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NUMERICAL SOLUTION OF TIME FRACTIONAL VOLTERRA INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this work, we develop and analyze a finite difference-based numerical scheme for a system of time-fractional Volterra-type integro-differential equations. The fractional derivative of order α with $\alpha \in (0,1)$ is considered in the Caputo sense. We provide sufficient conditions to ensure the existence of a unique solution. To construct the difference scheme, classical L1 discretization is employed for the fractional operator, and the integral part is approximated using the composite trapezoidal rule. The convergence analysis and error estimates are discussed. Having a mild singularity in the solution of the system, the proposed scheme achieves only α order accuracy. In the case of sufficiently smooth solutions, it achieves $(2-\alpha)$ order of accuracy. Finally, several numerical experiments are presented to support our theoretical findings and validate the proposed scheme.

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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary orders and has become an essential branch of mathematics. The non-local properties of fractional calculus allow us to model physical phenomena that have a dependence on time instants and previous histories simultaneously. In the last few years, fractional order differential equations (FDEs) and fractional order integro-differential equations (FIDEs) have attracted great attention due to their frequent appearance in various fields such as physics [6], electrochemistry [12], biology [13], viscoelasticity [16], engineering, and other area of applied mathematics. Likewise systems of integro-differential equations play a pivotal role in mathematical modeling of physical phenomena of various field of science and engineering, such as magnetic field, electric-circuit analysis, nano-hydrodynamics, control of systems, glass-forming process, dropwise condensation, statistics of polymer chains, wind ripple in the desert, modeling the competition between tumor cells and the immune system, modeling of fluid waves in to oceanography, the activity of interacting inhibitory and excitatory neurons, hereditary properties of various materials and processes etc. for more details, one may refer [2, 3, 5, 7] and references therein. Consequently, enormous attention has been paid by the researchers to develop approximate and numerical techniques for the solutions of FIDEs of physical interest since most FIDEs has yet to have an exact analytical form of solutions. For instance: Rawashdeh, in [20], applied the spline collocation method to obtain the numerical solution of FIDEs. In [18], Nazari and Shahmorad used the fractional differential transform method to approximate FIDEs with nonlocal boundary conditions. Arqub et al. developed a novel high-order algorithm that reproduces kernel approximation to solve Volterra integro-differential equations involving Atangana–Baleanu (ABC)-fractional derivative in [4]. A class of FIDEs was solved by Panda et al. in [19] using Adomian decomposition and homotopy perturbation methods. The solution of a fractional order differential/integro-differential equation generally has singularity in time [14]. However, the nature of the singularity is dependent on the regularity of the solution. We called the singularity weak/mild if the first derivative of the solution blows up as $t \to 0^+$. Recently, in [21] the authors proposed a finite difference scheme to approximate the Volterra type FIDEs having weak singularity. In [8], Ghosh and Mohapatra extended the previous work and showed a comparison study using two schemes based on the L1 and L1-2 discretization techniques.

In this article, we consider the following system of fractional order Volterra integro-differential equations (SFVIDEs):

$$\begin{cases} D_t^{\alpha}\mathbf{U}(t) + \mathbf{A}(t)\mathbf{U}(t) + \lambda \int_0^t \mathbf{K}(t,s)\mathbf{U}(s)ds = \mathbf{F}(t), \quad t \in (0,T], \\ \mathbf{U}(0) = \mathbf{U}_0, \end{cases} \tag{1.1}$$

where D_t^{α} denotes the Caputo fractional differential operator of order $\alpha \in (0,1)$, $T \in \mathbb{R}^+$, $\mathbf{U}(t) = [u^{(1)}(t), u^{(2)}(t), \ldots, u^{(n)}(t)]$ is the unknown vector function and

$$\begin{cases} \mathbf{F}(t) = [f^{(1)}(t), f^{(2)}(t), \ldots, f^{(n)}(t)],\ \lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n],\ \mathbf{U}_0 = [U_{01}, U_{02}, \ldots, U_{0n}], \\ \mathbf{A}(t) = [a_{ij}(t)],\ \mathbf{K}(t,s) = [K_{ij}(t,s)],\ \ i,j = 1(1)n. \end{cases} \tag{1.2}$$

$a_{ij} : [0,T] \to \mathbb{R}, K_{ij} : [0,T] \times [0,T] \to \mathbb{R}$ for $i,j = 1(1)n$, and the vector function $\mathbf{F}(t)$ are known and sufficiently smooth. The parameter $\lambda_i \in \mathbb{R}$. The model problem (1.1) in the present work is an academic example. However, system of integro-differential equations appear in many physical applications [5, 7].

In recent years, a large number of papers have been devoted to approximating a system of linear and nonlinear integro-differential equations numerically; in most cases, finding an exact analytic solution is usually not accessible. To name a few: Momani and Qaralleh [17] implemented the Adomian decomposition method for approximating the solution of system of fractional integro-differential equations. In

[25], Zurigat et al. used the homotopy analysis method to solve a system of linear and nonlinear fractional integro-differential equations. Akbar et al. employed an extension of the optimal homotopy asymptotic method to the fractional order integro-differential equations system in [1]. In [24], Youbi et al. investigated the approximate analytic solutions with the help of reproducing kernel function for a system of Volterra type FIDEs involving Caputo-Fabrizio derivative. In [11], Chebyshev pseudo-spectral collocation method is used to numerically solve a system of linear and nonlinear fractional integro-differential equations of Volterra type. Heydari et al. [10] presented the Chebyshev wavelet computational method by developing the operational matrix with the help of shifted Chebyshev polynomials for solving a class of system of nonlinear singular fractional Volterra integro-differential equations. Hesameddini and Shahbazi obtained the numerical solution of a system of fractional integro–differential equations based on the hybrid Bernstein Block–Pulse functions in [9]. In their study, the Riemann–Liouville fractional integral operator for hybrid Bernstein Block–Pulse functions were proposed and approximated by the Gauss quadrature formula. In [23], Wang et al. established an operational matrix using Bernoulli wavelets for solving coupled system of nonlinear fractional order integro-differential equations.

Observing the literature, we find that the majority of the solution methods are semi-analytical or spectral/collocation based and have limitations. Meanwhile, a little attempt has been made to solve a system of FIDEs using the finite difference technique. The article aims to introduce a numerical scheme by combining the well-known L1 technique and the composite trapezoidal rule to solve the fractional order system (1.1). The advantage of using the L1 technique is it uses the linear interpolation approximation for the integrand function. As a result, twice continuous differentiability is enough to achieve the optimal convergence rate. The simplicity of implementation and the well established convergence and stability analysis are also significant advantages of the L1 technique. The rest of this paper is organized as follows: In Section 2, we introduce few basic definitions of fractional calculus and establish the uniqueness of the solution. The difference scheme is derived in Section 3. Section 4 deals with the convergence analysis and error bounds. In Section 5, numerical experiments have been conducted, and concluding remarks are drawn in Section 6.

Notation 1. On a domain Ω, $\mathcal{C}^k(\Omega)$ represents the set of k times continuously differentiable functions. The positive constant C appears in several inequalities. $\Gamma(\cdot)$ denotes Euler Gamma function.

2. Preliminaries

Definition 1. Let $Y = \mathcal{C}([0,T],\mathbb{R})$ be the Banach space of all real valued continuous functions, then the supremum norm is defined as:

$$\|y(t)\| := \sup_{t\in[0,T]} |y(t)|.$$

Definition 2. A real function $\phi(x)$ is said to be in the space $\mathcal{C}_\mu$, $\mu \in \mathbb{R}$ if there exists a real number $\xi > \mu$ such that $\phi(x) = x^\xi \phi_1(x)$ where $\phi_1(x) \in \mathcal{C}(\mathbb{R})$. Clearly, $\mathcal{C}_\mu \subset \mathcal{C}_\gamma$ if $\gamma \leq \mu$. A function $\phi(x)$ is said to be in the space $\mathcal{C}_\mu^k$, $k \in \mathbb{N} \cup \{0\}$, if $\phi^{(k)}(x) \in \mathcal{C}_\mu$.

Definition 3. The Riemann-Liouville fractional integral of $\varphi(x) \in \mathcal{C}[a,b]$ of order $\widetilde{\alpha} \in \mathbb{R}^+$ is defined as:

$$J_x^{\widetilde{\alpha}} \varphi(x) = \frac{1}{\Gamma(\widetilde{\alpha})} \int_a^x (x-s)^{\widetilde{\alpha}-1} \varphi(s) ds.$$

Definition 4. For $m-1 < \widetilde{\alpha} \leq m$, the Caputo fractional derivative of $\varphi(x) \in \mathcal{C}^m[a,b]$ of order $\widetilde{\alpha} \in \mathbb{R}^+$ is defined as:

$$D_x^{\widetilde{\alpha}} \varphi(x) = J_x^{m-\widetilde{\alpha}} \varphi^{(m)}(x) = \begin{cases} \dfrac{1}{\Gamma(m-\widetilde{\alpha})} \displaystyle\int_a^x \frac{\varphi^{(m)}(s)}{(x-s)^{1-m+\widetilde{\alpha}}} ds, & \text{if} \quad m-1 < \widetilde{\alpha} < m, \\ \varphi^{(m)}(x), & \text{if} \quad \widetilde{\alpha} = m. \end{cases}$$

Theorem 1. *If* $\dfrac{(\alpha+1)\|\boldsymbol{A}\| + \lambda \|\boldsymbol{K}\|}{\Gamma(2+\alpha)} < 1$ *then* (1.1) *has unique solution in* $[0,T]$.

Proof. The Banach fixed point theorem is used to prove this theorem. Applying J_x^α from both sides of (1.1), we have $\mathbf{U}(t) = \Psi \mathbf{U}(t)$ for $t \in [0,T]$, where the operator $\Psi : \mathcal{C}([0,T], \mathbb{R}^n) \to \mathbb{R}^n$ for each $\mathbf{U} \in \mathcal{C}([0,T], \mathbb{R}^n)$ and $t \in [0,T]$ is defined as:

$$\Psi \mathbf{U}(t) \equiv \mathbf{U}_0 + J_t^\alpha \mathbf{F}(t) - J_t^\alpha \{\mathbf{A}(t)\mathbf{U}(t)\} - \lambda J_t^\alpha \left\{ \int_0^t \mathbf{K}(t,s) \mathbf{U}(s) ds \right\}.$$

Our aim is to show that Ψ is a contraction mapping. For any $\mathbf{U}_1(t), \mathbf{U}_2(t) \in \mathcal{C}([0,T], \mathbb{R}^n)$, we have

$$\begin{aligned}
&\left\| \Psi \mathbf{U}_1(t) - \Psi \mathbf{U}_2(t) \right\| \\
&= \left\| J_t^\alpha \{\mathbf{A}(t)(\mathbf{U}_1(t) - \mathbf{U}_2(t))\} + \lambda J_t^\alpha \left\{ \int_0^t \mathbf{K}(t,s)(\mathbf{U}_1(s) - \mathbf{U}_2(s)) ds \right\} \right\| \\
&\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \left\| \mathbf{A}(s) \right\| \left\| \mathbf{U}_1(s) - \mathbf{U}_2(s) \right\| ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s \left\| \mathbf{K}(s,z) \right\| \left\| \mathbf{U}_1(z) - \mathbf{U}_2(z) \right\| dz ds \\
&\leq \frac{\|\mathbf{A}\|}{\Gamma(\alpha+1)} \|\mathbf{U}_1(t) - \mathbf{U}_2(t)\| + \frac{\lambda \|\mathbf{K}\|}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \int_0^s \|\mathbf{U}_1(z) - \mathbf{U}_2(z)\| dz ds \\
&\leq \frac{(\alpha+1)\|\mathbf{A}\| + \lambda \|\mathbf{K}\|}{\Gamma(2+\alpha)} \left\| \mathbf{U}_1(t) - \mathbf{U}_2(t) \right\| \leq \left\| \mathbf{U}_1(t) - \mathbf{U}_2(t) \right\|.
\end{aligned}$$

Hence, Ψ is a contraction mapping. The Banach fixed point theorem implies the problem (1.1) has a unique solution for all $t \in [0,T]$. □

3. Proposed scheme

The problem (1.1) is discretized by the combination of the L1 technique and the composite trapezoidal rule. Set $N \in \mathbb{N}$. The uniform mesh $t_j = j\tau$ for $j = 0(1)N$ with equal step length $\tau = T/N$. The computed solution of $u^{(p)}$ at the mesh point t_j is denotes $U_j^{(p)}$.

The Caputo derivative $D_t^\alpha u^{(p)}(t)$ at t_j for $j = 1(1)N$ can be written as:

$$D_t^\alpha u^{(p)}(t_j) = \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \int_{s=t_k}^{t_{k+1}} (t_j - s)^{-\alpha} (u^{(p)}(s))' ds,$$

for $p = 1(1)n$, which can be discretized as follows [8, 15, 22]:

$$\begin{aligned} D_N^\alpha u^{(p)}(t_j) &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{j-1} \frac{u^{(p)}(t_{k+1}) - u^{(p)}(t_k)}{\tau} \int_{s=t_k}^{t_{k+1}} (t_j - s)^{-\alpha} ds \\ &= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{j-1} \left[u^{(p)}(t_{k+1}) - u^{(p)}(t_k)\right] b_{j-k} + \varepsilon_j^{(p)}, \end{aligned} \tag{3.1}$$

where $\varepsilon_j^{(p)} = (D_t^\alpha - D_N^\alpha) u^{(p)}(t_j)$ and $b_q = q^{1-\alpha} - (q-1)^{1-\alpha}$ for $q = 1(1)N$. The integral for $j = 1(1)N$ is

$$I_t u^{(p)}(t_j) = \int_{t_0}^{t_j} \sum_{i=1}^{n} K_{pi}(t_j, s) u^{(i)}(s) ds = \sum_{i=1}^{n} \sum_{m=0}^{j-1} \int_{t_m}^{t_{m+1}} K_{pi}(t_j, s) u^{(i)}(s) ds,$$

which can be approximated by applying the composite trapezoidal rule as:

$$I_N u^{(p)}(t_j) = \frac{\tau}{2} \sum_{i=1}^{n} \sum_{m=0}^{j-1} \left[K_{pi}(t_j, t_m) u^{(i)}(t_m) + K_{pi}(t_j, t_{m+1}) u^{(i)}(t_{m+1})\right] + R_j^{(p)}, \tag{3.2}$$

where $R_j^{(p)} = (I_t - I_N) u^{(p)}(t_j)$. Using (3.1) and (3.2), the model (1.1) transformed to

$$\begin{cases} D_N^\alpha u^{(p)}(t_j) + \sum\limits_{i=1}^{n} a_{pi}(t_j) u^{(i)}(t_j) + \lambda_p I_N u^{(p)}(t_j) = f^{(p)}(t_j) + \mathcal{E}_j^{(p)}, \\ u^{(p)}(t_0) = \eta_p, \end{cases} \tag{3.3}$$

for $j = 1(1)N$, and each $p = 1(1)n$. The remainder term $\mathcal{E}_j^{(p)}$ is given by $\mathcal{E}_j^{(p)} = \varepsilon_j^{(p)} + R_j^{(p)}$. Neglecting $\mathcal{E}_j^{(p)}$, the discrete problem (3.3) is reduced to

$$\begin{cases} f_j^{(p)} = \mu \sum\limits_{k=0}^{j-1} \left(U_{k+1}^{(p)} - U_k^{(p)}\right) d_{j-k} + \sum\limits_{i=1}^{n} (a_{pi})_j U_j^{(i)} \\ \qquad + \dfrac{\lambda_p \tau}{2} \sum\limits_{m=0}^{j-1} \sum\limits_{i=1}^{n} \left\{K_{pi}(t_j, t_m) U_m^{(i)} + K_{pi}(t_j, t_{m+1}) U_{m+1}^{(i)}\right\}, \\ U_0^{(p)} = \eta_p, \end{cases} \tag{3.4}$$

for $j = 1(1)N$, for each $p = 1(1)n$. The difference scheme (3.4) is explicit since one can easily find the solution from the following recursive relation:

$$\begin{cases} U_j^{(p)} = \prod^{-1}\Big[f_j^{(p)} + \mu\Big[d_1 U_{j-1}^{(p)} + \sum_{k=0}^{j-2}\Big(U_k^{(p)} - U_{k+1}^{(p)}\Big)d_{j-k}\Big] \\ \qquad - \sum_{i=1, i\neq p}^{n}(a_{pi})_j U_j^{(i)} - \frac{\lambda_p\tau}{2}K_{pp}(t_j, t_{j-1})U_{j-1}^{(p)} \\ \qquad - \frac{\lambda_p\tau}{2}\sum_{m=0}^{j-2}\sum_{i=1}^{n}\Big\{K_{pi}(t_j,t_m)U_m^{(i)} + K_{pi}(t_j,t_{m+1})U_{m+1}^{(i)}\Big\} \\ \qquad - \frac{\lambda_p\tau}{2}\sum_{i=1,i\neq p}^{n}\Big\{K_{pi}(t_j,t_{j-1})U_{j-1}^{(i)} + K_{pi}(t_j,t_j)U_j^{(i)}\Big\}\Big], \\ U_0^{(p)} = \eta_p, \end{cases} \tag{3.5}$$

where $\mu = \frac{h^{-\alpha}}{\Gamma(2-\alpha)}$, $\prod = \mu d_1 + (a_{pp})_j + \frac{\lambda_p\tau}{2}K_{pp}(t_j,t_j)$. Alternatively, it is possible to express (3.4) as a $N \times N$ linear system of equations:

$$G^{(p)}U^{(p)} = F^{(p)} \quad \text{for } \; p = 1(1)n, \tag{3.6}$$

where $U^{(p)} = [U_1^{(p)}, U_2^{(p)}, \ldots, U_N^{(p)}]$, $F^{(p)} = [F_1^{(p)}, F_2^{(p)}, \ldots, F_N^{(p)}]$ and for $i = 1(1)N$,

$$\begin{cases} G_{i,i}^{(p)} = \mu b_1 + (a_{ii})_j + \frac{\lambda_p\tau}{2}K_{ii}(x_j,x_j), \\ G_{i,k}^{(p)} = (a_{ik})_j + \frac{\lambda_p\tau}{2}K_{ik}(x_j,x_j), \quad k = i+1, i+2, \ldots, n, \\ G_{l,i}^{(j)} = (a_{il})_j + \frac{\lambda_i h}{2}K_{il}(x_j,x_j), \quad l = 1,2,\ldots,i-1, \\ F_i^{(j)} = f_j^{(i)} + \mu b_1 U_{j-1}^{(i)} + \sum_{k=0}^{j-2}(U_k^{(i)} - U_{k+1}^{(i)})b_{j-k} \\ \qquad - \frac{\lambda_i h}{2}\sum_{k=1}^{n}\Big[\sum_{m=0}^{j-1}K_{ik}(x_j,x_m)U_m^{(i)} - \sum_{m=0}^{j-2}K_{ik}(x_j,x_{m+1})U_{m+1}^{(i)}\Big]. \end{cases}$$

4. Error analysis

This section establishes the convergence result and estimates the global error of the proposed scheme (3.4).

Lemma 1. *The truncation error for the discretization of the Caputo fractional derivative for $j = 1(1)N$ satisfies*

$$|\varepsilon_j^{(p)}| \leq \begin{cases} C\tau^{2-\alpha}, & \text{if} \quad u^{(p)}(t) \in C^2[0,T], \\ C\tau^{\alpha}, & \text{if} \quad u^{(p)}(t) \in C_\alpha[0,T]. \end{cases}$$

Proof. Let $u^{(p)}(t) \in \mathcal{C}^2[0,T]$, then we have

$$\begin{aligned}\left|\varepsilon_j^{(p)}\right| &= \left|(D_t^\alpha - D_N^\alpha)u^p(t_j)\right| \\ &= \left|\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}(t_j-s)^{-\alpha}\left[\frac{du^{(p)}(s)}{ds} - \frac{u^{(p)}(t_{k+1})-u^{(p)}(t_k)}{\tau}\right]ds\right| \\ &\le C\left|\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}\frac{t_j+t_{j-1}-2s}{(t_j-s)^\alpha}ds + O(\tau^2)\right|.\end{aligned}$$

From [15], we get $\left|\frac{1}{\Gamma(1-\alpha)}\sum_{k=0}^{j-1}\int_{t_k}^{t_{k+1}}\frac{t_j+t_{j-1}-2s}{(t_j-s)^\alpha}ds\right| \le 2\tau^{2-\alpha}$. This however means $|\varepsilon_j^{(p)}| \le C\tau^{2-\alpha}$.

If $u^{(p)}(t) \in \mathcal{C}_\alpha[0,T]$, from [22] we have $\left|\varepsilon_j^{(p)}\right| = \left|(D_t^\alpha - D_N^\alpha)u^p(t_j)\right| \le C\tau^\alpha$. Hence, completes the proof. □

Lemma 2. *For each $p = 1(1)n$ the remainder $R_j^{(p)}$ satisfies $|R_j^{(p)}| \le CT\tau^2$ for $j = 1(1)N$.*

Proof. Since $K_{ij}(t,s) \in \mathcal{C}^\infty([0,T]\times[0,T])$ for $i,j = 1(1)n$ and $u^{(p)}(t) \in \mathcal{C}^2[0,T] \cap \mathcal{C}^\infty(0,T]$, there exist constants $C_1, C_2 > 0$ such that $|K_{ij}(t,s)| \le C_1$and $\left|\frac{d^q u^{(p)}}{dt^q}(t)\right| \le C_2$ for $q = 0,1,2$. Now, we have

$$\begin{aligned}|R_j^{(p)}| = |\,(I_t - I_N)\,u^{(p)}(t_j)| &= \sum_{i=1}^{n}\left|\sum_{m=0}^{j-1}\int_{t_m}^{t_{m+1}}K_{pi}(t_j,s)u^{(i)}(s)ds\right. \\ &\quad\left.-\frac{\tau}{2}\sum_{m=0}^{j-1}\left[K_{pi}(t_j,t_m)U_m^{(i)} + K_{pi}(t_j,t_{m+1})U_{m+1}^{(i)}\right]\right| \\ &\le C_1\sum_{i=1}^{n}\left|\sum_{m=0}^{j-1}\int_{t_m}^{t_{m+1}}u^{(i)}(s)ds - \frac{\tau}{2}\sum_{m=0}^{j-1}\left(U_m^{(i)} + U_{m+1}^{(i)}\right)\right|.\end{aligned}$$

Using the Taylor series, we get

$$\begin{aligned}|R_j^{(p)}| &\le C_1\sum_{i=1}^{n}\left|\sum_{m=0}^{j-1}\left[\left(\tau U_m^{(i)} + \frac{\tau^2}{2!}(U^{(i)})_m' + \frac{\tau^3}{3!}(U^{(i)})_m'' + \frac{\tau^4}{4!}(U^{(i)})_m''' + \cdots\right)\right.\right. \\ &\quad\left.\left.-\frac{\tau}{2}\left(U_m^{(i)} + U_m^{(i)} + \tau(U^{(i)})_m' + \frac{\tau^2}{2!}(U^{(i)})_m'' + \frac{\tau^3}{3!}(U^{(i)})_m''' + \cdots\right)\right]\right| \\ &\le C_1\sum_{i=1}^{n}\left|\sum_{m=0}^{j-1}\left[\frac{\tau^3}{12}(U^{(i)})_m'' - \frac{\tau^4}{24}(U^{(i)})_m''' + \ldots\right]\right|\end{aligned}$$

$$\leq C_1 C_2 \sum_{i=1}^{n} j\tau^3 \leq CN\tau^3 \leq CT\tau^2,$$

which is our desired bound. □

Denote $e_j^{(p)} = \left|u^{(p)}(t_j) - U_j^{(p)}\right|$ for $j = 1(1)N$. From (3.3) and (3.4), we get

$$\begin{cases} D_N^{\alpha} e_j^{(p)} + \sum_{i=1}^{n} a_{pi}(t_j) e_j^{(i)} + \lambda_p I_N e_j^{(p)} = \mathcal{E}_j^{(p)} & \text{for} \quad j = 1(1)N, \\ e_0^{(p)} = 0. \end{cases} \tag{4.1}$$

Lemma 3. *For any mesh function $\{U_j^{(p)}\}_{j=0}^{N}$ with $U_0^{(p)} = 0$, we have*

$$\left|U_j^{(p)}\right| \leq \max_{k=1(1)j} \left\{ \frac{\tau^{\alpha}\Gamma(2-\alpha)}{b_k} D_N^{\alpha} \left|U_k^{(p)}\right| \right\} \quad \textit{for} \quad j = 1(1)N.$$

Proof. Let $\max_{k=1(1)j} \left|U_k^{(p)}\right| = \left|U_q^{(p)}\right|$ for some $q \in \{1, 2, \ldots, j\}$. Since $U_0^{(p)} = 0$, (3.1) yields

$$D_N^{\alpha} \left|U_q^{(p)}\right| = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left\{ \left|U_q^{(p)}\right| - \sum_{k=1}^{q-1} (b_k - b_{k+1}) \left|U_{q-k}^{(p)}\right| \right\}.$$

Now, $b_1 = 1$ and $b_k > b_{k+1}$ for all $k \geq 1$ implies that

$$D_N^{\alpha} \left|U_q^{(p)}\right| \geq \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left\{ \left|U_q^{(p)}\right| - \sum_{k=1}^{q-1} (b_k - b_{k+1}) \left|U_q^{(p)}\right| \right\} \geq \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} b_q \left|U_q^{(p)}\right|.$$

Therefore, $\left|U_q^{(p)}\right| \leq \dfrac{\tau^{\alpha}\Gamma(2-\alpha)}{b_q} D_N^{\alpha} \left|U_q^{(p)}\right|$. Hence, the desired result. □

Theorem 2. *If $\{u^{(p)}(t_j)\}_{j=1}^{N}$ be the true solution and $\{U_j^{(p)}\}_{j=1}^{N}$ be the approximate solution of the problem (1.1) for each $p = 1(1)n$, then we have*

$$\left|e_j^{(p)}\right| \leq \begin{cases} CT^{\alpha}(\tau^{2-\alpha} + T\tau^2), & \textit{if} \quad u^{(p)}(t) \in \mathcal{C}^2[0,T], \\ CT^{\alpha}(\tau^{\alpha} + T\tau^2), & \textit{if} \quad u^{(p)}(t) \in \mathcal{C}_{\alpha}[0,T]. \end{cases}$$

Proof. Multiplying (4.1) by $e_j^{(p)}$, then using (3.1), we obtain

$$\begin{aligned} \mu\left|e_j^{(p)}\right|^2 + \sum_{i=1}^{n} \left|a_{pi}(t_j)\right| \left|e_j^{(i)}\right| \left|e_j^{(p)}\right| + \frac{\lambda_p \tau}{2} \sum_{i=1}^{n} \sum_{m=0}^{j-1} \Big[\left|K_{pi}(t_j, t_m)\right| \left|e_m^{(i)}\right| \left|e_j^{(p)}\right| \\ + \left|K_{pi}(t_j, t_{m+1})\right| \left|e_{m+1}^{(i)}\right| \left|e_j^{(p)}\right| \Big] \\ \leq \left|\mathcal{E}_j^{(p)}\right| \left|e_j^{(p)}\right| + \mu \sum_{k=1}^{j-1} (b_k - b_{k+1}) \left|e_{j-k}^{(p)}\right| \left|e_j^{(p)}\right|. \end{aligned}$$

Therefore, $\mu|e_j^{(p)}|^2 \le |\mathcal{E}_j^{(p)}||e_j^{(p)}| + \mu \sum_{k=1}^{j-1}(b_k - b_{k+1})|e_{j-k}^{(p)}||e_j^{(p)}|$. Dividing by $|e_j^{(p)}|$, we get $D_N^\alpha |e_j^{(p)}| \le |\mathcal{E}_j^{(p)}|$. Lemmas 1 and 2 gives

$$|\mathcal{E}_j^{(p)}| \le |\varepsilon_j^{(p)}| + |R_j^{(p)}| \le \begin{cases} C(\tau^{2-\alpha} + T\tau^2), & \text{if} \quad u^{(p)}(t) \in \mathcal{C}^2[0,T], \\ C(\tau^\alpha + T\tau^2), & \text{if} \quad u^{(p)}(t) \in \mathcal{C}_\alpha[0,T]. \end{cases}$$

Since $b_1 = 1$ and $b_q > b_{q+1}$ for all $q \ge 1$. The mean value theorem implies that

$$(1-\alpha)q^{-\alpha} \le b_q \le (1-\alpha)(q-1)^{-\alpha} \quad \text{for} \quad q \ge 2.$$

Using Lemma 3, we obtain

$$\begin{aligned} |e_j^{(p)}| &\le \max_{k=1,2,\ldots,j} \left\{ \frac{\tau^\alpha \Gamma(2-\alpha)}{b_k} |\mathcal{E}_j^{(p)}| \right\} \\ &\le \frac{\tau^\alpha \Gamma(2-\alpha)}{(1-\alpha)N^{-\alpha}} \begin{cases} C(\tau^{2-\alpha} + T\tau^2), & \text{if} \quad u^{(p)}(t) \in \mathcal{C}^2[0,T], \\ C(\tau^\alpha + T\tau^2), & \text{if} \quad u^{(p)}(t) \in \mathcal{C}_\alpha[0,T]. \end{cases} \\ &\le CT^\alpha \begin{cases} \tau^{2-\alpha} + T\tau^2, & \text{if} \quad u^{(p)}(t) \in \mathcal{C}^2[0,T], \\ \tau^\alpha + T\tau^2, & \text{if} \quad u^{(p)}(t) \in \mathcal{C}_\alpha[0,T]. \end{cases} \end{aligned}$$

This proves the theorem. □

5. NUMERICAL EXPERIMENT

The following two examples are in good agreement with theoretical establishments. The computational results are presented through several tables and graphs.

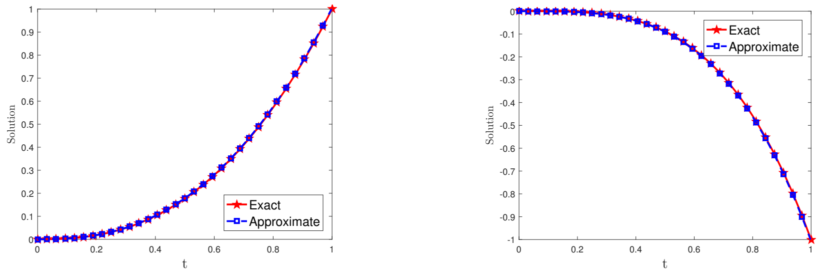
Example 1. Consider the following system:

$$\begin{cases} D_t^\alpha u^{(1)}(t) + u^{(1)}(t) + u^{(2)}(t) - \int_0^t (t-s)(u^{(1)}(s) + u^{(2)}(s))ds = f^{(1)}(t), \\ D_t^\alpha u^{(2)}(t) + u^{(1)}(t) + u^{(2)}(t) + \int_0^t (s-t)(u^{(1)}(s) - u^{(2)}(s))ds = f^{(2)}(t), \\ \text{for} \ \ t \in (0,1], \quad IC: u^{(1)}(0) = 0 = u^{(2)}(0). \end{cases}$$

We choose $f^{(i)}(t),\ i = 1,2$, so that the exact solution is $u^{(1)}(t) = t^{p+\alpha}$, $u^{(2)}(t) = -t^{q+\alpha}$ for $p,q \in \mathbb{R}$. Compute the maximum error and the corresponding rate of convergence defined as: $\Sigma_N = \max_{0\le j\le N} |u^{(i)}(t_j) - u_j^{(i)}|$; $\rho_N = \log_2\left(\frac{\Sigma_N}{\Sigma_{2N}}\right)$.

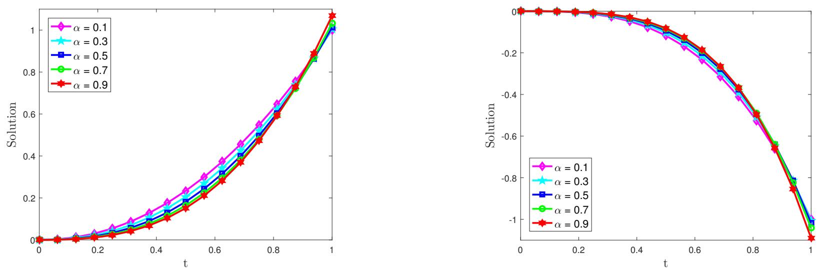
Case-I **(p = 2,q = 3):** Figure 1 shows the comparison between the exact and the approximate solution with $\alpha = 0.5, N = 32$ for Example 1. The graphs of approximate solutions for different α with $N = 16$ are presented in Figure 2. The log-log plots of the numerical error is displayed in Figure 3. Σ_M and ρ_M are shown in Table 1 for $u^{(1)}(t)$ and $u^{(2)}(t)$.

Case-II **(p = 0,q = 0):** Figure 4 shows the comparison between the exact and approximate solution with $\alpha = 0.5, N = 32$ for Example 1. The graphs of approximate



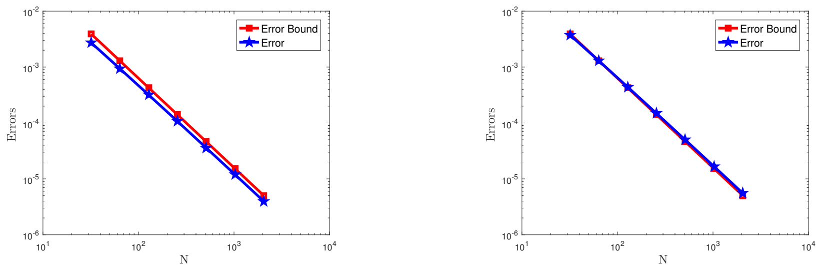
(A) $u^{(1)}(t)$ (B) $u^{(2)}(t)$

FIGURE 1. The exact and approximate solutions with $N = 32$, $\alpha = 0.5$ for Example 1



(A) $u^{(1)}(t)$ (B) $u^{(2)}(t)$

FIGURE 2. The approximate solutions with $N = 16$ for Example 1



(A) For $u^{(1)}(t)$ (B) For $u^{(2)}(t)$

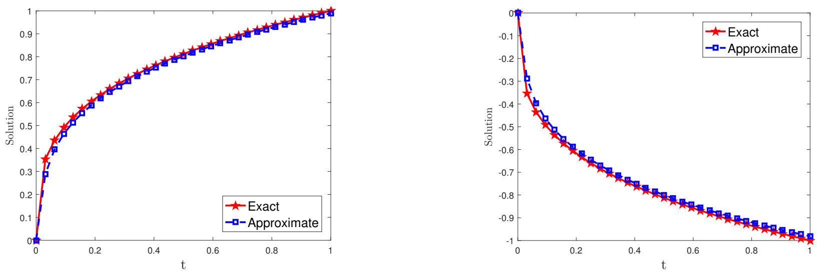
FIGURE 3. Log–log plots of Σ_M with $\alpha = 0.4$ for Example 1

solutions for different α with $N = 16$ are presented in Figure 5. From these graphs, it is clear that the solutions have mild singularity near $t = 0$. The log-log plots of the numerical errors displayed in Figure 6. Σ_N and ρ_N are shown in Table 2 for $u^{(1)}(t)$

TABLE 1. Σ_M and ρ_M with $p = 2,\ q = 3$ for Example 1

α/M		32	64	128	256	512	1024
0.1	$u^{(1)}(t)$	2.054E-04	6.219E-05	1.845E-05	5.387E-06	1.554E-06	4.442E-07
		1.723	1.753	1.776	1.793	1.807	1.818
	$u^{(2)}(t)$	2.563E-04	8.013E-05	2.431E-05	7.223E-06	2.112E-06	6.101E-07
		1.678	1.721	1.751	1.774	1.792	1.806
0.3	$u^{(1)}(t)$	1.440E-03	4.684E-04	1.504E-04	4.784E-05	1.511E-05	4.746E-06
		1.620	1.639	1.653	1.663	1.671	1.677
	$u^{(2)}(t)$	1.970E-03	6.469E-04	2.091E-04	6.683E-05	2.119E-05	6.674E-06
		1.607	1.629	1.646	1.657	1.666	1.673
0.7	$u^{(1)}(t)$	1.302E-02	5.359E-03	2.196E-03	8.969E-04	3.655E-04	1.488E-04
		1.280	1.287	1.292	1.295	1.297	1.298
	$u^{(2)}(t)$	1.752E-02	7.244E-03	2.975E-03	1.217E-03	4.963E-04	2.021E-04
		1.274	1.284	1.290	1.294	1.296	1.298
0.9	$u^{(1)}(t)$	3.196E-02	1.491E-02	6.962E-03	3.250E-03	1.517E-03	7.080E-04
		1.100	1.099	1.099	1.099	1.099	1.100
	$u^{(2)}(t)$	4.221E-02	1.974E-02	9.231E-03	4.313E-03	2.014E-03	9.401E-04
		1.096	1.097	1.098	1.099	1.099	1.100

and $u^{(2)}(t)$. These results show that the approximate solutions converge with the convergence rate α only.



(A) $u^{(1)}(t)$ (B) $u^{(2)}(t)$

FIGURE 4. The exact and approximate solutions with $N = 32,\ \alpha = 0.3$ for Example 1

(A) $u^{(1)}(t)$ (B) $u^{(2)}(t)$

FIGURE 5. The approximate solutions with $N = 16$ for Example 1

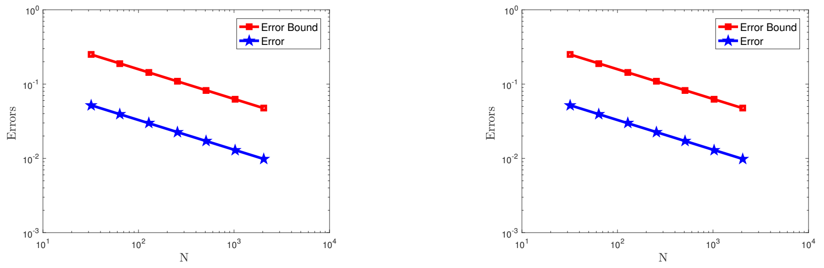
TABLE 2. Σ_M and ρ_M with $p = 0 = q$ for Example 1

α/M		32	64	128	256	512	1024
0.1	$u^{(1)}(t)$	6.024E-02	5.612E-02	5.234E-02	4.883E-02	4.556E-02	4.251E-02
		0.102	0.100	0.100	0.100	0.100	0.100
	$u^{(2)}(t)$	6.041E-02	5.616E-02	5.235E-02	4.884E-02	4.556E-02	4.251E-02
		0.105	0.101	0.100	0.100	0.100	0.100
0.3	$u^{(1)}(t)$	6.525E-02	5.299E-02	4.304E-02	3.496E-02	2.840E-02	2.307E-02
		0.300	0.300	0.300	0.300	0.300	0.300
	$u^{(2)}(t)$	6.530E-02	5.300E-02	4.304E-02	3.496E-02	2.840E-02	2.307E-02
		0.301	0.300	0.300	0.300	0.300	0.300
0.7	$u^{(1)}(t)$	1.631E-02	1.004E-02	6.180E-03	3.804E-03	2.342E-03	1.442E-03
		0.700	0.700	0.700	0.700	0.700	0.700
	$u^{(2)}(t)$	1.631E-02	1.004E-02	6.180E-03	3.804E-03	2.342E-03	1.442E-03
		0.700	0.700	0.700	0.700	0.700	0.700
0.9	$u^{(1)}(t)$	6.477E-03	3.521E-03	1.895E-03	1.011E-03	5.357E-04	2.819E-04
		0.879	0.893	0.906	0.917	0.926	0.934
	$u^{(2)}(t)$	7.833E-03	4.256E-03	2.297E-03	1.230E-03	6.537E-04	3.451E-04
		0.880	0.890	0.901	0.912	0.922	0.930

Example 2. Consider the following test problem:

$$\begin{cases} D_t^{\alpha}u^{(1)}(t) + (1-t)u^{(3)}(t) + \int_0^t \{stu^{(1)}(s) + u^{(2)}(t)\}dt = f^{(1)}(t), \\ D_t^{\alpha}u^{(2)}(t) + tu^{(1)}(t) + \int_0^t \{u^{(2)}(s) + u^{(3)}(t)\}dt = f^{(2)}(t), \\ D_t^{\alpha}u^{(3)}(t) + t^2u^{(2)}(t) + \int_0^t \{ts^2u^{(1)}(s) + u^{(3)}(t)\}dt = f^{(3)}(t), \\ \text{for } t \in (0,1], \quad IC: \ u^{(1)}(0) = u^{(2)}(0) = u^{(3)}(0) = 0. \end{cases}$$

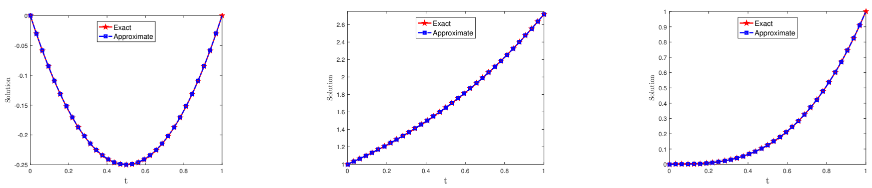
The exact solution is $u^{(1)}(t) = t(t-1)$, $u^{(2)}(t) = e^t$, $u^{(3)}(t) = t^3$ for the suitable choice of $f^{(i)}(t)$. The exact and approximate solutions for Example 2 are shown in Figure 7 and Figure 8 for $\alpha = 0.3$ and $\alpha = 0.7$, respectively. Figure 9 display the



(A) For $u^{(1)}(t)$ (B) For $u^{(2)}(t)$

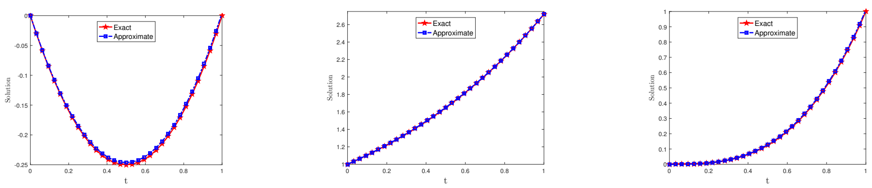
FIGURE 6. Log–log plots of Σ_M with $\alpha = 0.4$ for Example 1

log-log plots of the computational errors for Example 2. Figure 10 illustrates surface plots of the computational error. Table 3 shows Σ_N and ρ_N for $u^{(1)}(t)$, $u^{(2)}(t)$ and $u^{(3)}(t)$. The tabular data shows that the approximate solutions converge with $(2-\alpha)$ rate of accuracy.



(A) $u^{(1)}(t)$ (B) $u^{(2)}(t)$ (C) $u^{(3)}(t)$

FIGURE 7. Computed solutions with $N = 32$, $\alpha = 0.3$ for Example 2

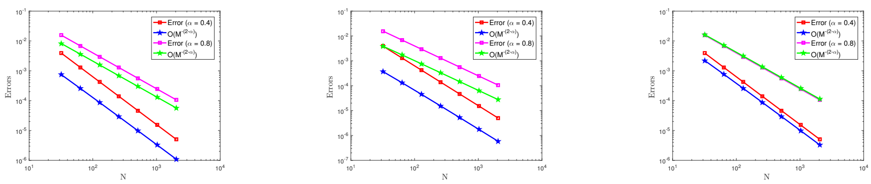


(A) $u^{(1)}(t)$ (B) $u^{(2)}(t)$ (C) $u^{(3)}(t)$

FIGURE 8. Computed solutions with $N = 32$, $\alpha = 0.7$ for Example 2

TABLE 3. Σ_M and ρ_M for Example 2

α/M		32	64	128	256	512	1024
0.1	$u^{(1)}(t)$	6.300E-05	1.909E-05	5.689E-06	1.672E-06	4.854E-07	1.396E-07
		1.722	1.747	1.767	1.784	1.798	1.810
	$u^{(2)}(t)$	2.021E-04	5.228E-05	1.353E-05	3.507E-06	9.096E-07	2.362E-07
		1.951	1.950	1.948	1.947	1.945	1.943
	$u^{(3)}(t)$	1.502E-04	4.822E-05	1.495E-05	4.512E-06	1.335E-06	3.894E-07
		1.640	1.689	1.728	1.757	1.778	1.795
0.5	$u^{(1)}(t)$	1.440E-03	5.201E-04	1.867E-04	6.671E-05	2.376E-05	8.447E-06
		1.469	1.478	1.485	1.489	1.492	1.495
	$u^{(2)}(t)$	6.896E-04	2.564E-04	9.368E-05	3.385E-05	1.215E-05	4.340E-06
		1.427	1.453	1.468	1.478	1.485	1.490
	$u^{(3)}(t)$	3.812E-03	1.394E-03	5.041E-04	1.810E-04	6.470E-05	2.305E-05
		1.451	1.467	1.478	1.484	1.489	1.492
0.9	$u^{(1)}(t)$	1.385E-02	6.511E-03	3.049E-03	1.425E-03	6.654E-04	3.106E-04
		1.089	1.095	1.097	1.099	1.099	1.100
	$u^{(2)}(t)$	6.710E-03	3.203E-03	1.511E-03	7.090E-04	3.317E-04	1.550E-04
		1.067	1.084	1.092	1.096	1.098	1.099
	$u^{(3)}(t)$	2.517E-02	1.182E-02	5.533E-03	2.587E-03	1.208E-03	5.640E-04
		1.091	1.095	1.097	1.098	1.099	1.099

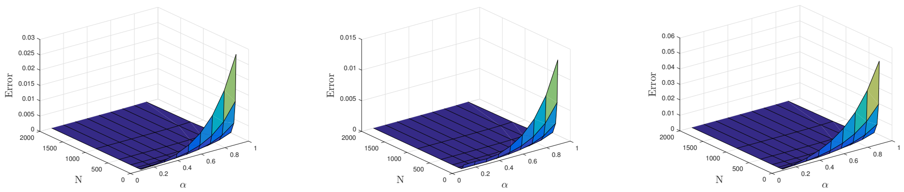


(A) $u^{(1)}(t)$ (B) $u^{(2)}(t)$ (C) $u^{(3)}(t)$

FIGURE 9. Log–log plots for Example 2

6. CONCLUSION

In this work, a novel finite difference scheme is developed for solving a system of fractional order Volterra-type integro-differential equations. The L1 technique is employed to discretize the Caputo fractional derivative, in which the integral component is approximated with the help of the composite trapezoidal rule. The convergence analysis and the error estimation are provided. The theoretical and experimental results confirm the efficiency and the applicability of the proposed scheme.



(A) $u^{(1)}(t)$ (B) $u^{(2)}(t)$ (C) $u^{(3)}(t)$

FIGURE 10. Surface plots of pointwise error for Example 2

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