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EXISTENCE AND BLOW UP OF SOLUTIONS OF A VISCOELASTIC $m(x)$-BIHARMONIC EQUATION WITH LOGARITHMIC SOURCE TERM

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Abstract. In this paper, we are concerned with a logarithmic nonlinear viscoelastic $m(x)$-biharmonic equation. Firstly, we proved the local existence of solutions by using the Faedo-Galerkin method. Later, we proved the blow up of solutions by using the concavity method.

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1. INTRODUCTION

In this paper, we discuss a viscoelastic $m(x)$-biharmonic equation with logarithmic source term

$$\begin{cases} u_t + \Delta^2 u + \Delta^2_{m(x)} u - \int\limits_0^t g(t-z)\, \Delta^2 u dz = |u|^{p-2} u \ln|u|, (x,t) \in \Omega \times (0,T), \\ u(x,t) = \frac{\partial u}{\partial v}(x,t) = 0, \quad x \in \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), \quad x \in \Omega. \end{cases} \tag{1.1}$$

Here Ω denotes a bounded domain in $\mathbb{R}^n$ $(n \geq 1)$, with smooth boundary $\partial\Omega$. The variable exponents $m(.)$ are provided as measurable functions defined within Ω. The operator $\Delta^2_{m(x)}$ is the so-called $m(x)$-biharmonic operator and is defined by

$$\Delta^2_{m(x)} u = \Delta\left(|\Delta u|^{m(x)-2} \Delta u\right).$$

Also, $m(x)$ is given continuous and measurable function on $\overline{\Omega}$ such that

$$m_- = ess \inf_{x\in\Omega} m(x), \quad m_+ = ess \sup_{x\in\Omega} m(x).$$

We consider the following hypotheses:

$A_1)$ $2 < m_- \leq m(x) \leq m_+ < p < \frac{Nm(x)}{N-m(x)}$.

$A_2)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded C^1 function with the following assumptions:

$$g(z) \geq 0,\ g'(z) \leq 0,\ \ 1 - \int_0^t g(z)\,dz = l > 0.$$

$A_3)$ $E(0) > 0$, is the energy functional defined by (4.2).

Up till now, there are so many results about the parabolic-type differential equations. Our main objective in the present paper is to consider the equation both with viscoelastic term $(\int_0^t g(t-z)\,\Delta^2 u dz)$, variable exponent term $(\Delta^2_{m(x)} u)$ and the logarithmic source term $(|u|^{p-2} u \ln |u|)$ which make the problem different from those considered in the literature. The problem we deal with is a very general problem:

- The equation with variable exponents arises in many branches in sciences such as image processing, electrorheological fluids and nonlinear elasticity theory (see [5, 8, 23]).
- The equation with logarithmic source term arises in many branches in quantum field theory, optics, inflation cosmology, nuclear physics and geophysics (see [2, 4, 10]).
- The fourth-order equation has its origin in the canonical model introduced by Petrovsky [18, 19]. This type equations arises in many branches in sciences such as acoustics, geophysics, ocean acoustics and optics [9].

Qu et al. [21] studied the fourth-order parabolic equation

$$u_t + \Delta^2 u = |u|^{p(x)}.$$

They established the asymptotic behavior of solutions. Later, Liu [15] demonstrated the local existence and blow-up of solutions for the same equation.

Han [11] investigated the fourth-order parabolic equation

$$u_t + \Delta^2 u - \nabla f(\nabla u) = h(x,t,u).$$

The author exhibited the global existence and finite-time blow-up of solutions.

Abita examined in [22] the pseudo-parabolic equation within the context of a linear memory term and a logarithmic nonlinear source term

$$u_t - \Delta u_t + \int_0^t g(t-s)\,\Delta u(x,s)\,ds - \Delta u = |u|^{p(x)-2} u \ln(|u|). \tag{1.2}$$

The author demonstrated that a solution to equation (1.2) blows up in finite time T, and gave the upper bound for the blow-up time.

Chuong et al. [7] investigated the following Cahn-Hilliard equation

$$u_t + \Delta^2 u - \Delta_{p(x)} u = |u|^{q(x)-2} u.$$

The authors established the existence and nonexistence of global solutions.

Chuong et al. [6] studied the pseudo-parabolic equations involving $p(x)-$ Laplacian and logarithmic nonlinearity

$$u_t - \Delta u_t - \Delta_{p(x)} u = |u|^{q(x)-2} u \ln(|u|).$$

They obtained some threshold results on existence or nonexistence of global weak solutions when the initial energy is subcritical.

Wang and Liu [24] studied the $p(x)$-biharmonic parabolic equation with the logarithmic nonlinearity

$$u_t + \Delta\left(|\Delta u|^{p(x)-2} \Delta u\right) = |u|^{q-2} u \ln|u|.$$

They proved the existence of the local weak solutions and the existence of the global weak solutions.

Narayanan and Soundararajan [16] studied the viscoelastic $p(x)-$ Laplacian equation with logarithmic nonlinearity

$$u_t - \Delta u - \beta \nabla\left(|\nabla u|^{p(x)-2} \nabla u\right) + \int_0^t g(t-z) \Delta u(x,z)\, dz = |u|^{q-2} u \ln|u|.$$

They proved the existence and finite time blow up of solutions of the problem.

This work consists of four sections. Firstly, in Section 2, we give some theories needed about Lebesgue and Sobolev space with variable exponents. Then, Section 3 is about the existence of weak solutions by using the Faedo Galerkin approximation method. Moreover, in Section 4, we obtain the analysis of blow-up phenomena and the determination of an upper bound for the blow-up time.

2. Preliminaries

Throughout this study, we represent the $L^p(\Omega)$ norm as $\|.\|_p$. Additionally, we provide essential background on Lebesgue spaces and Sobolev spaces with variable exponents (for detailed, see [1, 8, 20]).

Let $p\colon \Omega \to [1,\infty]$ be a measurable function. We introduce the Lebesgue space with variable exponent $p(.)$

$$L^{p(.)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ measurable in } \Omega,\ \ \rho_{p(.)}(\lambda u)\ < \infty,\ \text{ for some } \lambda > 0\right\},$$

where

$$\rho_{p(.)}(u) = \int_\Omega |u(x)|^{p(x)}\, dx.$$

The norm, known as Luxemburg's norm, is defined as follows:

$$\|u\|_{p(x)} = \inf\left\{\lambda > 0 : \int_{\Omega} \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\},$$

$L^{p(.)}(\Omega)$ is a Banach space.

Next we proceed to define the variable-exponent Sobolev space $W^{m,p(.)}(\Omega)$ as

$$W^{m,p(.)}(\Omega) = \left\{u \in L^{p(.)}(\Omega) \text{ such that } D^{\alpha}u \text{ exists and } D^{\alpha}u \in L^{p(.)}(\Omega),\ |\alpha| \le m\right\}.$$

Lemma 1 ([8]). *Assume that*

$$1 \le m_- := ess \inf_{x\in\Omega} m(x) \le m(x) \le m_+ := ess \sup_{x\in\Omega} m(x) < \infty,$$

and we get

$$\min\left\{\|u\|_{m(.)}^{m_-}, \|u\|_{m(.)}^{m_+}\right\} \le \rho_{m(.)}(u) \le \max\left\{\|u\|_{m(.)}^{m_-}, \|u\|_{m(.)}^{m_+}\right\}, \tag{2.1}$$

for any $u \in L^{m(.)}$.

Definition 1 ([8]). For any given points x and y belonging to the bounded domain Ω, there is a constant $M > 0$ such that the subsequent inequality is satisfied:

$$|p(x) - p(y)| \le \frac{M}{\ln|x-y|},\ \text{for } x, y \in \Omega, \text{ with } |x-y| < \delta,\ 0 < \delta < 1,$$

then we say that $p(x)$ is log-Holder continuous.

Lemma 2 ([8]). *Suppose that* $q\colon \Omega \longrightarrow [1,\infty)$ *is a measurable function that fulfills inequality (2.1) and the continuity and compactness of the embedding* $L^{q(.)} \hookrightarrow L^{p(.)}$ *can be established. In this context, the variable exponents* $p(x)$ *and* $q(x)$ *adhere to the condition* $p(x) \le q(x)$ *almost everywhere within the domain* Ω.

Lemma 3 ([8]). *Assume that* $q\colon \Omega \to [1,\infty)$ *is a measurable function that fulfills inequality (2.1) and there exists a continuous and compact Sobolev embedding from* $W_0^{1,m(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$, *where the exponents* $m(x) \in C(\overline{\Omega})$, $q\colon \Omega \to [1,\infty)$,

$$ess \inf_{x\in\Omega} (m^*(x) - q(x)) > 0,$$

and

$$m^*(x) = \left\{\frac{Nm(x)}{N - m(x)},\ \text{if } m(x) < N, \infty,\ \text{if } m(x) \ge N.\right.$$

Lemma 4 ([12]). *Suppose that* v *be a positive number. We get*

$$\alpha^q \ln\alpha \le (ev)^{-1}\alpha^{q+v},$$

holds for all $\alpha \in [1,\infty)$.

Lemma 5 ([13, 17], Concavity method)**.** *Suppose that* $\beta > 0$*, let* $\psi(t) \geq 0$ *be weakly twice-differentiable on* $(0,\infty)$ *such that* $\psi(0) > 0$*,* $\psi'(0) > 0$ *and*

$$\psi''(t)\psi(t) - (1+\beta)\left(\psi'(t)\right)^2 \geq 0,$$

for all $t \in (0,\infty)$*. Then there exists a* $T > 0$ *such that*

$$\lim_{t\to T^-} \psi(t) = \infty,$$

and

$$T \leq \frac{\psi(0)}{\beta\psi'(0)}.$$

Here we define a energy functional, which will be used in further calculations:

$$E(t) = \frac{1}{2}\left(1 - \int_0^t g(z)\,dz\right)\|\Delta u\|_2^2 + \frac{1}{2}\int_0^t g(t-z)\|\Delta u(x,z) - \Delta u(x,t)\|_2^2\,dz$$
$$- \int_\Omega \frac{|\Delta u|^{m(x)}}{m(x)}dx + \frac{1}{p^2}\|u\|_p^p - \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx. \tag{2.2}$$

3. Weak solutions

In this section, we discuss the existence of weak solutions for the problem (1.1), by using the Faedo Galerkin approximation method.

Lemma 6. *A function* $u(x,t)$ *is considered a weak solution for the problem (1.1) if it satisfies the following conditions:*

$$u(x,t) \in L^2\left(0.T; W_0^{2,m(x)}(\Omega) \cap L^p(\Omega)\right) \cap L^2\left(0,T; H_0^2(\Omega)\right) \cap C\left(0,T; L^2(\Omega)\right),$$
$$u_t(x,t) \in L^2\left(0,T; L^2(\Omega)\right) \text{ and } u(x,0) = u_0(x),$$

which also satisfies:

$$\int_0^T\int_\Omega u_t\theta dxdt + \int_0^T\int_\Omega \Delta u\Delta\theta dxdt + \int_0^T\int_\Omega |\Delta u|^{m(x)-2}\Delta u\Delta\theta dxdt$$
$$- \int_0^T\int_0^t g(t-z)\int_\Omega \Delta u(x,z)\Delta\theta dxdzdt = \int_0^T\int_\Omega |u|^{p-2}u\ln|u|\,\theta dxdt, \tag{3.1}$$

$\forall\theta \in C^\infty(0,T; C_0^\infty(\Omega))$*, where the maximal interval of existence is defined as* $[0,T]$*.*

Theorem 1. *Assuming that* $u_0 \in W_0^{2,m(x)}(\Omega) \cap L^p(\Omega) \setminus \{0\}$*, the problem (1.1) admits a weak solution* $u(x,t)$*, which is provided as described in Lemma 6.*

Proof. We deal with an orthonormal basis of $L^2(\Omega)$, which is orthogonal in $H_0^2(\Omega)$ given by $\{\theta_i\}_{i=1}^{\infty}$. $\{\theta_i\}_{i=1}^{\infty}$ is a sequence of eigenfunctions of $-\Delta$ corresponding to the eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$. Now, we proceed to seek finite-dimensional approximations of equation (1.1) denoted as $\{u_n\}$ where

$$u_n(x,t) = \sum_{i=1}^{n} a_{ni}(t)\,\theta_i(x), \tag{3.2}$$

so that

$$\begin{aligned}&\int_\Omega u_n'\theta_i dx + \int_\Omega \Delta u_n \Delta\theta_i dx + \int_\Omega |\Delta u_n|^{m(x)-2}\Delta u_n \Delta\theta_i dx - \int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\,\Delta\theta_i dxdz\\ &= \int_\Omega |u_n|^{p-2}u_n \ln|u_n|\,\theta_i dx,\end{aligned} \tag{3.3}$$

and

$$u_n(x,0) = \sum_{i=1}^{n} a_{ni}(0)\,\theta_i(x) \longrightarrow u_0(x) \text{ in } W_0^{2,m(x)}(\Omega)\cap L^p(\Omega)\diagdown\{0\}. \tag{3.4}$$

Now, we must obtain the coefficients $\{a_{ni}\}_{i=1}^{n}$. For $k = 1,2,...n$,

$$\begin{aligned}a_{nk}'(t) = &-\int_\Omega \Delta u_n \Delta\theta_k dx - \int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\,\Delta\theta_k dxdz\\ &+\int_\Omega |\Delta u_n|^{m(x)-2}\Delta u_n \Delta\theta_k dx + \int_\Omega |u_n|^{p-2}u_n \ln|u_n|\,\theta_k dx.\end{aligned} \tag{3.5}$$

Here we take into account the maximum interval of existence as $[0,T]$. When we multiply equation (3.3) by $a_{ni}^{'}(t)$ and sum for $i = 1,2,...n$, we obtain

$$\begin{aligned}&\int_\Omega |u_n'|^2 dx + \int_\Omega \Delta u_n \Delta\theta_n' dx - \int_\Omega |\Delta u_n|^{m(x)-2}\Delta u_n \Delta\theta_n' dx\\ &+\int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\,\Delta\theta_n' dxdz = \int_\Omega |u_n|^{p-2}u_n \ln|u_n|\,\theta_n' dx.\end{aligned}$$

Consequently,

$$\begin{aligned}&\int_\Omega |u_n'|^2 dx - \frac{1}{2}\int_0^t g'(t-z)\,\|\Delta u_n(x,z)-\Delta u_n(x,t)\|_2^2\,dz + \frac{1}{2}g(t)\,\|\Delta u_n\|_2^2\\ &= -\frac{d}{dt}E(u_n(t)).\end{aligned} \tag{3.6}$$

Considering E as the functional defined by equation (2.2), we proceed with the implications of hypothesis (A_2) to obtain

$$E'(u_n(t)) = -\int_\Omega |u_n'|^2 dx - \frac{1}{2}g(t)\|\Delta u_n\|_2^2 + \frac{1}{2}\int_0^t g'(t-z)\|\Delta u_n(x,z) - \Delta u_n(x,t)\|_2^2 dz \\ \leq 0. \tag{3.7}$$

This leads to the implication that $E(u_n)$ is non-increasing. Consequently, we establish $E(u_n(t)) \leq E(u_n(0))\ \forall t \in [0,T)$ and for every $n \in \mathbb{N}$. Multiplying (3.3) by $a_{ni}(t)$ and then taking the sum over i, we have

$$\int_\Omega u_n' u_n dx + \int_\Omega |\Delta u_n|^2 dx - \int_\Omega |\Delta u_n|^{m(x)} dx - \int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\Delta u_n(x,t)\,dxdz \\ = \int_\Omega |u_n|^p \ln|u_n|\,dx.$$

We get

$$\frac{d}{dt}\frac{1}{2}\int_\Omega |u_n|^2 dx + \frac{d}{dt}\int_0^t\int_\Omega |\Delta u_n|^2 dxdz - \frac{d}{dt}\int_0^t\int_\Omega |\Delta u_n|^{m(x)} dxdz \\ = \int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\Delta u_n(x,t)\,dxdz + \int_\Omega |u_n|^p \ln|u_n|\,dx. \tag{3.8}$$

We call

$$H_n(t) = \frac{1}{2}\int_\Omega |u_n|^2 dx + \int_0^t\int_\Omega |\Delta u_n|^2 dxdz - \int_0^t\int_\Omega |\Delta u_n|^{m(x)} dxdz. \tag{3.9}$$

So,

$$\frac{d}{dt}H_n(t) = \int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\Delta u_n(x,t)\,dxdz + \int_\Omega |u_n|^p \ln|u_n|\,dx. \tag{3.10}$$

To proceed, we utilize Young's inequality. Consequently, we obtain

$$\int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\Delta u_n(x,t)\,dxdz \\ = \int_0^t g(t-z)\int_\Omega \Delta u_n(x,t)\left(\Delta u_n(x,z) - \Delta u_n(x,t)\right)dxdz + \int_0^t g(t-z)\|\Delta u_n\|_2^2 dz$$

$$\leq \frac{1}{2}\int_0^t g(t-z)\|\Delta u_n(x,z)-\Delta u_n(x,t)\|_2^2 dz$$
$$+\frac{1}{2}\int_0^t g(t-z)\|\Delta u_n\|_2^2 dz+\int_0^t g(t-z)\|\Delta u_n\|_2^2 dz.$$

This results in

$$\int_0^t g(t-z)\int_\Omega \Delta u_n(x,z)\Delta u_n(x,t)\,dxdz$$
$$\leq \frac{1}{2}\int_0^t g(t-z)\|\Delta u_n(x,z)-\Delta u_n(x,t)\|_2^2 dz+\frac{3}{2}\int_0^t g(t-z)\|\Delta u_n\|_2^2 dz.$$

Therefore, based on hypothesis (A_2) and equation (3.7), we get

$$\frac{d}{dt}H_n(t)\leq E(u_n(t))-\frac{1}{2}\left(1-\int_0^t g(z)\,dz\right)\|\Delta u_n\|_2^2$$
$$+\int_\Omega \frac{|\Delta u_n|^{m(x)}}{m(x)}dx+\frac{1}{p}\int_\Omega |u_n|^p\ln|u_n|\,dx$$
$$-\frac{1}{p^2}\|u_n\|_p^p+\frac{3}{2}\int_0^t g(t-z)\|\Delta u_n\|_2^2 dz+\int_\Omega |u_n|^p\ln|u_n|\,dx, \tag{3.11}$$

and

$$\frac{d}{dt}H_n(t)\leq E(u_n(t))+2\int_0^t g(z)\,dz\,\|\Delta u_n\|_2^2-\frac{1}{2}\|\Delta u_n\|_2^2+\int_\Omega \frac{|\Delta u_n|^{m(x)}}{m(x)}dx$$
$$-\frac{1}{p^2}\|u_n\|_p^p+\left(1+\frac{1}{p}\right)\int_\Omega |u_n|^p\ln|u_n|\,dx,$$

$$\frac{d}{dt}H_n(t)\leq E(u_n(0))+2(1-l)\|\Delta u_n\|_2^2+\left(1+\frac{1}{p}\right)\int_\Omega |u_n|^p\ln|u_n|\,dx. \tag{3.12}$$

Now, utilizing Lemma 6, we arrive at

$$\int_\Omega |u_n|^p\ln|u_n|\,dx\leq \int_{\{x\in\Omega:|u_n|\geq 1\}} |u_n|^p\ln|u_n|\,dx\leq (ev)^{-1}.\int_{\{x\in\Omega:|u_n|\geq 1\}} |u_n|^{p+v}dx$$
$$\leq (ev)^{-1}.\|u_n\|_{p+v}^{p+v}, \tag{3.13}$$

here $0 < v < p^* - p$, $p^* = \frac{Np}{N-p}$. By using the Gagliardo-Nirenberg interpolation inequality as presented in [3], we obtain

$$\|u_n\|_{p+v}^{p+v} \le C\|\Delta u_n\|_m^{\beta(p+v)}\|u_n\|_2^{(1-\beta)(p+v)}, \tag{3.14}$$

where $\beta = \frac{Nm(p+v-2)}{(p+v)(Nm-2N+2m)}$. By employing Young's inequality with ε we get

$$\|u_n\|_{p+v}^{p+v} \le \varepsilon\|\Delta u_n\|_m^m + C(\varepsilon)\|u_n\|_2^{\frac{m_-(1-\beta)(p+v)}{p_- - \beta(p+v)}}. \tag{3.15}$$

Consequently, equation (3.12) implies

$$\frac{d}{dt}H_n(t) \le \left(1+\frac{1}{p}\right)(ev)^{-1}\left[\varepsilon\|\Delta u_n\|_m^m + C(\varepsilon)\|u_n\|_2^{\frac{m_-(1-\beta)(p+v)}{p_- - \beta(p+v)}}\right] + E(u_n(0)) + 2(1-l)\|\Delta u_n\|_2^2. \tag{3.16}$$

We obtain

$$\frac{d}{dt}H_n(t) \le \left(1+\frac{1}{p}\right)(ev)^{-1}\left[\varepsilon\|\Delta u_n\|_m^m + C(\varepsilon)\|u_n\|_2^{\frac{m_-(1-\beta)(p+v)}{p_- - \beta(p+v)}}\right] + E(u_n(0)) + 2(1-l)\|\Delta u_n\|_2^2.$$

Set $\psi = \frac{m_-(1-\beta)(p+v)}{2(m-\beta)(p+v)} > 1$ and suppose $\min\left\{\|\Delta u_n\|_{m(x)}^{m_-}, \|\Delta u_n\|_{m(x)}^{m_+}\right\} = \|\Delta u_n\|_{m(x)}^{m_-}$, then by (2.1) we obtain

$$\frac{d}{dt}H_n(t) \le \left(1+\frac{1}{p}\right)(ev)^{-1}\left[C(\varepsilon)\|u_n\|_2^{2\psi} + \varepsilon\int_\Omega |\Delta u_n|^{m(x)}dx\right] + E(u_n(0)) + 2(1-l)\|\Delta u_n\|_2^2.$$

By integrating this inequality from 0 to t, we obtain

$$H_n(t) \le H_n(0) + E(u_n(0))t + \left(1+\frac{1}{p}\right)(ev)^{-1}2C(\varepsilon)\int_0^t \frac{1}{2}\|u_n\|_2^{2\psi}dz + \left(1+\frac{1}{p}\right)(ev)^{-1}\varepsilon\int_0^t\int_\Omega |\Delta u_n|^{m(x)}dxdz + 2(1-l)\int_0^t \|\Delta u_n\|_2^2 dz. \tag{3.17}$$

We define the constant $H_n(0) + E(u_n(0))t = c_1$, which is dependent on t, within the interval $t \in [0,T]$. Now, we select $l \ge \frac{3}{4}$ and $\varepsilon = \frac{evp}{2(p+1)}$, in order to derive

$$H_n(t) \le c_1(t) + \frac{1}{2}\int_0^t \|\Delta u_n\|_2^2 dz - \frac{1}{2}\int_0^t\int_\Omega |\Delta u_n|^{m(x)}dxdz + c_2\int_0^t \frac{1}{2}\|u_n\|_2^{2\psi}dz$$

$$\leq c_1(t) + \frac{1}{2} H_n(t) + c_2 \int_0^t H_n^{\psi} dz,$$

here $c_2 = 2C(\varepsilon)\left(1+\frac{1}{p}\right)(ev)^{-1}$. Consequently, we get

$$H_n(t) \leq c_1(t) + c_2 \int_0^t H_n^{\psi} dz. \tag{3.18}$$

Applying the Gronwall-Bellman-Bihari type inequality, we will obtain

$$H_n(t) = \frac{1}{2}\int_\Omega |u_n|^2 dx + \int_0^t \int_\Omega |\Delta u_n|^2 dxdz - \int_0^t \int_\Omega |\Delta u_n|^{m(x)} dxdz \leq C_T, \tag{3.19}$$

where, C_T depends on T. Our assumption $\min\left\{\|\Delta u_n\|_{m(x)}^{m_-}, \|\Delta u_n\|_{m(x)}^{m_+}\right\} = \|\Delta u_n\|_{m(x)}^{m_-}$, combined with (A_1) gives $\|\Delta u_n\|_{m(x)}^2 \leq \|\Delta u_n\|_{m(x)}^{m_-}$. Now, referring to equation (2.1) we obtain

$$\int_0^T \|\Delta u_n\|_{m(x)}^2 dz \leq \int_0^T \|\Delta u_n\|_{m(x)}^{m_-} dz \leq \int_0^T \int_\Omega |\Delta u_n|^{m(x)} dxdz \leq C_T. \tag{3.20}$$

Given the existence of equation (3.20), we can apply the Sobolev embedding to deduce

$$\int_0^T \|u_n\|_m^2 dx \leq C \int_0^T \|\Delta u_n\|_{m(x)}^2 dz \leq C_T. \tag{3.21}$$

The continuity of E yields a constant C and considering that $u_n(x,0) \longrightarrow u_0(x)$ in $W_0^{2,m(x)}(\Omega) \cap L^p(\Omega)$, we get

$$E(u_n(x,0)) \leq C, \text{ for any } n. \tag{3.22}$$

Additionally, we have as derived from equation (3.6)

$$\int_\Omega |u_{n_t}|^2 dx \leq -\frac{d}{dt} E(u_n),$$

integrating from 0 to t yields

$$\int_0^t \|u_n'(z)\|_2^2 dz + E(u_n) \leq E(u_n(x,0)) \leq C. \tag{3.23}$$

Together with the standard compactness results, the estimates (3.19), (3.20), (3.21) and (3.23), will lead to

$$u_n \longrightarrow u \text{ weakly in } L^2\left(0,T; W_0^{2,m(x)}(\Omega) \cap L^p(\Omega)\right) \cap L^2\left(0,T; H_0^2(\Omega)\right),$$

$$u_n' \longrightarrow u' \text{ weakly in } L^2\left(0,T;L^2(\Omega)\right),$$
$$|\Delta u_n|^{m(x)-2}\Delta u_n \longrightarrow \xi \text{ weakly in } L^2\left(0,T;L^{m'(x)}(\Omega)\right). \tag{3.24}$$

Now, utilizing the Aubin-Lions lemma, we obtain

$$u_n \longrightarrow u \text{ in } C\left(0,T;L^2(\Omega)\right). \tag{3.25}$$

The operator $|k|^{m(x)-2}k$ combined with the Minty-Browder condition leads to $\xi = |\Delta u|^{m(x)-2}\Delta u$. This concludes the proof. □

4. Upper bound for blow-up time

We obtain an upper bound for the blow-up time of solutions to equation (1.1).

Theorem 2. *Assume the presence of a weak solution $u(x,t)$ as defined in (1.1) for the problem indicated in equation (1.1). Also $(A_1)-(A_3)$ satisfied. We can conclude that the solution u blows up in finite time T^*,which is determined by*

$$T^* \leq \frac{(1+\gamma)\left(1+\frac{1}{\phi}\right)\|u_0\|_2^2}{2\mu\gamma J(0)}, \tag{4.1}$$

where ϕ, γ and μ are positive constants.

Proof. We multiply equation (1.1) by u_t and integrate over Ω, to obtain

$$\int_\Omega u_t^2 dx = -\int_\Omega \Delta u \Delta u_t dx + \int_\Omega |\Delta u|^{m(x)-2}\Delta u \Delta u_t dx + \int_\Omega |u|^{p-2} u \ln|u|\, u_t dx$$
$$+\int_\Omega \int_0^t g(t-z)\Delta^2 u(x,z)\, u_t(x,t)\, dz dx,$$

consequently

$$\int_\Omega u_t^2 dx - \frac{1}{2}\int_0^t g'(t-z)\|\Delta u(x,z)-\Delta u(x,t)\|_2^2 dz + \frac{1}{2}g(t)\|\Delta u\|_2^2$$
$$= -\frac{1}{2}\frac{d}{dt}\|\Delta u\|_2^2 + \frac{d}{dt}\int_\Omega \frac{|\Delta u|^{m(x)}}{m(x)}dx$$
$$-\frac{1}{2}\frac{d}{dt}\int_0^t g(t-z)\|\Delta u(x,z)-\Delta u(x,t)\|_2^2 dz + \frac{d}{dt}\int_0^t g(z)dz\|\Delta u\|_2^2$$
$$-\frac{1}{p^2}\frac{d}{dt}\|u\|_p^p + \frac{1}{p}\frac{d}{dt}\int_\Omega |u|^p \ln|u|\, dx.$$

Now, we set a functional

$$J(t) = -\frac{1}{2}\int_0^t g(t-z)\|\Delta u(x,z) - \Delta u(x,t)\|_2^2 dz$$
$$-\frac{1}{2}\left(1 - \int_0^t g(z)\,dz\right)\|\Delta u\|_2^2 + \int_\Omega \frac{|\Delta u|^{m(x)}}{m(x)}dx + \frac{1}{p}\int_\Omega |u|^p \ln|u|\,dx - \frac{1}{p^2}\|u\|_p^p, \tag{4.2}$$

where by utilizing hypothesis (A_2), we derive

$$\frac{dJ(t)}{dt} = \int_\Omega u_t^2 dx + \frac{1}{2}g(t)\|\Delta u\|_2^2 - \frac{1}{2}\int_0^t g'(t-z)\|\Delta u(x,z) - \Delta u(x,t)\|_2^2 dz \geq 0. \tag{4.3}$$

We define an auxiliary functional,

$$K(t) = \int_0^t \int_\Omega u^2(x,z)\,dxdz + A. \tag{4.4}$$

Subsequently

$$K'(t) = 2\int_\Omega \int_0^t u(x,z)\,u_t(x,z)\,dzdx + \int_\Omega u_0^2(x)\,dx, \tag{4.5}$$

and

$$K''(t) = 2\int_\Omega uu_t dx$$
$$= -2\int_\Omega u(x,t)\,\Delta\left(|\Delta u|^{m(x)-2}\Delta u\right)dx + 2\int_\Omega u(x,t)\int_0^t g(t-z)\,\Delta^2 u(x,z)\,dzdx$$
$$-2\int_\Omega u(x,t)\,\Delta^2 u dx + 2\int_\Omega u(x,t)\,|u|^{p-2}u\ln|u|\,dx.$$

So that

$$K''(t) = -2\|\Delta u\|_2^2 - 2\int_\Omega |\Delta u|^{m(x)}dx + 2\int_\Omega |u|^p \ln|u|\,dx + 2\int_0^t g(t-z)\,dz\,\|\Delta u\|_2^2.$$
$$+2\int_0^t g(t-z)\int_\Omega \Delta u(x,t)\left(\Delta u(x,z) - \Delta u(x,t)\right)dxdz.$$

Next, select a constant μ such that $m_+ < \mu < p$ and $\mu(\mu-2) > \frac{1-l}{l}$. This yields

$$
\begin{aligned}
K''(t) = {} & 2\mu J(t) + \mu \int_0^t g(t-z)\left\|\Delta u(x,z) - \Delta u(x,t)\right\|_2^2 dz \\
& + 2\mu \int_\Omega \frac{|\Delta u|^{m(x)}}{m(x)} dx + \mu\left(1 - \int_0^t g(z)\,dz\right)\|\Delta u\|_2^2 \\
& - \frac{2\mu}{p}\int_\Omega |u|^p \ln|u|\,dx - 2\int_\Omega |\Delta u|^{m(x)}\,dx + \frac{2\mu}{p^2}\|u\|_p^p \\
& - 2\left(1 - \int_0^t g(t-z)\,dz\right)\|\Delta u\|_2^2 + 2\int_\Omega |u|^p \ln|u|\,dx \\
& + 2\int_0^t g(t-z)\int_\Omega \Delta u(x,t)\left(\Delta u(x,z) - \Delta u(x,t)\right) dx dz.
\end{aligned}
$$

Such that

$$
\begin{aligned}
K''(t) \geq {} & 2\mu J(t) + \mu \int_\Omega g(t-z)\left\|\Delta u(x,z) - \Delta u(x,t)\right\|_2^2 dx dz \\
& + \frac{2\mu}{p^2}\|u\|_p^p + (\mu-2)\left(1 - \int_0^t g(t-z)\,dz\right)\|\Delta u\|_2^2 \\
& + \left(\frac{2\mu}{p} - 2\right)\int_\Omega |u|^p \ln|u|\,dx + \left(\frac{2\mu}{m_+} - 2\right)\int_\Omega |\Delta u|^{m(x)}\,dx \\
& + \int_0^t g(t-z)\int_\Omega \Delta u(x,t)\left(\Delta u(x,z) - \Delta u(x,t)\right) dx dz.
\end{aligned}
$$

Utilizing (A_2) and assuming $|u| \geq 1$, we obtain

$$
\begin{aligned}
K''(t) \geq {} & 2\mu J(t) + \left[(\mu-2)\,l + \frac{(1-l)}{\mu}\right]\|\Delta u\|_2^2 + \left(\frac{2\mu}{m_+} - 2\right)\int_\Omega |\Delta u|^{m(x)}\,dx + \frac{2\mu}{p^2}\|u\|_p^p \\
& + 2\left(1 - \frac{\mu}{p}\right)\int_\Omega |u|^p \ln|u|\,dx \\
& \geq 2\mu J(t). \qquad (4.6)
\end{aligned}
$$

From equation (4.3), we deduce

$$J(t) \geq J(0) + \int_0^t \int_\Omega u_t^2(x,z)\,dxdz. \tag{4.7}$$

Now, considering $\phi > 0$

$$\left(K'(t)\right)^2 \leq 4(1+\phi)\left[\int_\Omega \int_0^t u(x,z)\,u_t(x,z)\,dzdx\right]^2 + \left(1+\frac{1}{\phi}\right)\left[\int_\Omega u_0^2(x)\,dx\right]^2.$$

By utilizing Hölder's inequality, we obtain

$$K'^2 \leq 4(1+\phi)\left(\int_0^t \int_\Omega u^2(x,z)\,dxdz\right)\left(\int_0^t \int_\Omega u_t^2(x,z)\,dxdz\right) + \left(1+\frac{1}{\phi}\right)\left[\int_\Omega u_0^2(x)\,dx\right]^2.$$

As ϕ is arbitrary and positive, we select $\gamma = \phi = \sqrt{\frac{\mu}{2}} - 1 > 0$, we get

$$K''(t)K(t) - (1+\gamma)\left(K'(t)\right)^2 \geq -(1+\gamma)\left(1+\frac{1}{\phi}\right)\left[\int_\Omega u_0^2(x)\,dx\right]^2 + 2\mu A J(0). \tag{4.8}$$

Since we possess equation (4.7) and $J(0) > 0$, in equation (4.8) we select $A > 0$ to be sufficiently large such that

$$K''(t)K(t) - (1+\gamma)\left(K'(t)\right)^2 \geq 0. \tag{4.9}$$

From this we infer the finite time blow up of the solution u at a time T^*, using Levine's [14] concavity method as detailed. By selecting A as

$$A = \frac{(1+\gamma)\left(1+\frac{1}{\phi}\right)\left[\int_\Omega u_0^2(x)\,dx\right]^2}{2\mu J(0)},$$

we obtain an upper bound for T^* as provided below:

$$0 < T^* \leq \frac{(1+\gamma)\left(1+\frac{1}{\phi}\right)\|u_0\|_2^2}{2\mu\gamma J(0)}. \tag{4.10}$$

□

5. Conclusion

As far as we know, there have not been any existence and blow up results in the literature known for the variable exponent biharmonic equation with logarithmic source term. Our work extends the works for some parabolic type equations with variable exponents treated in the literature to the variable exponent biharmonic equation with logarithmic source term.

References

[1] S. Antontsev and S. Shmarev, *Evolution PDEs with nonstandard growth conditions.* Springer, 2015, vol. 4, doi: 10.2991/978-94-6239-112-3.

[2] I. Bialynicki-Birula and J. Mycielski, "Wave equations with logarithmic nonlinearities." *Bulletin de l'Academie Polonaise des Sciences. Serie des Sciences, Mathematiques, Astronomiques et Physiques*, vol. 23, no. 4, pp. 461–466, 1975, doi: 10.1109/PROC.1973.9296.

[3] H. Brezis, *Functional analysis, Sobolev spaces and partial differential equations.* New York: Springer, 2011, vol. 2, no. 3, doi: 10.1007/978-0-387-70914-7.

[4] T. Cazenave and A. Haraux, "Équations d'évolution avec non linéarité logarithmique." *Annales de la Faculté des sciences de Toulouse: Mathématiques*, vol. 2, no. 1, pp. 21–51, 1980.

[5] Y. Chen, S. Levine, and M. Rao, "Variable exponent, linear growth functionals in image restoration." *SIAM journal on Applied Mathematics*, vol. 66, no. 4, pp. 1383–1406, 2006, doi: 10.1137/050624522.

[6] Q. V. Chuong, L. C. Nhan, and L. X. Truong, "Existence and non-existence of global solutions of pseudo-parabolic equations involving p (x)-Laplacian and logarithmic nonlinearity." *Journal of Elliptic and Parabolic Equations*, vol. 8, no. 1, pp. 483–512, 2022, doi: 10.58997/ejde.2022.46.

[7] Q. V. Chuong, L. C. Nhan, and L. X. Truong, "Existence and nonexistence of global solutions to the Cahn-Hilliard equation with variable exponent sources." *Electronic Journal of Differential Equations*, vol. 2022, no. 01-87, pp. 46–22, 2022, doi: 10.58997/ejde.2022.46.

[8] L. Diening, P. Harjulehto, P. Hästö, and M. Ruzicka, *Lebesgue and Sobolev spaces with variable exponents.* Springer, 2011. doi: 10.1007/978-3-642-18363-8.

[9] J. Ferreira, N. Irkıl, E. Pişkin, C. Raposo, and M. Shahrouzi, "Blow up of solutions for a Petrovsky type equation with logarithmic nonlinearity." *Bulletin of the Korean Mathematical Society*, vol. 59, no. 6, pp. 1495–1510, 2022, doi: 10.4134/BKMS.b210853.

[10] P. Górka, "Logarithmic Klein-Gordon Equation." *Acta Physica Polonica B*, vol. 40, no. 1, 2009.

[11] Y. Han, "A class of fourth-order parabolic equation with arbitrary initial energy." *Nonlinear Analysis: Real World Applications*, vol. 43, pp. 451–466, 2018, doi: 10.1016/j.nonrwa.2018.03.009.

[12] Y. Han, C. Cao, and P. Sun, "A p-Laplace equation with logarithmic nonlinearity at high initial energy level." *Acta Applicandae Mathematicae*, vol. 164, no. 1, pp. 155–164, 2019, doi: 10.1007/s10440-018-00230-4.

[13] V. Kalantarov and L. Olga, "The occurrence of collapse for quasilinear equations of parabolic and hyperbolic types." *Journal of Soviet Mathematics*, vol. 10, pp. 53–70, 1978, doi: 10.1007/BF01109723.

[14] H. A. Levine, "Some nonexistence and instability theorems for solutions of formally parabolic equations of the form $Pu_t = -Au + \mathfrak{F}(u)$." *Archive for Rational Mechanics and Analysis*, vol. 51, no. 5, pp. 371–386, 1973, doi: 10.1007/BF00263041.

[15] Y. Liu, W. Zhou, and B. Liang, "Existence and blow-up of solutions to a parabolic equation with nonstandard growth conditions." *Bulletin of the Australian Mathematical Society*, vol. 99, no. 2, pp. 242–249, 2019, doi: 10.1017/S0004972718001181.

[16] L. Narayanan and G. Soundararajan, "Nonexistence of global solutions of a viscoelastic p(x)-Laplacian equation with logarithmic nonlinearity." *AIP Conference proceedings*, vol. 2451, no. 1, 2022, doi: 10.1063/5.0095323.

[17] L. E. Payne, *Improperly posed problems in partial differential equations.* SIAM, 1975. doi: 10.1137/1.9781611970463.ch1.

[18] I. G. Petrowsky, "Uber das Cauchysche Problem fur Systeme von partiellen Differentialgleichungen." *Matematicheskii Sbornik*, vol. 2, no. 5, pp. 815–870, 1937.

[19] I. G. Petrowsky, "Sur l'analyticité des solutions des systèmes d'équations différentielles." *Matematicheskii Sbornik*, vol. 5, no. 1, pp. 3–70, 1939.

[20] E. Pişkin and B. Okutmuştur, *An introduction to Sobolev spaces.* Bentham Science Publishers, 2021. doi: 10.2174/97816810891331210101.

[21] C. Qu, W. Zhou, and B. Liang, "Asymptotic behavior for a fourth-order parabolic equation modeling thin film growth." *Applied Mathematics Letters*, vol. 78, pp. 141–146, 2018, doi: 10.1016/j.aml.2017.11.015.

[22] A. Rahmoune, "Bounds for blow-up solutions of a semilinear pseudo-parabolic equation with a memory term and logarithmic nonlinearity in variable space." *Mathematica Scandinavica*, vol. 128, no. 3, 2022, doi: 10.7146/math.scand.a-133418.

[23] M. Ruzicka, *Electrorheological fluids: modeling and mathematical theory.* Springer, 2007. doi: 10.1007/BFb0104029.

[24] J. Wang and C. Liu, "p-biharmonic parabolic equations with logarithmic nonlinearity." *Electronic Journal of Differential Equations*, vol. 2019, no. 08, pp. 1–18, 2019.

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