right)$$
$$\leq \frac{\sum_{i=1}^{m-2} \eta_i}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]} \tau_i^{\alpha+\beta-\gamma-1}.$$

The following inequality is obtained from both cases:

$$G_2(t,s) \leq \frac{\sum_{i=1}^{m-2} \eta_i}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]} \tau_i^{\alpha+\beta-\gamma-1}.$$

□

Finally, by means of equation (3.4), we get:

Lemma 6. *The $G(t,s)$ function defined by (3.6) provides the following inequality:*

$$G(t,s) \leq \frac{1}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i \tau_i^{\alpha+\beta-\gamma-1}\right]}$$

for any $(t,s) \in [0,1] \times [0,1]$.

Proof. This Lemma is obvious from Lemma 4 and Lemma 5. □

4. Fixed point theorems

The following fixed point theorems are fundamental and essential to the proofs our main results.

Theorem 1 (Banach contraction mapping principle, [10]). *Let (X,d) be a nonempty complete metric space, and let $T: X \to X$ be a contraction, i.e., there exists a number $0 < \rho < 1$ such that $d(Tx,Ty) \leq \rho d(x,y)$. Then the operator T has a unique fixed point $x^* \in X$.*

Theorem 2 (Krasnoselskii's fixed point theorem[7]). *Let M be a closed, convex, bounded and nonempty subset of a Banach space E. Let A,B be the operators such that*

(i) *A is compact and continuous,*
(ii) *B is a contraction mapping,*
(iii) *$Ax+By \in M$ whenever $x,y \in M$.*

Then there exists $z \in M$ such that $z = Az + Bz$.

Theorem 3. *Suppose that $f\colon [0,1]\times[0,\infty) \to [0,\infty)$ is a continuous function, for all $t \in [0,1]$ and $f(t,0)$ is not identically null on $[0,1]$.*

(H3) *For all $t \in [0,1]$ and $u,v \in [0,\infty)$, we have*

$$|f(t,u) - f(t,v)| \leq \sigma(t)\cdot|u-v|,$$

where $\sigma(t) \in C([0,1];[0,\infty))$. Then, there exists a unique positive solution for problem (3.1) under the following condition $r_1 < 1$, where

$$r_1 = \frac{1}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\cdot\sigma^* \text{ and } \sigma^* = \int_0^1 \sigma(s)\mathrm{d}s.$$

Proof. We consider the Banach space $E = C[0,1]$ with the maximum norm. Define $T\colon E \to E$ by

$$(Tu)(t) = \int_0^1 G(t,s)f(s,u(s))\mathrm{d}s$$

and setting $\sup_{0\leq t\leq 1} f(t,0) = F$.

We consider the following set

$$B_r = \{u \in E : \| u \| \leq r\},$$

where

$$r \geq \frac{r_2}{1-r_1}, \text{ with } r_2 = \frac{1}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\cdot F.$$

For each $t \in [0,1]$ and $u \in B_r$, we have

$$\begin{aligned}
|\,(Tu)(t)\,| &\leq \left[\frac{1}{\Gamma(\alpha+\beta)} + \frac{\sum_{i=1}^{m-2}\eta_i}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\cdot\tau_i^{\alpha+\beta-\gamma-1}\right] \\
&\quad\cdot\left[\int_0^1 |\,f(s,u(s)) - f(s,0)\,|\,\mathrm{d}s\right] + F \\
&\leq \frac{1}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\cdot\left[\int_0^1 \sigma(s)\mathrm{d}s\cdot \| u \| + F\right]
\end{aligned}$$

$$= \frac{1}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\cdot[\sigma^*\cdot \parallel u \parallel +F].$$

This means that $\parallel Tu \parallel \leq r$. Therefore $TB_r \subseteq B_r$. Next, we prove that T is a contraction mapping for $u,v \in B_r$. We have

$$\begin{aligned} \mid (Tu)(t)-(Tv)(t) \mid &\leq \int_0^1 G(t,s)\cdot \mid f(s,u(s))-f(s,v(s)) \mid \mathrm{d}s \\ &\leq \frac{1}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\cdot\int_0^1 \sigma(s)\mid u(s)-v(s)\mid \mathrm{d}s \\ &\leq \frac{1}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\cdot\sigma^*\cdot \parallel u-v \parallel . \end{aligned}$$

Since $r_1<1$, T is a contraction. Therefore, by Theorem 1, problem (3.1) has a unique positive solution. □

Example 1. Let be $\alpha=\frac{18}{11}$, $\beta=\frac{15}{11}$, $\gamma=\frac{1}{2}$, $m=3$, $\eta_1=1$ and $\tau_1=\frac{1}{2}$ for the boundary value problem (3.1).

Consider the following boundary value problem:

$$\begin{cases} D_{0^+}^{\frac{18}{11}}\,{}^cD_{0^+}^{\frac{15}{11}}u+\frac{t^3}{400}.(1+u)=0, \quad t\in(0,1), \\ u(0)=u'(0)={}^cD_{0^+}^{\frac{15}{11}}u(0)=0, \\ D_{0^+}^{\frac{1}{2}}u(1)=D_{0^+}^{\frac{1}{2}}u(\frac{1}{2}), \end{cases}$$

where $\sigma(t)=\frac{t^3}{400}$ and $f(t,u)=\frac{t^3}{400}.(1+u)$.

We obtain $\sigma^*=\frac{1}{1600}$ and $r_1\cong 0,000483$ by some simple calculations. In this case all the conditions of Theorem 3 are satisfied. Hence, by Theorem 3, we prove that the boundary value problem has one positive solution.

Theorem 4. *Assume that (H1)-(H3) hold and* $f\colon [0,1]\times[0,\infty)\to[0,\infty)$ *is a continuous function. Furthermore, we suppose*

(H4) $f(t,u)\leq\theta(t)$*, for all* $(t,u)\in[0,1]\times[0,\infty)$ *and* $\theta\in C([0,1],\mathbb{R}^+)$.

Then problem (3.1) *has at least one positive solution on* $[0,1]$ *if* $R<1$*, where*

$$R=\frac{\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\cdot\sigma^*.$$

Proof. We consider the closed ball

$$B_\rho=\{u\in E:\parallel u \parallel\leq\rho\}$$

with fixed radius ρ:

$$\rho \geq \frac{\| \theta \|}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}.$$

We define the operators T_1 and T_2 on B_ρ as

$$(T_1u)(t) = \int_0^1 G_1(t,s)f(s,u(s))\mathrm{d}s,$$
$$(T_2u)(t) = \int_0^1 G_2(t,s)f(s,u(s))\mathrm{d}s.$$

For $u, v \in B_\rho$, we have

$$(T_1)u)(t) \leq \frac{\| \theta \|}{\Gamma(\alpha+\beta)},$$
$$(T_2v)(t) \leq \frac{\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\cdot \| \theta \|}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}.$$

Consequently,

$$\| T_1u+T_2v \| \leq \frac{\| \theta \|}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}.$$

Then $T_1u+T_2v \in B_\rho$.

In the following, we prove that T_2 is a contraction.

$$\begin{aligned}
\mid (T_2u)(t)-(T_2v)(t) \mid &\leq \int_0^1 G_2(t,s) \mid f(s,u(s))-f(s,v(s)) \mid ds \\
&\leq \frac{\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\int_0^1 \sigma(s) \mid u(s)-v(s) \mid \mathrm{d}s \\
&\leq \frac{\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}}{\Gamma(\alpha+\beta)\left[1-\sum_{i=1}^{m-2}\eta_i\tau_i^{\alpha+\beta-\gamma-1}\right]}\cdot\sigma^*\cdot \| u-v \| .
\end{aligned}$$

Since $R<1$, we conclude that T_2 is a contraction. Now, we show that T_1 is a completely continuous operator. Continuity of f implies that the operator T_1 is continuous. Also, T_1 is uniformly bounded on B_ρ as

$$(T_1u)(t) \leq \frac{\| \theta \|}{\Gamma(\alpha+\beta)}.$$

Now, we prove the compactness of operator $(T_1 u)$. We have for $u \in B_\rho$, $t_1, t_2 \in [0,1]$ and $t_1 < t_2$.

$$
\begin{aligned}
\left|(T_1 u)(t_1) - (T_1 u)(t_2)\right| &\leq \int_0^1 \mid G_1(t_1,s) - G_1(t_2,s) \mid \cdot \mid f(s,u(s)) \mid \mathrm{d}s \\
&\leq \frac{\| \theta \|}{\Gamma(\alpha+\beta)} \bigg| \int_0^{t_1} (t_1^{\alpha+\beta-1}(1-s)^{\alpha+\beta-\gamma-1} - (t_1-s)^{\alpha+\beta-1} \\
&\quad - (t_2^{\alpha+\beta-1}(1-s)^{\alpha+\beta-\gamma-1} + (t_2-s)^{\alpha+\beta-1}))\mathrm{d}s \\
&\quad + \int_{t_1}^{t_2} (t_1^{\alpha+\beta-1}(1-s)^{\alpha+\beta-\gamma-1} - t_2^{\alpha+\beta-1}(1-s)^{\alpha+\beta-\gamma-1} \\
&\quad + (t_2-s)^{\alpha+\beta-1})ds + \int_{t_2}^1 (t_1^{\alpha+\beta-1}(1-s)^{\alpha+\beta-\gamma-1} \\
&\quad - t_2^{\alpha+\beta-1}(1-s)^{\alpha+\beta-\gamma-1})\mathrm{d}s \bigg|.
\end{aligned}
$$

So, $\mid T_1 u(t_1) - T_1 u(t_2) \mid \to 0$ tends to zero when $t_1 \to t_2$, thus T_1 is relatively compact on B_ρ. Hence, by the Arzela Ascoli theorem, T_1 is compact on B_ρ. Thus, by Krasnoselskii's fixed point theorem, problem (3.1) has one positive solution in E. □

Example 2. Let be $\alpha = \frac{15}{11}$, $\beta = \frac{18}{11}$, $\gamma = \frac{1}{2}$, $m = 3$, $\eta_1 = 1$ and $\tau_1 = \frac{1}{2}$ for the boundary value problem (3.1).

Consider the following boundary value problem:

$$
\begin{cases}
D_{0^+}^{\frac{15}{11}}\, {}^c D_{0^+}^{\frac{18}{11}} u + \frac{t^3}{600} . \frac{u}{1+u} = 0, t \in (0,1) \\
u(0) = u'(0) = {}^c D_{0^+}^{\frac{18}{11}} u(0) = 0, \\
D_{0^+}^{\frac{1}{2}} u(1) = D_{0^+}^{\frac{1}{2}} u(\frac{1}{2}),
\end{cases}
$$

where $\sigma(t) = \frac{t^3}{600}$, $f(t,u) = \frac{t^3}{600} . \frac{u}{1+u}$ and $\theta(t) = \frac{t^3}{600}$.

Through calculation, we get $\sigma^* = \frac{1}{2400}$ and $R \cong 0,000113$. We conclude that all the assumptions of Theorem 4 are verified. Thus, problem has at least one positive solution.

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