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$B-$MAXIMAL AND $B-$SINGULAR INTEGRAL OPERATORS IN $B-$LOCAL MORREY-LORENTZ SPACES

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Abstract. In this article, the boundedness of $B-$maximal M_γ, $B-$singular integral T_γ and $B-$maximal singular integral $\mathcal{T}_\gamma$ operators on $B-$local Morrey-Lorentz spaces are obtained with the use of rearrangement inequalities and Hardy inequality.

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1. INTRODUCTION

Lorentz spaces have first been introduced by G. G. Lorentz and Lorentz spaces are very useful in the theory of interpolation. These spaces are Banach spaces and generalizations of Lebesgue spaces. The Lorentz space $L_{p,q}(\mathbb{R}^n)$ is known as the set all of measurable functions f such that

$$\|f\|_{L_{p,q}(\mathbb{R}^n)} := \|t^{\frac{1}{p}-\frac{1}{q}} f^*(t)\|_{L_q(0,\infty)} < \infty, \quad 0 < p,q \leq \infty.$$

Here f^* denotes non-increasing rearrangement of f and defined by

$$f^*(t) = \inf\{\lambda > 0\colon \ |\{y \in \mathbb{R}^n\colon \ |f(y)| > \lambda\}| \leq t\}, \quad t \in (0,\infty).$$

Functional $\|\cdot\|_{L_{p,q}}$ is a norm if and only if $1 \leq q \leq p$ or $p = q = \infty$. If $p = q = \infty$, then $L_{\infty,\infty}(\mathbb{R}^n) \equiv L_\infty(\mathbb{R}^n)$. One can easily see that $L_{p,p}(\mathbb{R}^n) \equiv L_p(\mathbb{R}^n)$ and $L_{p,\infty}(\mathbb{R}^n) \equiv WL_p(\mathbb{R}^n)$. It is clear that $L_{p,q} \subset L_p \subset L_{p,r} \subset WL_p$ for $0 < q \leq p \leq q \leq r \leq \infty$. For more details, we refer to [8, 22, 23].

On Lorentz spaces, boundedness problem of classical singular integral operators and singular integral operators related to Laplace-Bessel differential operator

$$\Delta_B := \sum_{i=1}^{k} \frac{\partial^2}{\partial x_i^2} + \frac{\gamma_i}{x_i}\frac{\partial}{\partial x_i} + \sum_{i=k+1}^{n} \frac{\partial^2}{\partial x_i^2} \quad 1 \leq k \leq n,$$

have been studied by many researchers [1, 3, 5, 15–19, 22, 23, 26, 29, 30]. Also, boundedness of singular integral operators related to Laplace-Bessel differential operator defined on variable exponent Lorentz spaces have been obtained by Aykol and Kaya [7]. Convolution type operators are generated by generalized shift operator

$$T^y f(x) := C_{\gamma,k}\int_0^\pi \cdots \int_0^\pi f\left[(x_1,y_1)_{\alpha_1},\ldots,(x_k,y_k)_{\alpha_k},x''-y''\right] \mathrm{d}\gamma(\alpha),$$

where $C_{\gamma,k} = \pi^{-\frac{k}{2}}\Gamma(\frac{\gamma_i+1}{2})[\Gamma(\frac{\gamma_i}{2})]^{-1}$, $(x_i,y_i)_{\alpha_i} = (x_i^2-2x_iy_i\cos\alpha_i+y_i^2)^{\frac{1}{2}}$, $1\le i\le k$, $1\le k\le n$, and $\mathrm{d}\gamma(\alpha)=\prod_{i=1}^{k}\sin^{\gamma_i-1}\alpha_i\,\mathrm{d}\alpha_i$ ([20,21]).

The $B-$convolution operator is defined as:

$$(f\otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)T^y g(x)(y')^\gamma \mathrm{d}y.$$

Here, $\mathbb{R}^n_{k,+} = \{x\in\mathbb{R}^n\colon\ x_1>0,\ldots,x_k>0,\ 1\le k\le n\}$ denotes the part of the Euclidean space $\mathbb{R}^n$ and $\gamma=(\gamma_1,\ldots,\gamma_k)$, $\gamma_1>0,\ldots,\gamma_k>0$, $|\gamma|=\gamma_1+\cdots+\gamma_k$. Denote $x=(x',x'')$, $x'=(x_1,\ldots,x_k)\in\mathbb{R}^k$, and $x''=(x_{k+1},\ldots,x_n)\in\mathbb{R}^{n-k}$.

$M^{\mathrm{loc}}_{p,q,\lambda}(\mathbb{R}^n)$ local Morrey-Lorentz spaces have first been introduced by Aykol et al. ([4]) with a finite quasi-norm

$$\|f\|_{M^{\mathrm{loc}}_{p,q,\lambda}(\mathbb{R}^n)} = \sup_{r>0} r^{-\frac{\lambda}{q}}\left\|t^{\frac{1}{p}-\frac{1}{q}}f^*(t)\right\|_{L_q(0,r)}.$$

These spaces are generalizations of Lorentz spaces and $M^{\mathrm{loc}}_{p,q,0}(\mathbb{R}^n)=L_{p,q}(\mathbb{R}^n)$. The boundedness of maximal operator in $M^{\mathrm{loc}}_{p,q,\lambda}(\mathbb{R}^n)$ spaces have been obtained in [4]. Then, in [14], authors have proved that Hardy-Littlewood maximal, Calderón-Zygmund and maximal Calderón-Zygmund operators are bounded in these spaces. Also, in [6], Aykol et al. have shown that Hilbert transform is bounded in local and weak local Morrey-Lorentz spaces. On the other hand, generalized versions of these operators with the Laplace-Bessel differential operator have been studied in various function spaces by many mathematicians [2,3,5,11,15–18,20,21,24,25]. The above results inspire us to investigate the boundedness of $B-$maximal, $B-$singular integral and $B-$maximal singular integral operator defined on $B-$local Morrey-Lorentz spaces. Maximal operators have a crucial role in PDEs, singular integrals and the differentiability properties of functions. In the present study, we consider $B-$maximal operator defined by (see [13])

$$M_\gamma f(x) = \sup_{r>0}|B_+(0,r)|_\gamma^{-1}\int_{B_+(0,r)} T^y|f(x)|(y')^\gamma \mathrm{d}y,$$

where $B_+(x,r)=\{y\in\mathbb{R}^n_{k,+}\colon\ |x-y|<r\}$. For a measurable set $B_+(0,r)\subset\mathbb{R}^n_{k,+}$, we have

$$|B_+(0,r)|_\gamma = \int_{B_+(0,r)}(x')^\gamma \mathrm{d}x = \omega(n,k,\gamma)r^Q,$$

where $\omega(n,k,\gamma) = \dfrac{\pi^{\frac{n-k}{2}}}{2^k}\displaystyle\prod_{i=1}^{k}\frac{\Gamma\left(\frac{\gamma_i+1}{2}\right)}{\left(\frac{\gamma_i}{2}\right)}$, $Q = n + |\gamma|$.

$B-$singular integral operator is defined as

$$\begin{aligned}T_\gamma f(x) &= p.v.\int_{\mathbb{R}^n_{k,+}} \frac{\Omega(\theta)}{|y|^Q}[T^y f(x)](y')^\gamma \mathrm{d}y \\ &= \lim_{\varepsilon\to 0}\int_{\{y\in\mathbb{R}^n_{k,+}:\,|y|>\varepsilon\}} \frac{\Omega(\theta)}{|y|^Q}[T^y f(x)](y')^\gamma \mathrm{d}y = \lim_{\varepsilon\to 0} T_{\gamma,\varepsilon} f(x), \qquad (1.1)\end{aligned}$$

where $\theta = y/|y|$, and $\Omega(\theta)$ belong to some function spaces on the semi-sphere $S_{k,+} = \{x\in\mathbb{R}^n_{k,+}:\ |x| = 1\}$ and also satisfy the "cancellation" condition with $\mathrm{d}\sigma(\theta)$ is the area element of the sphere $|\theta| = 1$,

$$\int_{S_{k,+}} \Omega(\theta)(\theta')^\gamma \mathrm{d}\sigma(\theta) = 0.$$

$B-$singular integral operator is a convolution type operator, where the kernel of this operator is $K(y) = \dfrac{\Omega(\theta)}{|y|^Q}$ and thus it can be written as $T_\gamma f(x) = (K\otimes f)(x)$. Kernel of $B-$singular integral operator $K(x)$ satisfies the followings:

- $\sup\limits_{\theta\in S_{k,+}} |K(\theta)| < \infty$;
- $\displaystyle\int_{S_{k,+}} |K_\varepsilon(x)|(x')^\gamma \mathrm{d}\sigma \le C$ for all ε and $x\in\mathbb{R}^n_{k,+}$, where C is a constant independent of ε and $\mathrm{d}\sigma$;
- $\displaystyle\int_{|x|\ge\delta} |K_\varepsilon(x)|(x')^\gamma \mathrm{d}x \to 0$ as $\varepsilon\to 0$ for any fixed $\delta > 0$;
- The kernel of the $B-$singular integral operator satisfies Hörmander's condition:

  $$\int_{B_+(0,2|x|)} |K(y) - T^y K(x)|(y')^\gamma \mathrm{d}y \le C, \quad y\ne 0,$$

  where C is an absolute constant;
- There exist $\delta > 0$ and $C > 0$ for all distinct $x,y\in\mathbb{R}^n_{k,+}$ with $2|x| < |y|$ such that $|K(x)| \le C|x|^{-Q}$ and

  $$|T^y K(x) - K(y)| \le C|x|^\delta |y|^{-Q-\delta}.$$

Lemma 1 ([11]). *Let* $\sup\limits_{x\in\mathbb{R}^n_{k,+}} |\Omega(\theta)| = M < +\infty$. *If* $|x| \ge |y|/2$, *then*

$$|T^y K(x) - K(y)| \le CM\,|y|^{-Q},$$

where $C > 0$ *is a constant.*

$B-$maximal singular integral operator $\mathcal{T}_\gamma$ is one of the important tools of harmonic analysis and its applications that is defined by

$$\mathcal{T}_\gamma f(x) = \sup_{\varepsilon>0} |T_{\gamma,\varepsilon} f(x)|,$$

and behavior of this operator in Lebesgue spaces have been investigated in [11].

The purpose of this study is to obtain that $B-$maximal M_γ, $B-$singular integral T_γ and $B-$maximal singular integral $\mathcal{T}_\gamma$ operators are bounded on $B-$local Morrey-Lorentz spaces with the use of rearrangement inequalities and Hardy inequality.

The draft of this study is as follows: Section 1 is devoted to introduction. In Section 2, we recall some basic notions and some known results which we need throughout the paper. In Section 3, we have obtained that $B-$maximal operators in $B-$local Morrey-Lorentz spaces are bounded. Section 4 is devoted to the boundedness of $B-$singular integral and $B-$maximal singular integral operators.

Throughout the paper, C denotes a positive constant independent of appropriate parameters and not necessary the same at each occurrence.

2. Preliminaries

Given any measurable set E with $|E|_\gamma = \int_E (x')^\gamma \mathrm{d}x$ and a measurable function $f\colon \mathbb{R}^n_{k,+} \to \mathbb{R}$, the $\gamma-$rearrangement of f in decreasing order is defined by

$$f^*_\gamma(t) = \inf\left\{s>0 : f_{*,\gamma}(s) \le t\right\}, \quad \forall t \in (0,\infty),$$

where $f_{*,\gamma}(s)$ denotes the $\gamma-$distribution function of f given by

$$f_{*,\gamma}(s) = \left|\left\{x\in\mathbb{R}^n_{k,+} : |f(x)|>s\right\}\right|_\gamma.$$

The average function of f^{**}_γ is defined as

$$f^{**}_\gamma(t) = \frac{1}{t}\int_0^t f^*_\gamma(s)\mathrm{d}s, \quad t>0,$$

and

$$(f+g)^{**}_\gamma(t) \le f^{**}_\gamma(t) + g^{**}_\gamma(t)$$

is valid [10].

Now, we give some characteristics of $\gamma-$rearrangement of functions:

- if $0<p<\infty$, then

$$\int_{\mathbb{R}^n_{k,+}} |f(x)|^p (x')^\gamma \mathrm{d}x = \int_0^\infty (f^*_\gamma(t))^p \mathrm{d}t;$$

- for any $t>0$,

$$\sup_{|E|_\gamma = t} \int_E |f(x)|(x')^\gamma \mathrm{d}x = \int_0^t f^*_\gamma(s)\mathrm{d}s; \tag{2.1}$$

- $$\int_{\mathbb{R}^n_{k,+}} |f(x)g(x)|(x')^{\gamma}\mathrm{d}x \le \int_0^{\infty} f^*_{\gamma}(t)g^*_{\gamma}(t)\mathrm{d}t;$$
- it is well known that
$$(f+g)^*_{\gamma}(t) \le f^*_{\gamma}(t/2) + g^*_{\gamma}(t/2) \tag{2.2}$$
holds [8, 10, 28].

Definition 1 ([22]). $L_{p,q,\gamma}(\mathbb{R}^n_{k,+})$ Lorentz space is the set of all measurable functions $f \in \mathbb{R}^n_{k,+}$ such that

$$\|f\|_{L_{p,q,\gamma}(\mathbb{R}^n_{k,+})} = \left\| t^{\frac{1}{p}-\frac{1}{q}} f^*_{\gamma}(t) \right\|_{L_q(0,\infty)} < \infty.$$

If $0 < p \le \infty$, $q = \infty$, then $L_{p,\infty,\gamma}(\mathbb{R}^n_{k,+}) = WL_{p,\gamma}(\mathbb{R}^n_{k,+})$, where $WL_{p,\gamma}(\mathbb{R}^n_{k,+})$ is weak Lebesgue space of all measurable functions f such that

$$\|f\|_{WL_{p,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{t>0} t^{1/p} f^*_{\gamma}(t) < \infty, \quad 1 \le p < \infty.$$

If $p = q = \infty$ or $1 \le q \le p$, then functional $\|f\|_{p,q,\gamma}$ is a norm [8, 15, 28]. However if $p = q = \infty$, then $L_{\infty,\infty,\gamma}(\mathbb{R}^n_{k,+}) = L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$.

In case $0 < p, q \le \infty$, the functional $\|\cdot\|^*_{L_{p,q,\gamma}}$ is given by

$$\|f\|^*_{L_{p,q,\gamma}} = \|f\|^*_{L_{p,q,\gamma}(0,\infty)} = \|t^{\frac{1}{p}-\frac{1}{q}} f^{**}_{\gamma}(t)\|_{L_q(0,\infty)},$$

which is a norm on $L_{p,q,\gamma}(\mathbb{R}^n_{k,+})$ for $1 < p < \infty$, $1 \le q \le \infty$ or $p = q = \infty$.

If $1 < p \le \infty$, $1 \le q \le \infty$, then

$$\|f\|_{p,q,\gamma} \le \|f\|^*_{p,q,\gamma} \le \frac{p}{p-1}\|f\|_{p,q,\gamma},$$

that is, $\|f\|_{p,q,\gamma}$ and $\|f\|^*_{p,q,\gamma}$ are equivalent.

Definition 2 ([12]). Let $1 \le p < \infty$, and $0 \le \lambda \le Q$. $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ $B-$Morrey space is the set of all measurable functions with $f \in L^{\text{loc}}_{p,\gamma}(\mathbb{R}^n_{k,+})$ such that

$$\|f\|_{L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{x\in\mathbb{R}^n_{k,+}, r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_{p,\gamma}(B_+(x,r))} < \infty.$$

If $\lambda = 0$, then $L_{p,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,\gamma}(\mathbb{R}^n_{k,+})$; if $\lambda < 0$ or $\lambda > Q$, then $L_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n_{k,+}$. Also, the weak $B-$Morrey space $WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ is the set of all functions $f \in WL^{\text{loc}}_{p,\gamma}(\mathbb{R}^n_{k,+})$ with the following norm

$$\|f\|_{WL_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{x\in\mathbb{R}^n_{k,+}, r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_{p,\gamma}(B_+(x,r))} < \infty.$$

Definition 3 ([9])**.** Let $0 \le p < \infty$ and $0 \le \lambda \le 1$. $LM_{p,\lambda} \equiv LM_{p,\lambda}(0,\infty)$ local Morrey space is the set of all functions $f \in L_p^{\mathrm{loc}}(0,\infty)$ such that

$$\|f\|_{LM_{p,\lambda}(0,\infty)} = \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{L_p(0,r)} < \infty.$$

Moreover, $WLM_{p,\lambda} \equiv WLM_{p,\lambda}(0,\infty)$ denotes the weak local Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(0,\infty)$ such that

$$\|f\|_{WLM_{p,\lambda}(0,\infty)} = \sup_{r>0} r^{-\frac{\lambda}{p}} \|f\|_{WL_p(0,r)} < \infty.$$

Definition 4. Let $0 < p,q \le \infty$ and $0 \le \lambda \le 1$. $M^{\mathrm{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ $B-$local Morrey-Lorentz space is the set of all measurable functions with quasi-norm

$$\|f\|_{M^{\mathrm{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f^*_\gamma(t)\|_{L_q(0,r)} < \infty.$$

If $\lambda < 0$ or $\lambda > 1$, then $M^{\mathrm{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+}) = \Theta$, where Θ is the set of all functions equivalent to 0 on $\mathbb{R}^n_{k,+}$. Also,

$$M^{\mathrm{loc}}_{p,q,0,\gamma}(\mathbb{R}^n_{k,+}) = L_{p,q,\gamma}(\mathbb{R}^n_{k,+}) \quad \text{and} \quad M^{\mathrm{loc}}_{p,p,\lambda,\gamma}(\mathbb{R}^n_{k,+}) \equiv M^{\mathrm{loc}}_{p,0,\gamma}(\mathbb{R}^n_{k,+}).$$

$WM^{\mathrm{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ weak $B-$local Morrey-Lorentz space is the set of all measurable functions with the quasinorm

$$\|f\|_{WM^{\mathrm{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})} = \sup_{r>0} r^{-\frac{\lambda}{q}} \|t^{\frac{1}{p}-\frac{1}{q}} f^*_\gamma(t)\|_{WL_q(0,r)} < \infty.$$

Definition 5 ([27])**.** Let $\beta \in \mathbb{R}$ and φ be a measurable function on $(0,\infty)$. The weighted Hardy operators H^β and $\mathcal{H}^\beta$, on which the power weights acts, are defined as

$$H^\beta\varphi(t) = t^{\beta-1}\int_0^t \frac{\varphi(y)}{y^\beta}\mathrm{d}y, \qquad \mathcal{H}^\beta\varphi(t) = t^\beta \int_t^\infty \frac{\varphi(y)}{y^{\beta+1}}\mathrm{d}y.$$

For the proof of our main theorems, we need the following Hardy inequalities:

Theorem 1 ([6, 27])**.** *Let* $\beta \in \mathbb{R}$, $0 \le \lambda < 1$.

i. *If* $1 \le q < \infty$ *and* $\beta < \frac{1}{q'} + \frac{\lambda}{q}$, *then*

$$\left\|H^\beta\varphi\right\|_{LM_{q,\lambda}(0,\infty)} \le C\|\varphi\|_{LM_{q,\lambda}(0,\infty)}.$$

ii. *If* $1 < q < \infty$ *and* $\beta = \frac{1}{q'} + \frac{\lambda}{q}$, *then*

$$\left\|H^\beta\varphi\right\|_{WLM_{q,\lambda}(0,\infty)} \le C\|\varphi\|_{LM_{q,\lambda}(0,\infty)}.$$

iii. *If* $1 \le q < \infty$ *and* $\beta > \dfrac{\lambda-1}{q}$, *then*

$$\left\| \mathcal{H}^{\beta}\varphi \right\|_{LM_{q,\lambda}(0,\infty)} \le C \left\| \varphi \right\|_{LM_{q,\lambda}(0,\infty)}.$$

iv. *If* $1 < q < \infty$ *and* $\beta = \dfrac{\lambda-1}{q}$, *then*

$$\left\| \mathcal{H}^{\beta}\varphi \right\|_{WLM_{q,\lambda}(0,\infty)} \le C \left\| \varphi \right\|_{LM_{q,\lambda}(0,\infty)}.$$

3. $B-$MAXIMAL OPERATORS IN $B-$LOCAL MORREY-LORENTZ SPACES

We first give sharp rearrangement inequality for $B-$maximal operator. By using this inequality, we get $B-$maximal operator is bounded in $M_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Lemma 2 ([3]). *Sharp rearrangement inequality for $B-$maximal operator is given by*

$$(M_\gamma f)^*_\gamma(t) \le C f^{**}_\gamma(t), \quad t > 0, \tag{3.1}$$

where $C = C(n,\gamma) > 0$ *is a constant.*

Theorem 2. *Let* $1 \le q \le \infty$, $0 \le \lambda < 1$ *and* $\frac{q}{q+\lambda} \le p \le \infty$.

(i) *If* $\frac{q}{q+\lambda} < p < \infty$, *then $B-$maximal operator M_γ is bounded in $B-$local Morrey-Lorentz space* $M^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

(ii) *If* $p = \frac{q}{q+\lambda}$, *then $B-$maximal operator M_γ is bounded from $B-$local Morrey-Lorentz space $M^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to weak $B-$local Morrey-Lorentz space* $WM^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

(iii) *If* $p = q = \infty$, *then $B-$maximal operator M_γ is bounded in* $L_{\infty,\gamma}(\mathbb{R}^n_{k,+})$.

Proof. (i) Let $\frac{q}{q+\lambda} < p < \infty$ and $f \in M^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. By Definition 4 and the inequality (3.1), we obtain

$$\begin{aligned}
\|M_\gamma f\|_{M^{\text{loc}}_{p,q,\lambda,\gamma}} &= \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} (M_\gamma f)^*_\gamma(t) \right\|_{L_q(0,r)} \\
&\le C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}} f^{**}_\gamma(t) \right\|_{L_q(0,r)} \\
&= C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\frac{1}{p}-\frac{1}{q}-1} \int_0^t f^*_\gamma(s)ds \right\|_{L_q(0,r)}.
\end{aligned}$$

Let $\beta = \frac{1}{p} - \frac{1}{q}$ and $\varphi = t^{\frac{1}{p}-\frac{1}{q}} f^*_\gamma(t)$. Then from Theorem 1, we get

$$\|M_\gamma f\|_{M^{\text{loc}}_{p,q,\lambda,\gamma}} \le C \sup_{r>0} r^{-\frac{\lambda}{q}} \left\| t^{\beta} \int_0^t \frac{\varphi(s)}{s^{\beta}} ds \right\|_{L_q(0,r)}$$

$$
\begin{aligned}
&= C\left\|H^{\beta}\varphi\right\|_{LM_{q,\lambda}(0,\infty)} \\
&\le C\|\varphi\|_{LM_{q,\lambda}(0,\infty)} \\
&= C\sup_{r>0} r^{-\frac{\lambda}{q}}\|\varphi\|_{L_q(0,r)} \\
&= C\sup_{r>0} r^{-\frac{\lambda}{q}}\left\|t^{\frac{1}{p}-\frac{1}{q}}f_{\gamma}^{*}(t)\right\|_{L_q(0,r)} \\
&= \|f\|_{M^{\mathrm{loc}}_{p,q,\lambda,\gamma}}.
\end{aligned}
$$

Hence, the desired result is obtained.

(ii) Let $p=\frac{q}{q+\lambda}$ and $f\in M^{\mathrm{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. From Definition 4 and inequality (3.1), we have

$$\|M_{\gamma}f\|_{WM^{\mathrm{loc}}_{\frac{q}{q+\lambda},q,\lambda,\gamma}} = C\sup_{r>0} r^{-\frac{\lambda}{q}}\left\|t^{1+\frac{\lambda-1}{q}}\int_0^t f_{\gamma}^{*}(s)\mathrm{d}s\right\|_{WL_q(0,r)} = C\left\|H^{\beta}\varphi\right\|_{WLM_{q,\lambda}(0,\infty)},$$

where $\beta=1+\frac{\lambda-1}{q}$ and $\varphi=t^{1+\frac{\lambda-1}{q}}f_{\gamma}^{*}(t)$. Therefore, we obtain by Theorem 1

$$\left\|H^{\beta}\varphi\right\|_{WLM_{q,\lambda}(0,\infty)} \le C\|\varphi\|_{LM_{q,\lambda}(0,\infty)} = C\|f\|_{M^{\mathrm{loc}}_{\frac{q}{q+\lambda},q,\lambda}}.$$

Then we get the operator M_{γ} is bounded from $B-$local Morrey-Lorentz space $M^{\mathrm{loc}}_{\frac{q}{q+\lambda},q,\lambda,\gamma}$ to weak $B-$local Morrey-Lorentz space $WM^{\mathrm{loc}}_{\frac{q}{q+\lambda},q,\lambda,\gamma}$.

(iii) Let $p=\infty$ and $f\in M^{\mathrm{loc}}_{\infty,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. From [14, Remark 2.1], it is well known that $M^{\mathrm{loc}}_{\infty,q,\lambda,\gamma}=\Theta$ for any $0<q<\infty$. Hence, we now consider the case of $q=\infty$. Since $M^{\mathrm{loc}}_{\infty,q,\lambda,\gamma}=WL_{p,\gamma}$ for $q=\infty$ and M_{γ} is bounded in $L_{\infty,\gamma}$, we obtain the desired result. □

Corollary 1. (i) *If $1\le p<\infty$, $0<\lambda\le 1$, then $B-$maximal operator M_{γ} is bounded in $M^{\mathrm{loc}}_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.*

(ii) *If $1<p<\infty$, $1\le q\le\infty$, then $B-$maximal operator M_{γ} is bounded in $L_{p,q,\gamma}(\mathbb{R}^n_{k,+})$.*

(iii) *If $1<q<\infty$, then $B-$maximal operator M_{γ} is bounded in $WL_{q,\gamma}(\mathbb{R}^n_{k,+})$.*

4. $B-$SINGULAR INTEGRAL OPERATORS IN $B-$LOCAL MORREY-LORENTZ SPACES

We first give sharp rearrangement inequality for $B-$singular integral operator. By using this inequality, we obtain that $B-$singular integral is bounded in $M^{\mathrm{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Theorem 3 ([18])**.** *Let $f,g\in\mathbb{R}^n_{k,+}$. Then*

$$(f\otimes g)^{**}_{\gamma}(t)\le C_{\gamma,k}\left(f^{**}_{\gamma}(t)\int_0^t g^{**}_{\gamma}(u)\mathrm{d}u+\int_t^{\infty} f^{*}_{\gamma}(u)g^{**}_{\gamma}(u)\mathrm{d}u\right),\quad t>0.$$

Theorem 4 ([18]). *If $K_\alpha \in WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})$, $0<\alpha<Q$, then*

$$(K_\alpha \otimes g)^*_\gamma(t) \le (K_\alpha \otimes g)^{**}_\gamma(t) \le C\left(t^{\alpha/Q-1}\int_0^t f^*_\gamma(s)\mathrm{d}s + \int_t^\infty s^{\alpha/Q-1} f^*_\gamma(s)\mathrm{d}s\right),$$

where $C = C_{\gamma,k}(Q/\alpha)^2\|K_\alpha\|_{WL_{Q/(Q-\alpha),\gamma}(\mathbb{R}^n_{k,+})}$.

Since $T_\gamma f$ is convolution type operator and from the above theorem, we can write

$$(T_\gamma f)^*_\gamma(t) \le (T_\gamma f)^{**}_\gamma(t) \le C\left(t^{-1}\int_0^t f^*_\gamma(s)\mathrm{d}s + \int_t^\infty s^{-1} f^*_\gamma(s)\mathrm{d}s\right). \tag{4.1}$$

Theorem 5. *Suppose that $f \in M^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1\le q\le\infty$, $0\le\lambda<1$, $\frac{q}{q+\lambda}\le p\le\frac{q}{\lambda}$ and $B-$singular integral operator $T_\gamma f$ exists a.e. $x\in\mathbb{R}^n_{k,+}$. Moreover,*

(i) *If $1\le q<\infty$, $\frac{q}{q+\lambda}<p<\frac{q}{\lambda}$, then $B-$singular integral operator is bounded in $B-$local Morrey-Lorentz space $M^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.*

(ii) *If $1<q<\infty$, $p=\frac{q}{q+\lambda}$, then $B-$singular integral operator is bounded from $M^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ to weak $B-$local Morrey-Lorentz space $WM^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.*

Proof. Let $1\le q\le\infty$, $0\le\lambda<1$, and $\frac{q}{q+\lambda}\le p\le\frac{q}{\lambda}$. Since f satisfies (4.1), $B-$singular integral operator $T_\gamma f$ exists a.e. $x\in\mathbb{R}^n_{k,+}$.

(i) Let $1\le q<\infty$, $0\le\lambda<1$, and $\frac{q}{q+\lambda}\le p<\frac{q}{\lambda}$ and $f\in M^{\text{loc}}_{p,q,\lambda,\gamma}$. From Definition 4 and by using inequality (4.1) and Minkowski's inequality, we have

$$\begin{aligned}
\|T_\gamma f\|_{M^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})} &= C\sup_{r>0} r^{-\lambda/q}\left\|t^{\frac{1}{p}-\frac{1}{q}}(T_\gamma f)^*_\gamma(t)\right\|_{L_q(0,r)}\\
&\le C\sup_{r>0} r^{-\lambda/q}\left\|t^{\frac{1}{p}-\frac{1}{q}}(T_\gamma f)^{**}_\gamma(t)\right\|_{L_q(0,r)}\\
&\le C\sup_{r>0} r^{-\lambda/q}\left\|t^{\frac{1}{p}-\frac{1}{q}}\left[t^{-1}\int_0^t f^*_\gamma(s)\mathrm{d}s + \int_t^\infty \frac{f^*_\gamma(s)}{s}\mathrm{d}s\right]\right\|_{L_q(0,r)}\\
&\le C\sup_{r>0} r^{-\lambda/q}\left\|t^{\frac{1}{p}-\frac{1}{q}-1}\int_0^t f^*_\gamma(s)\mathrm{d}s\right\|_{L_q(0,r)}\\
&\quad + \sup_{r>0} r^{-\lambda/q}\left\|t^{\frac{1}{p}-\frac{1}{q}}\int_t^\infty \frac{f^*_\gamma(s)}{s}\mathrm{d}s\right\|_{L_q(0,r)}\\
&= I_1 + I_2.
\end{aligned}$$

I_1 can be estimated using the same method as in the proof of Theorem 2. Let us estimate I_2:

$$\begin{aligned}
I_2 &= \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{1}{p}-\frac{1}{q}} \int_t^\infty \frac{f_\gamma^*(s)}{s} \mathrm{d}s \right\|_{L_q(0,r)} \\
&= \sup_{r>0} r^{-\lambda/q} \left\| t^{\beta} \int_t^\infty \frac{\varphi(s)}{s^{\beta+1}} \mathrm{d}s \right\|_{L_q(0,r)} \\
&= \sup_{r>0} r^{-\lambda/q} \left\| \mathcal{H}^{\beta}\varphi \right\|_{L_q(0,r)} \\
&= \left\| \mathcal{H}^{\beta}\varphi \right\|_{LM_{q,\lambda}(0,\infty)} \\
&= \|\varphi\|_{LM_{q,\lambda}(0,\infty)} \\
&= C \sup_{r>0} r^{-\lambda/q} \|\varphi\|_{LM_q(0,r)} \\
&= C \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t) \right\|_{L_q(0,r)} \\
&= \|f\|_{M^{\mathrm{loc}}_{p,q,\lambda,\gamma}},
\end{aligned} \tag{4.2}$$

where $\varphi(t) = t^{\frac{1}{p}-\frac{1}{q}} f_\gamma^*(t)$. Since $\frac{1}{p} - \frac{\lambda}{q} > 0$, the inequality $\beta > \frac{\lambda}{q} - \frac{1}{q}$ is valid for $\beta = \frac{1}{p} - \frac{1}{q}$. Therefore, we obtain $I_2 \le C\|f\|_{M^{\mathrm{loc}}_{p,q,\lambda,\gamma}}$. As a consequence, we get the desired result.

(ii) Let $p = \dfrac{q}{q+\lambda}$, $1 < q < \infty$ and $f \in M^{\mathrm{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$. From Definition 4 and with the use of inequality (4.1) and Minkowski's inequality, we have

$$\begin{aligned}
\|T_\gamma f\|_{WM^{\mathrm{loc}}_{q/(q+\lambda),q,\lambda,\gamma}(\mathbb{R}^n_{k,+})} &= C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\frac{\lambda-1}{q}} (T_\gamma f)_\gamma^*(t) \right\|_{WL_q(0,\infty)} \\
&\le C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\frac{\lambda-1}{q}} (T_\gamma f)_\gamma^{**}(t) \right\|_{WL_q(0,\infty)} \\
&\le C \sup_{r>0} r^{-\lambda/q} \left\| t^{\frac{\lambda-1}{q}} \int_0^t f_\gamma^*(s) \mathrm{d}s \right\|_{WL_q(0,\infty)} \\
&\quad + C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\frac{\lambda-1}{q}} \int_t^\infty \frac{f_\gamma^*(s)}{s} \mathrm{d}s \right\|_{WL_q(0,\infty)} \\
&= J_1 + J_2.
\end{aligned}$$

J_1 can be estimated using the same method as in the proof of Theorem 2. Let us estimate J_2. By Theorem 1, we have

$$J_2 = C \sup_{r>0} r^{-\lambda/q} \left\| t^{1+\frac{\lambda-1}{q}} \int_t^\infty \frac{f_\gamma^*(s)}{s} \mathrm{d}s \right\|_{WL_q(0,\infty)}$$

$$
\begin{aligned}
&= C\sup_{r>0} r^{-\lambda/q}\left\| t^{\beta}\int_t^{\infty}\frac{\varphi(s)}{s^{\beta+1}}ds\right\|_{WL_q(0,\infty)}\\
&= \sup_{r>0} r^{-\lambda/q}\left\|\mathcal{H}^{\beta}\varphi\right\|_{WL_q(0,\infty)}\\
&= C\left\|\mathcal{H}^{\beta}\varphi\right\|_{WLM_{q,\lambda}(0,\infty)}\\
&= \|\varphi\|_{LM_{q,\lambda}(0,\infty)}\\
&= C\sup_{r>0} r^{-\lambda/q}\|\varphi\|_{L_q(0,\infty)}\\
&= C\sup_{r>0} r^{-\lambda/q}\left\| t^{1+\frac{\lambda-1}{q}} f^*_{\gamma}(t)\right\|_{L_q(0,\infty)}\\
&= \|f\|_{M^{\text{loc}}_{\frac{q}{q+\lambda},q,\lambda,\gamma}},
\end{aligned}
$$

where $\beta = 1+\frac{(\lambda-1)}{q}$ and $\varphi(t) = t^{1+\frac{\lambda-1}{q}} f^*_{\gamma}(t)$. Therefore, we obtain that T_{γ} is bounded from $M^{\text{loc}}_{q/(q+\lambda),q,\lambda,\gamma}$ to $WM^{\text{loc}}_{q/(q+\lambda),q,\lambda,\gamma}$. Thus, the proof is completed. □

Theorem 6. *Suppose that* $f \in M^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$, $1 \le q \le \infty$, $0 \le \lambda < 1$, $\dfrac{q}{q+\lambda} \le p \le \dfrac{q}{\lambda}$ *and* (4.1) *is valid, then* $B-$*maximal singular integral operator* $\mathcal{T}_{\gamma} f$ *is finite a.e.* $x \in \mathbb{R}^n_{k,+}$. *Furthermore,*

(i) *If* $1 \le q < \infty$, $\dfrac{q}{q+\lambda} < p < \dfrac{q}{\lambda}$, *then* $\mathcal{T}_{\gamma}$ *is bounded on* $B-$*local Morrey-Lorentz space* $M^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

(ii) *If* $1 < q < \infty$, $p = \dfrac{q}{q+\lambda}$, *then* $\mathcal{T}_{\gamma}$ *is bounded from* $M^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$ *to weak* $B-$*local Morrey-Lorentz space* $WM^{\text{loc}}_{p,q,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Proof. Let $1 \le q \le \infty$, $0 \le \lambda < 1$, and $\dfrac{q}{q+\lambda} \le p \le \dfrac{q}{\lambda}$. Since f satisfies (4.1), the $B-$maximal singular integral operator $\mathcal{T}_{\gamma} f(x)$ is finite almost everywhere for $x \in \mathbb{R}^n_{k,+}$. By using inequality (4.1), the proof of theorem can be easily obtained in a similar manner of method of Theorem 5. □

REFERENCES

[1] R. P. Agarwal, S. Gala, and M. A. Ragusa, "A regularity criterion of the 3D MHD equations involving one velocity and one current density component in Lorentz space," *Z. Angew. Math. Phys.*, vol. 3, p. Art. No. 95, 2020, doi: 10.1007/s00033-020-01318-4.

[2] I. A. Aliev and A. D. Gadjiev, "Weighted estimates of multidimensional singular integrals generated by the generalized shift operator," *Mat. Sb.*, vol. 183, pp. 45–66, 1992, doi: 10.1070/SM1994v077n01ABEH003428.

[3] C. Aykol and A. Şerbetçi, "On the boundedness of fractional B-maximal operators in the Lorentz spaces $L_{p,q,\gamma}(\mathbb{R}^n_+)$," *An. St. Univ. Ovidius Constanta*, vol. 17, pp. 27–38, 2009.
[4] C. Aykol, V. S. Guliyev, and A. Şerbetçi, "Boundedness of the maximal operator in the local Morrey-Lorentz spaces," *J. Inequal Appl*, vol. 1, pp. 1–11, 2013, doi: 10.1186/1029-242X-2013-346.
[5] C. Aykol, V. S. Guliyev, and A. Şerbetçi, "The O'Neil inequality for the Hankel convolution operator and some applications," *Eurasian Math. J.*, vol. 4, pp. 8–191, 2013.
[6] C. Aykol, V. S. Guliyev, A. Küçükaslan, and A. Şerbetçi, "The boundedness of Hilbert transform in the local Morrey-Lorentz spaces," *Integral Transforms Spec. Funct.*, vol. 4, pp. 318–330, 2016, doi: 10.1080/10652469.2015.1121483.
[7] C. Aykol and E. Kaya, "B-maximal operators, B-singular integral operators and B-Riesz potentials in variable exponent Lorentz spaces," *Filomat*, vol. 17, pp. 5765–5774, 2023, doi: 10.2298/FIL2317765A.
[8] C. Bennett and R. Sharpley, *Interpolation of Operators*. Academic Press, Boston, 1988.
[9] V. I. Burenkov and H. V. Guliyev, "Necessary and sufficient conditions for boundedness of the maximal operator in the local Morrey-type spaces," *Stud. Math.*, vol. 163, no. 2, pp. 157–176, 2004.
[10] D. E. Edmunds and W. D. Evans, *Hardy operators, function spaces and embeddings*. Iger Monographs in Math., Springer-Verlag-Berlin, Heidelberg, 2004.
[11] I. Ekincioglu and E. Kaya, "Bessel type Kolmogorov inequalities on weighted Lebesgue spaces," *Applicable Analysis*, vol. 8, pp. 1634–1643, 2021, doi: 10.1080/00036811.2019.1659953.
[12] V. S. Guliyev, "Sobolev theorems for anisotropic Riesz-Bessel potentials on Morrey-Bessel spaces," *Dokl. Akad. Nauk.*, vol. 365, no. 2, pp. 155–156, 1999.
[13] V. S. Guliyev, "On maximal function and fractional integral, associated with the Bessel differential operator," *Mathematical Inequalities and Applications*, vol. 6, pp. 317–330, 2003.
[14] V. S. Guliyev, C. Aykol, A. Küçükaslan, and A. Şerbetçi, "Maximal operator and Calderón-Zygmund operators in local Morrey-Lorentz spaces," *Integral Transforms Spec. Funct.*, vol. 11, pp. 866–877, 2016, doi: 10.1080/10652469.2016.1227329.
[15] V. S. Guliyev, A. Şerbetçi, and I. Ekincioglu, "Necessary and sufficient conditions for the boundedness of rough B-fractional integral operators in the Lorentz spaces," *J. Math. Anal. Appl.*, vol. 336, pp. 425–437, 2007, doi: 10.1016/j.jmaa.2007.02.080.
[16] V. S. Guliyev, A. Şerbetçi, and I. Ekincioglu, "On boundedness of the generalized B-potential integral operators in the Lorentz spaces," *Int. Trans. and Special Func.*, vol. 18, pp. 885–895, 2007, doi: 10.1080/10652460701510980.
[17] V. S. Guliyev, A. Şerbetçi, and Z. V. Safarov, "Inequality of O'Neil-type for convolutions associated with the Laplace-Bessel differential operator and applications," *Mathematical Ineq. and Appl.*, vol. 11, pp. 99–112, 2007, doi: 10.7153/mia-11-06.
[18] V. S. Guliyev, Z. V. Safarov, and A. Şerbetçi, "On the rearrangement estimates and the boundedness of the generalized fractional integrals associated with the Laplace-Bessel differential operator," *Acta Math. Hung.*, vol. 119, pp. 201–217, 2008, doi: 10.1007/s10474-007-6107-5.
[19] R. Hunt, "On $L(p,q)$ spaces," *Enseign. Math.*, vol. 12, pp. 249–276, 1966.
[20] M. I. Klyuchantsev, "On singular integrals generated by the generalized shift operator I," *Sibirsk. Math. Zh.*, vol. 11, pp. 810–821, 1970, doi: 10.1007/BF00969676.
[21] B. M. Levitan, "Bessel function expansions in series and Fourier integrals," *Uspekhi Mat. Nauk 6.*, vol. 42, pp. 102–143, 1951.
[22] G. G. Lorentz, "Some new function spaces," *Ann. Math.*, vol. 51, pp. 37–55, 1950, doi: 10.2307/1969496.
[23] G. G. Lorentz, "On the theory of spaces λ," *Pac. J. Math.*, vol. 1, pp. 411–429, 1951.

[24] L. N. Lyakhov, "Multipliers of the mixed Fourier-Bessel transform," *Proc. Steklov Inst. Math.*, vol. 214, pp. 227–242, 1996.
[25] G. Mingione, "Gradient estimates below the duality exponent," *Math. Ann.*, vol. 346, pp. 571–627, 2010, doi: 10.1007/s00208-009-0411-z.
[26] J. Peetre, "On the theory of $L_{p,\lambda}$ spaces," *J. Funct. Anal.*, vol. 4, pp. 71–87, 1969, doi: 10.1016/0022-1236(69)90022-6.
[27] N. Samko, "Weighted Hardy and potential operators in Morrey spaces," *J. Funct. Spaces Appl.*, vol. 2012, 2012, doi: 10.1155/2012/678171.
[28] E. M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*. Princeton University Press, Princeton, 1971.
[29] M. Q. Wei and L. Y. Sun, "Boundedness for the modified fractional integral operator from mixed Morrey spaces to the Bounded Mean Oscillation space and Lipschitz spaces," *Journal of Function Spaces*, vol. 2022, p. Art. No. 4924127, 2022, doi: 10.1155/2022/4924127.
[30] T. L. Yee, K. L. Cheung, and K. P. Ho, "Integral operators on local Orlicz-Morrey spaces," *Filomat*, vol. 4, pp. 1231–1243, 2022, doi: 10.2298/FIL2204231Y.

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