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GENERALIZED STONEAN BE-ALGEBRAS

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Abstract. In this paper, the notion of a generalized Stonean BE-algebra is introduced. A set of equivalent conditions is given for every quasi-complemented BE-algebra to become a generalized Stonean BE-algebra. A necessary and sufficient condition is stated for a self-distributive and commutative BE-algebra to become a generalized Stonean BE-algebra. The concept of Stonean filters is introduced and then generalized Stonean BE-algebras are characterized by Stonean filters. The notion of hyper Stonean BE-algebras is introduced and then a characterization theorem in terms of prime filters is given.

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1. INTRODUCTION

The notion of BE-algebras was introduced and extensively studied by H.S. Kim and Y.H. Kim in [3]. These classes of BE-algebras were introduced as a generalization of the class of BCK-algebras of K. Iseki and S. Tanaka [2]. Some properties of filters of BE-algebras were studied by S.S. Ahn and Y.H. Kim in [1] and by B.L. Meng in [8]. In [12], A. Walendziak discussed some significant properties of commutative BE-algebras. He also investigated the relationship between BE-algebras, implicative algebras and J-algebras. In [8], Meng introduced the notion of prime filters in BCK-algebras, and then gave a description of the filter generated by a set, and obtained some of fundamental properties of prime filters. Some properties of prime ideals are investigated in BCK-algebras [2]. The first author studied some properties of prime filters in BE-algebras. Also, the author extensively studied the algebraic as well as the topological properties of prime filters of commutative BE-algebras [11]. The notion of dual annihilators of commutative BE-algebra is introduced and studied extensively the properties of these dual annihilators [4]. In 2020, the notions of regular filters [6] and O-filters [5] in commutative BE-algebras are introduced and studied the interconnection between those two special classes of filters. A. Soleimani

Nasab and A. Borumand Saeid introduced the notion of Stonean implicative filters in Hilbert algebras and characterized the Stonean Hilbert algebras with the help of Stonean implicative filters [9].

In this paper, certain properties of dual annihilator filters, prime filters, and minimal prime filters of commutative BE-algebras are investigated. The notion of generalized Stonean BE-algebras is introduced. It is proved that a self-distributive and commutative BE-algebra will become a generalized Stonean BE-algebra whenever every prime filter of the BE-algebra is minimal. It is observed that every generalized Stonean BE-algebra is a quasi-complemented BE-algebra, the other direction is not always true. However, a set of equivalent conditions is given for every quasi-complemented BE-algebra to become a generalized Stonean BE-algebra. Some necessary conditions of generalized Stonean BE-algebras are derived. Generalized Stonean BE-algebras are also characterized with the help of σ-filters, dual annihilator filters and regular filters.

Filters are important substructures in a BE-algebra and play an important role. It is well understood that filters are the kernels of congruences. Filter theory is crucial in the study of any class of logical algebras. From a logical standpoint, different filters correspond to different sets of valid formulas in an appropriate logic. Designing various types of filters in some logical algebra, on the other hand, is also algebraically interesting. With this motivation, we introduce the concept of Stonean filters is introduced in commutative BE-algebras. Some sufficient conditions are derived for every filter of a self-distributive and commutative BE-algebra to become a Stonean filter. The notion of hyper Stonean BE-algebras is introduced and observed that every hyper Stonean BE-algebra is generalized Stonean BE-algebra. Though every generalized Stonean BE-algebra need not to be a hyper BE-algebra, however, a sufficient condition is derived for every generalized Stonean BE-algebra to become a hyper BE-algebra. The class of hyper Stonean BE-algebras is characterized with the help of prime filters of self-distributive and commutative BE-algebras. It is observed that every maximal filter of a self-distributive and commutative BE-algebra is Stonean filter. Some equivalent conditions are given for every filter of a commutative BE-algebra to become a Stonean filter which leads to a characterization of generalized Stonean BE-algebras. Finally, an extension property of Stonean filters of commutative BE-algebras is derived.

2. Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [3], [11], [7], [4], [6], and [5] for the ready reference.

Definition 1. [3] An algebra $(X,*,1)$ of type $(2,0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x*x=1$,
(2) $x*1=1$,

(3) $1 * x = x$,
(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X$.

A BE-algebra X is called *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called *transitive* if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called *commutative* if $(x * y) * y = (y * x) * x$ for all $x, y \in X$. Every commutative BE-algebra is transitive. For any $x, y \in X$, define $x \vee y = (y * x) * x$. If X is commutative, then $(X, \vee)$ is a semilattice [12]. We introduce a relation $\leq$ on a BE-algebra X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$. Clearly $\leq$ is reflexive. If X is commutative, then $\leq$ is transitive, anti-symmetric and hence a partial order on X.

Theorem 1. *[3] Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) *$1 \leq x$ implies $x = 1$,*
(2) *$y \leq z$ implies $x * y \leq x * z$ and $z * x \leq y * x$.*

Definition 2. [1] A non-empty subset F of a BE-algebra X is called a filter of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,
(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

For any non-empty subset A of a transitive BE-algebra X, the set $\langle A \rangle = \{x \in X \mid a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) = 1$ for some $a_1, a_2, \ldots, a_n \in A\}$ is the smallest filter containing A. For any $a \in X$, $\langle a \rangle = \{x \in X \mid a^n * x = 1$ for some $n \in \mathbb{N}\}$, where $a^n * x = a * (a * (\cdots * (a * x) \cdots))$ with the repetition of a is n times, is called *the principal filter generated a*. If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$. If X is commutative and self-distributive, then $\langle a \rangle \cap \langle b \rangle = \langle a \vee b \rangle$ for any $a, b \in X$. Let F be a filter of a transitive BE-algebra and $a \in X$, then $\langle F \cup \{a\} \rangle = \{x \in X \mid a^n * x \in F$ for some $n \in \mathbb{N}\}$. A proper filter P of a BE-algebra X is called *prime* [11] if $F \cap G \subseteq P$ implies $F \subseteq P$ or $G \subseteq P$ for any two proper filters F, G of X. A proper filter P of a BE-algebra is prime if and only if $\langle x \rangle \cap \langle y \rangle \subseteq P$ implies $x \in P$ or $y \in P$ for any $x, y \in X$. A proper filter M of a transitive BE-algebra is called maximal if there exists no proper filter Q such that $M \subset Q$. Every maximal filter of a commutative BE-algebra is prime.

Theorem 2. *[11] Let F and G be two filters of a transitive BE-algebra X. Then*

$$F \vee G = \{x \in X \mid a * (b * x) = 1 \text{ for some } a \in F, b \in G\}$$

is the supremum of F and G. Hence the set $\mathcal{F}(X)$ of all filters of X is a lattice with respect to the operation $\vee$.

Lemma 1. *[4] Let X be a commutative BE-algebra. Then for any $x, y, a \in X$*

(1) *$y * z \leq (z * x) * (y * x)$,*
(2) *$(x * y) \vee a \leq (x \vee a) * (y \vee a)$.*

For any non-empty subset A of a BE-algebra X, the *dual annihilator* [4] of A is defined as $A^+ = \{x \in X \mid x \vee a = 1 \text{ for all } a \in A\}$. In a commutative BE-algebra X, the set A^+ forms a filter of X such that $A \cap A^+ = \{1\}$. In case of $A = \{a\}$, we have $(a)^+ = \{x \in X \mid a \vee x = 1\}$. Clearly $X^+ = \{1\}$ and $\{1\}^+ = X$. An element $a \in X$ is called *dual dense* if $(a)^+ = \{1\}$.

Proposition 1. *[4] Let X be a commutative BE-algebra and $\varnothing \neq A, B \subseteq X$. Then*
(1) *if $A \subseteq B$, then $B^+ \subseteq A^+$,*
(2) $A \subseteq A^{++}$,
(3) $A^+ = A^{+++}$.

Proposition 2. *[4] Let F, G be two filters of a commutative BE-algebra X. Then*
(1) *$F \cap G = \{1\}$ if and only if $F \subseteq G^+$,*
(2) $(F \vee G)^+ = F^+ \cap G^+$,
(3) $(F \cap G)^{++} = F^{++} \cap G^{++}$.

Proposition 3. *[4] Let X be a commutative BE-algebra and $a, b \in X$. Then*
(1) $\langle a \rangle \subseteq (a)^{++}$,
(2) *$a \leq b$ implies $(a)^+ \subseteq (b)^+$,*
(3) *$a \in (b)^{++}$ implies $(b)^+ \subseteq (a)^+$.*

A prime filter P of a commutative BE-algebra X is called *minimal* [7] if it is minimal in the class of all prime filters of X.

Theorem 3. *[7] Let X be a self-distributive and commutative BE-algebra. A prime filter P of X is minimal if and only if to each $x \in P$, there exists $y \notin P$ such that $x \vee y = 1$.*

Proposition 4. *[6] Let X be a self-distributive and commutative BE-algebra. Then for any $x \in X$, we have*

$$(x)^+ = \bigcap \{P \mid P \text{ is a minimal prime filter such that } x \notin P\}$$

A filter F of a commutative BE-algebra X is called a *dual annihilator filter* [4] if $F = F^{++}$. A filter F of a commutative BE-algebra X is called a *regular filter* [6] if $(x)^{++} \subseteq F$ whenever $x \in F$. Every dual annihilator filter is a regular filter.

Proposition 5. *[6] Every minimal prime filter of a self-distributive and commutative BE-algebra is a regular filter.*

A filter F of a commutative BE-algebra X is called an *O-filter* [5] if $F = O(S)$ for some $\vee$-closed subset S of X. Every O-filter of a commutative BE-algebra is a regular filter. A commutative BE-algebra X is called *quasi-complemented* [5] if to each $x \in X$, there exists $y \in X$ such that $x \vee y = 1$ and $(x)^+ \cap (y)^+ = \{1\}$.

Theorem 4. *[5] A commutative BE-algebra X is quasi-complemented if and only if to each $x \in X$, there exists $y \in X$ such that $(x)^{++} = (y)^+$*

A filter F of a commutative BE-algebra X is called a σ-*filter* [10] of X if $\sigma(F) = F$ where $\sigma(F) = \{x \in X \mid (x)^+ \vee F = X\}$. Every σ-filter of a commutative BE-algebra is a regular filter and every σ-filter of a commutative BE-algebra is an O-filter.

3. PROPERTIES OF GENERALIZED STONEAN BE-ALGEBRAS

In this section, the notion of generalized Stonean BE-algebras is introduced. A set of equivalent conditions is derived for a quasi-complemented BE-algebra to become a generalized Stonean BE-algebra. It is proved that every quasi-complemented BE-algebra is a generalized Stonean if and only if $\mathcal{D}^+(X) = \{(x)^+ \mid x \in X\}$ is a Boolean algebra.

Lemma 2. *Let X be a commutative BE-algebra and $a, b \in X$. Then the following properties hold:*

(1) $(a)^+ = (\langle a\rangle)^+$;
(2) $a \in (b)^+$ *if and only if* $\langle a\rangle \subseteq (b)^+$;
(3) $\langle a\rangle \cap \langle b\rangle = \{1\}$ *if and only if* $\langle a\rangle \subseteq (b)^+$;
(4) $(a)^+ \cap (a * b)^+ \subseteq (b)^+$;
(5) $(a)^{++} \cap (b)^{++} = (a \vee b)^{++}$.

Proof. (1) Since $\{a\} \subseteq \langle a\rangle$, we get $(\langle a\rangle)^+ \subseteq (a)^+$. Conversely, let $x \in (a)^+$. Then $a \vee x = 1$. For any $c \in \langle a\rangle$, we get $a^n * c = 1$ for some positive integer n. Now

$$1 = a^n * c \leq (a^n * c) \vee x \leq (a \vee x)^n * (c \vee x) = 1 \vee (c \vee x) = c \vee x$$

Hence $c \vee x = 1$ for any $c \in \langle a\rangle$. Then $x \in (\langle a\rangle)^+$. Therefore $(a)^+ \subseteq (\langle a\rangle)^+$.

(2) Assume that $a \in (b)^+$. Then $a \vee b = 1$. Let $x \in \langle a\rangle$. Then there exists a positive integer n such that $a^n * x = 1$. Now, we get

$$\begin{aligned} 1 &= a^n * x \leq (a^n * x) \vee b \leq (a \vee b)^n * (x \vee b) && \text{by Lemma 1(2)} \\ &= 1 * (x \vee b) = x \vee b \end{aligned}$$

which yields that $x \in (b)^+$. Therefore $\langle a\rangle \subseteq (b)^+$. Converse is clear since $a \in \langle a\rangle$.

(3) Let $a, b \in L$. Assume that $\langle a\rangle \cap \langle b\rangle = \{1\}$. Let $x \in \langle a\rangle$. For any $y \in \langle b\rangle$, we get $x \vee y \in \langle a\rangle \cap \langle b\rangle = \{1\}$. Hence $x \vee y = 1$ for any $y \in \langle b\rangle$. Thus $x \in (\langle b\rangle)^+ = (b)^+$. Therefore $\langle a\rangle \subseteq (b)^+$. Conversely, assume that $\langle a\rangle \subseteq (b)^+$. Let $x \in \langle a\rangle \cap \langle b\rangle$. Then $x \in \langle a\rangle \subseteq (b)^+$ and $x \in \langle b\rangle$. Hence $x \in \langle b\rangle \cap (b)^+ = \{1\}$. Thus $x = 1$. Therefore $\langle a\rangle \cap \langle b\rangle = \{1\}$.

(4) Let $x \in (a)^+ \cap (a * b)^+$. Then $a \vee x = 1$ and $(a * b) \vee x = 1$. Hence

$$\begin{aligned} 1 &= (a * b) \vee x \leq (a \vee x) * (b \vee x) && \text{by Lemma 1(2)} \\ &= 1 * (b \vee x) = b \vee x \end{aligned}$$

which means $b \vee x = 1$. Hence $x \in (b)^+$. Therefore $(a)^+ \cap (a * b)^+ \subseteq (b)^+$.

(5) Let $a,b \in X$. Since $a,b \leq a \vee b$, we get $(a)^+, (b)^+ \subseteq (a \vee b)^+$. Hence $(a \vee b)^{++} \subseteq (a)^{++}, (b)^{++}$. Thus $(a \vee b)^{++} \subseteq (a)^{++} \cap (b)^{++}$. Conversely, let $x \in (a)^{++} \cap (b)^{++}$. Suppose $y \in (a \vee b)^+$ be an arbitrary element. Since $y \in (a \vee b)^+$, we get

$$\begin{aligned} y \vee (a \vee b) = 1 &\Rightarrow y \vee a \in (b)^+ \Rightarrow x \vee y \vee a = 1 \qquad \text{since } x \in (b)^{++} \\ &\Rightarrow x \vee y \in (a)^+ \Rightarrow x \vee (x \vee y) = 1 \qquad \text{since } x \in (a)^{++} \\ &\Rightarrow x \vee y = 1 \qquad \text{for all } y \in (a \vee b)^+ \end{aligned}$$

which means that $x \in (a \vee b)^{++}$. Therefore $(a)^{++} \cap (b)^{++} \subseteq (a \vee b)^{++}$. □

Proposition 6. *Let X be a commutative BE-algebra and $a,b \in X$. Then the following assertions are equivalent:*

(1) $a \vee b = 1$;
(2) $(a)^{++} \cap \langle b \rangle = \{1\}$;
(3) $(a)^{++} \cap (b)^{++} = \{1\}$.

Proof. (1)⇒ (2): Let $a,b \in X$. Assume that $a \vee b = 1$. Then $b \in (a)^+$. By Lemma 2(1), we get $\langle b \rangle \subseteq (a)^+$. Hence $(a)^{++} \cap \langle b \rangle \subseteq (a)^{++} \cap (a)^+ = \{1\}$.
(2)⇒ (3): Assume that $(a)^{++} \cap \langle b \rangle = \{1\}$ for any $a,b \in X$. By Lemma 2(3), we get $(a)^{++} \subseteq (b)^+$. Therefore $(a)^{++} \cap (b)^{++} \subseteq (b)^+ \cap (b)^{++} = \{1\}$.
(3)⇒ (1): Assume that $(a)^{++} \cap (b)^{++} = \{1\}$ for any $a,b \in X$. By Lemma 2(3), we get $(a)^{++} \subseteq (b)^{+++} = (b)^+$. Hence $a \in (a)^{++} \subseteq (b)^+$, which means $a \vee b = 1$. □

Lemma 3. *Every prime filter of a BE-algebra contains a minimal prime filter.*

Proof. Let P be a prime filter of a BE-algebra X. Consider

$$\Im = \{Q \mid Q \text{ is a prime filter such that } Q \subseteq P\}.$$

Clearly $P \in \Im$ and hence $\Im \neq \varnothing$. By Zorn's Lemma, $\Im$ has a minimal element, say P_0. Clearly P_0 is the required minimal prime filter in X. □

Proposition 7. *The intersection of all minimal prime filters of a BE-algebra is* $\{1\}$.

Proof. Clearly $\{1\} \subseteq \bigcap\{P \mid P \text{ is a minimal prime filter }\}$. Let $x \neq 1$ or $x \notin \langle 1 \rangle$. Then there exists a prime filter P such that $x \notin P$. By Lemma 3, there exists a minimal prime filter P_0 of X such that $P_0 \subseteq P$. Since $x \notin P$, we must have $x \notin P_0$. Hence $x \notin \cap\{P \mid P \text{ is a minimal prime filter }\}$. Thus $\bigcap\{P \mid P \text{ is a minimal prime filter }\} \subseteq \{1\}$. Therefore $\{1\} = \bigcap\{P \mid P \text{ is a minimal prime filter }\}$. □

Theorem 5. *A prime filter P of a self-distributive and commutative BE-algebra X is minimal if and only if it satisfies the following condition:*

$$x \notin P \textit{ if and only if } (x)^+ \subseteq P$$

Proof. Assume that P is minimal. Let $x \notin P$. Then clearly $(x)^+ \subseteq P$. Conversely, let $(x)^+ \subseteq P$. Suppose $x \in P$. Since P is minimal, there exists $y \notin P$ such that $x \vee y = 1$. Hence $y \in (x)^+$ and $y \notin P$. Thus $(x)^+ \nsubseteq P$, which is a contradiction. Therefore $x \notin P$.

Conversely, assume that X satisfies in the above condition. Let $x \in P$. By the assumed condition, we get $(x)^+ \nsubseteq P$. Hence, there exists $y \in (x)^+$ such that $y \notin P$. Thus $y \vee x = 1$ where $y \notin P$. By Theorem 5, P is a minimal prime filter of X. □

Definition 3. A commutative BE-algebra X is called a generalized *Stonean BE-algebra* if $(x)^+ \vee (x)^{++} = X$ for all $x \in X$.

Example 1. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ on X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	b	c
b	1	a	1	c
c	1	a	b	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	1	1
b	1	1	b	1
c	1	1	1	c

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. Clearly $(a)^+ = \{1, b, c\}$; $(b)^+ = \{1, a, c\}$ and $(c)^+ = \{1, a, b\}$. Hence, it can be seen that $(a)^{++} = \{1, a\}$; $(b)^{++} = \{1, b\}$; $(c)^{++} = \{1, c\}$. It can be easily verified that $(a)^+ \vee (a)^{++} = X$; $(b)^+ \vee (b)^{++} = X$ and $(c)^+ \vee (c)^{++} = X$. Therefore X is a generalized Stonean BE-algebra.

Example 2. Let $X = \{1, a, b, c, d\}$ and define a binary operation $*$ on X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	c	b
b	1	a	1	b	a
c	1	a	1	1	a
d	1	1	1	b	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	1	1	a
b	1	1	b	d	b
c	1	1	d	c	b
d	1	a	b	b	d

Clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. It is easy to check that $(b)^+ = \{1, a\}$ and $(b)^{++} = \{1, b, c\}$. Hence $(b)^+ \vee (b)^{++} = \{1, a\} \vee \{1, b, c\} = X$. Similarly, we can see that $(a)^+ \vee (a)^{++} = X$; $(c)^+ \vee (c)^{++} = X$ and $(d)^+ \vee (d)^{++} = X$. Therefore X is a generalized Stonean BE-algebra.

Proposition 8. *If every prime filter of a self-distributive and commutative BE-algebra X is minimal, then X is a generalized Stonean BE-algebra.*

Proof. Assume that every prime filter of a self-distributive and commutative BE-algebra X is minimal. Let $x \in X$. Suppose $(x)^+ \vee (x)^{++} \neq X$. Then there exists a maximal filter P of X such that $(x)^+ \vee (x)^{++} \subseteq P$. Since every maximal filter is prime, P is a prime filter of X. Hence $(x)^+ \subseteq P$ and $(x)^{++} \subseteq P$. Since $(x)^+ \subseteq P$, by Theorem

5, we get $x \notin P$. Clearly $x \in (x)^{++} \subseteq P$. Hence $x \in P$, which is a contradiction. Thus $(x)^{+} \vee (x)^{++} = X$. Therefore X is a generalized Stonean BE-algebra. □

The converse of Proposition 8 is not true. That is, every generalized Stonean BE-algebra need not to have all of it's prime filters to be minimal. Indeed, consider the following example:

Example 3. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c
1	1	a	b	c
a	1	1	1	1
b	1	b	1	1
c	1	c	c	1

$\vee$	1	a	b	c
1	1	1	1	1
a	1	a	b	c
b	1	b	b	c
c	1	c	c	c

Then clearly $(X, *, \vee, 1)$ is a self-distributive and commutative BE-algebra. Observe that $(x)^{+} = \{1\}$ for all $x \in X$. Hence $(x)^{+} \vee (x)^{++} = X$ for all $x \in X$. Therefore X is a generalized Stonean BE-algebra. Clearly $\{1\}$ is a prime filter of X. It can be easily verified that $P = \{1, c\}$ is a prime filter of X which is not minimal.

Proposition 9. *Every generalized Stonean BE-algebra with a dual dense element is a quasi-complemented BE-algebra.*

Proof. Suppose X is generalized Stonean. Let $d \in X$ be such that $(d)^{+} = \{1\}$. Let $x \in X$. Then $(x)^{+} \vee (x)^{++} = X$. Hence $d \in (x)^{+} \vee (x)^{++}$, which implies $a * (b * d) = 1$ for some $a \in (x)^{+}$ and $b \in (x)^{++}$. Since $b \in (x)^{++}$, we get $(b)^{++} \subseteq (x)^{++}$. Thus

$$
\begin{aligned}
a * (b * d) = 1 \Rightarrow a \leq b * d &\Rightarrow (a)^{+} \subseteq (b * d)^{+} \\
&\Rightarrow (a)^{+} \cap (b)^{+} \subseteq (b)^{+} \cap (b * d)^{+} \\
&\Rightarrow (a)^{+} \cap (b)^{+} \subseteq (d)^{+} = \{1\} && \text{by Lemma 2(4)} \\
&\Rightarrow (a)^{+} \cap (b)^{+} = \{1\} \Rightarrow (a)^{+} \subseteq (b)^{++} && \text{by Lemma 2(3)} \\
&\Rightarrow (a)^{+} \subseteq (x)^{++} && \text{since } (b)^{++} \in (x)^{++} \\
&\Rightarrow (a)^{+} \cap (x)^{+} = \{1\} && \text{by Lemma 2(3)}
\end{aligned}
$$

Since $a \in (x)^{+}$, we get $a \vee x = 1$. Therefore L is quasi-complemented. □

The importance of the sufficient condition of having a dual-dense element can be seen in Example 1. Clearly X is a generalized Stonean BE-algebra. Observe that X has no dual-dense element. For $a \in X$, there exists no $x \in X$ such that $a \vee x = 1$ and $(a)^{+} \cap (x)^{+} = \{1\}$. Therefore X is not a quasi-complemented BE-algebra.

A filter F of a BE-algebra X is called a factor of X if there exists a proper filter G such that $F \cap G = \{1\}$ and $F \vee G = X$. Denote $\mathcal{D}^{++}(X) = \{(x)^{++} \mid x \in X\}$ and $\mathcal{D}^{+}(X) = \{(x)^{+} \mid x \in X\}$. The converse of Proposition 9 need not be true. However, in the following, a set of equivalent conditions is given to show that every quasi-complemented BE-algebra to become a generalized Stonean BE-algebra.

Theorem 6. *Let X be a quasi-complemented BE-algebra. Then the following assertions are equivalent:*

(1) *X is a generalized Stonean BE-algebra;*
(2) *each $(x)^+$ is a factor of X;*
(3) *for each $x \in X$, there exists $x' \in X$ such that $(x)^+ \vee (x')^+ = X$;*
(4) *for any $x, y \in X$, $(x)^+ \vee (y)^+ = (x \vee y)^+$;*
(5) *$\mathcal{D}^{++}(X)$ is a sublattice of the lattice $\mathcal{F}(X)$ of all filters of X.*

Proof. (1)$\Rightarrow$ (2): Assume that X is generalized Stonean. Let $x \in X$. Clearly $(x)^+ \cap (x)^{++} = \{1\}$. By (1), we get $(x)^+ \vee (x)^{++} = X$. Therefore $(x)^+$ is a factor of L.
(2)$\Rightarrow$ (3): Assume condition (2). Let $x \in X$. Since X is quasi-complemented, there exists $x' \in X$ such that $(x)^{++} = (x')^+$. Since $(x)^+$ is a factor of X, there exists a filter G such that $(x)^+ \cap G = \{1\}$ and $(x)^+ \vee G = X$. Since $(x)^+ \cap G = \{1\}$, we get $G \subseteq (x)^{++} = (x')^+$. Therefore $X = (x)^+ \vee G \subseteq (x)^+ \vee (x')^+$. Hence $(x)^+ \vee (x')^+ = X$.
(3)$\Rightarrow$ (4): Assume condition (3). Let $x, y \in X$. By (3), there exists $x' \in X$ such that $(x)^+ \vee (x')^+ = X$. Clearly $(x)^+ \vee (y)^+ \subseteq (x \vee y)^+$. Conversely, let $a \in (x \vee y)^+$. Then $a \vee x \vee y = 1$, which gives $a \vee y \in (x)^+$. By Proposition 1.7(2) and Lemma 3.1, we get

$$\begin{aligned}
a \vee y \in (x)^+ &\Rightarrow (x)^{++} \subseteq (a \vee y)^+ \Rightarrow (x)^{++} \cap (a \vee y)^{++} = \{1\} && \text{by Lemma 2(3)}\\
&\Rightarrow (x)^{++} \cap (a)^{++} \cap (y)^{++} = \{1\} && \text{by Lemma 2(5)}\\
&\Rightarrow (x)^{++} \cap (a)^{++} \subseteq (y)^+ && \text{by Lemma 2(3)}\\
&\Rightarrow (x')^+ \cap (a)^{++} \subseteq (y)^+ \Rightarrow (x')^+ \cap \langle a \rangle \subseteq (y)^+ && \text{by Proposition 3(1)}
\end{aligned}$$

Clearly $(x)^+ \cap \langle a \rangle \subseteq (x)^+$. Hence $a \in \langle a \rangle = X \cap \langle a \rangle = \{(x)^+ \vee (x')^+\} \cap \langle a \rangle = \{(x)^+ \cap \langle a \rangle\} \vee \{(x')^+ \cap \langle a \rangle\} \subseteq (x)^+ \vee (y)^+$. Therefore $(x \vee y)^+ \subseteq (x)^+ \vee (y)^+$.
(4) $\Rightarrow$ (5): For any $x, y \in X$, it is clear that $(x)^{++} \cap (y)^{++} = (x \vee y)^{++}$. Since X is quasi-complemented, there exist $x', y' \in X$ such that $(x)^{++} = (x')^+$ and $(y)^{++} = (y')^+$. Hence $(x)^{++} \vee (y)^{++} = (x')^+ \vee (y')^+ = (x' \vee y')^+ = (c)^{++}$ for some $c \in X$, as X is quasi-complemented. Therefore $\mathcal{D}^{++}(X)$ is a sublattice of $\mathcal{F}(X)$.
(5)$\Rightarrow$ (1): Assume that the condition (5) holds. Let $x \in X$. Since X is quasi-complemented, there exists $y \in X$ such that $(x)^{++} = (y)^+$. Since $\mathcal{D}^{++}(X)$ is a sublattice of $\mathcal{F}(X)$, we get $(x)^{++} \vee (y)^{++} = (t)^{++}$ for some $t \in X$. Thus $\langle x \rangle \vee \langle y \rangle \subseteq (x)^{++} \vee (y)^{++} = (t)^{++}$. Therefore, by Proposition 2(2), we get

$$(t)^+ = (t)^{+++} \subseteq (\langle x \rangle \vee \langle y \rangle)^+ = (x)^+ \cap (y)^+ = (x)^+ \cap (x)^{++} = \{1\}$$

which implies that $(t)^{++} = \{1\}^+ = X$. Hence $(x)^+ \vee (x)^{++} = (y)^{++} \vee (x)^{++} = (t)^{++} = X$. Therefore X is a generalized Stonean BE-algebra. $\square$

The following corollaries state the properties of generalized Stonean BE-algebra in terms of minimal prime filters. Two filters F and G of a BE-algebra X are called *comaximal* if $F \vee G = L$.

Corollary 1. *If X is a self-distributive and generalized Stonean BE-algebra, then any two distinct minimal prime filters of X are comaximal.*

Proof. Assume that X is a generalized Stonean BE-algebra. By condition (2) of the main theorem, each $(x)^+$ is a *factor* of X. Let P and Q be two distinct minimal prime filters of X. Choose $a \in P - Q$. Hence $(a)^+ \subseteq Q$. Since P is minimal, by Proposition 5, we get that $(a)^{++} \subseteq P$. Since $(a)^+$ is a factor of X, there exists a filter G such that $(a)^+ \cap G = \{1\}$ and $(a)^+ \vee G = X$. Hence $G \subseteq (a)^{++} \subseteq P$. Thus $X = (a)^+ \vee G \subseteq Q \vee P$. Therefore P and Q are comaximal. □

Corollary 2. *If X is a self-distributive and generalized Stonean BE-algebra, then every prime filter contains a unique minimal prime filter.*

Proof. Let P be a prime filter of X. Suppose P contains two distinct minimal prime filters, say Q_1 and Q_2. Then $Q_1 \vee Q_2 \subseteq P$. Since X is generalized Stonean, by Corollary 1, we get $Q_1 \vee Q_2 = X$. Hence $X = Q_1 \vee Q_2 \subseteq P$, which is a contradiction. Therefore P contains a unique minimal prime filter. □

From Theorem 6, it can be easily observed that $\mathcal{D}^+(X)$ is a semilattice with respect to operation $\vee$ of filters. We now define that $(\mathcal{D}^+(X), \vee)$ is complemented if to each $(a)^+ \in \mathcal{D}^+(X)$, there exists $(b)^+ \in \mathcal{D}^+(X)$ such that $(a)^+ \cap (b)^+ = \{1\}$ and $(a)^+ \vee (b)^+ = X$.

Theorem 7. *Let X be a quasi-complemented BE-algebra. Then X is a generalized Stonean BE-algebra if and only if $\mathcal{D}^+(X)$ is a complemented semilattice.*

Proof. Assume that X is a generalized Stonean BE-algebra. Let $(x)^+, (y)^+ \in \mathcal{D}^+(X)$. Since X is a generalized Stonean BE-algebra, by Theorem 6, we get $(x)^+ \vee (y)^+ = (x \vee y)^+$. Hence $(\mathcal{D}^+(X), \vee)$ is a semilattice. Let $(x)^+ \in \mathcal{D}^+(X)$ where $x \in X$. Since X is quasi-complemented, there exists $x' \in X$ such that $(x)^{++} = (x')^+$. Clearly $(x)^+ \cap (x')^+ = (x)^+ \cap (x)^{++} = \{1\}$. Since X is generalized Stonean, we get $(x)^+ \vee (x)^{++} = X$. Hence $(x)^+ \vee (x')^+ = X$. Thus $(x')^+$ is the complement of $(x)^+$ in $\mathcal{D}^+(X)$. Therefore $\mathcal{D}^+(X)$ is a complemented semilattice.

Conversely, assume that $\mathcal{D}^+(X)$ is a complemented semilattice. Let $x \in X$. Then $(x)^+ \in \mathcal{D}^+(X)$. Since $(\mathcal{D}^+(X), \vee)$ is complemented, there exists $(x')^+ \in \mathcal{D}^+(X)$ such that $(x)^+ \cap (x')^+ = \{1\}$ and $(x)^+ \vee (x')^+ = X$. Since $(x)^+ \cap (x')^+ = \{1\}$, we get $(x')^+ \subseteq (x)^{++}$. Hence $X = (x)^+ \vee (x')^+ \subseteq (x)^+ \vee (x)^{++}$, which gives $(x)^+ \vee (x)^{++} = X$. Therefore X is a generalized Stonean BE-algebra. □

In [10], the notion of σ-filters is introduced in commutative BE-algebras and the properties of σ-filters are studied. In the following theorem, generalized Stonean BE-algebras are characterized with the help of σ-filters and dual annihilator filters.

Theorem 8. *The following assertions are equivalent in a commutative BE-algebra:*
(1) *X is generalized Stonean;*
(2) *every regular filter is a σ-filter;*
(3) *every dual annihilator filter is a σ-filter.*

Proposition 10. *Every prime σ-filter of a self-distributive and commutative BE-algebra X is a minimal prime filter.*

Proof. Let P be a prime σ-filter of X. Since P is proper, there exists $a_0 \in X - P$. Let $x \in P$. Then $x \in \sigma(P)$ and hence $(x)^+ \vee P = X$. Since $a_0 \in X$, there exists $a \in (x)^+$ and $b \in P$ such that $a * (b * a_0) = 1 \in P$. Since $a \in (x)^+$, we get $a \vee x = 1$. Suppose $a \in P$. Since $b \in P$, we must have $a_0 \in P$ because of $a * (b * a_0) \in P$. But $a_0 \in P$ contradicts the fact that $a_0 \in X - P$. Hence $a \notin P$ such that $a \vee x = 1$. By Proposition 5, P is a minimal prime filter of X. □

The converse of Proposition 10 is true in particular condition.

Theorem 9. *If X is a self-distributive and generalized Stonean BE-algebra, then every minimal prime filter of X is a σ-filter.*

Proof. Suppose that X is a generalized Stonean BE-algebra. Let P be a minimal prime filter of X. By Proposition 5, P is a regular filter of X. Since X is generalized Stonean, by Theorem 8, P is a σ-filter of X. □

4. Stonean filters of BE-algebras

In this section, the notion of Stonean filters is introduced in commutative BE-algebras. Some properties of Stonean filters are derived in commutative BE-algebras. A set of equivalent conditions is given for every filter of a commutative BE-algebra to become a Stonean filter.

Definition 4. A proper filter F of a commutative BE-algebra X is called a *Stonean filter* of X if $F^+ \subseteq (x)^+ \vee (x)^{++}$ for all $x \in X$.

Example 4. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	b	1	b
b	1	a	1	1	a
c	1	a	b	1	d
d	1	1	1	1	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	b	c	1
b	1	b	b	1	1
c	1	c	1	c	1
d	1	1	1	1	d

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. It can be easily seen that $(a)^+ = \{1, d\}$, $(b)^+ = \{1, c, d\}$; $(c)^+ = \{1, b, d\}$ and $(d)^+ = \{1, a, b, c\}$. Now, we see that $(a)^{++} = \{1, a, b, c\}$; $(b)^{++} = \{1, b\}$; $(c)^{++} = \{1, c\}$ and $(d)^{++} = \{1, d\}$.

Consider the filter $F = \{1, c\}$. Then $F^+ = \{1, b, d\}$. It can be easily verified that $F^+ \subseteq (x)^+ \vee (x)^{++}$ for all $x \in X$. Therefore F is a Stonean filter of X.

Example 5. Let $X = \{1, a, b, c\}$ be a set. Define a binary operation $*$ X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	1	1	d
b	1	c	1	c	d
c	1	b	b	1	d
d	1	a	b	c	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	b	c	1
b	1	b	b	1	1
c	1	c	1	c	1
d	1	1	1	1	d

Then clearly $(X, *, \vee, 1)$ is a commutative BE-algebra. It can be easily seen that $(a)^+ = \{1, d\}$, $(b)^+ = \{1, c, d\}$; $(c)^+ = \{1, b, d\}$ and $(d)^+ = \{1, a, b, c\}$. Now, we see that $(a)^{++} = \{1, a, b, c\}$; $(b)^{++} = \{1, b\}$; $(c)^{++} = \{1, c\}$ and $(d)^{++} = \{1, d\}$. Consider the filter $F = \{1, c\}$. Then $F^+ = \{1, b, d\}$. It can be easily verified that $F^+ \subseteq (x)^+ \vee (x)^{++}$ for all $x \in X$. Therefore F is a Stonean filter of X.

Proposition 11. *Every prime filter of a self-distributive and commutative BE-algebra is a Stonean filter.*

Proof. Let P be a prime filter of a self-distributive and commutative BE-algebra X. Suppose $x \in P$. Then $\langle x \rangle \subseteq P$. Hence, by Proposition 1(1), we get $P^+ \subseteq (x)^+$. Thus $P^+ \subseteq (x)^+ \subseteq (x)^+ \vee (x)^{++}$. Suppose $x \notin P$. Since P is prime, we get $(x)^+ \subseteq P$. Hence $P^+ \subseteq (x)^{++}$. Thus $P^+ \subseteq (x)^{++} \subseteq (x)^+ \vee (x)^{++}$. Therefore P is Stonean. $\square$

The converse of Proposition 11 is not true, which means that every Stonean filter of a BE-algebra need not be prime. For, consider the self-distributive and commutative BE-algebra given in Example 5. It can be easily noticed that the filter $F = \{1, c\}$ is Stonean but not prime because of $a \vee d = 1 \in F$ but neither $a \in F$ nor $d \in F$.

Since every maximal filter of a self-distributive and commutative BE-algebra is a prime filter, the following corollary is a direct consequence of Proposition 11.

Corollary 3. *Every maximal filter of a self-distributive and commutative BE-algebra is a Stonean filter.*

Definition 5. A commutative BE-algebra X is called a *hyper Stonean BE-algebra* if $\langle x \rangle \vee (x)^+ = X$ for all $x \in X$.

Since $\langle x \rangle \subseteq (x)^{++}$ for all $x \in X$, it can be easily verified that every hyper Stonean BE-algebra is a generalized Stonean BE-algebra. However, every generalized Stonean BE-algebra need not be a hyper Stonean BE-algebra. For consider the following example:

Example 6. Let $X = \{1,a,b,c\}$ be a set. Define a binary operation $*$ X and then deduce the operation $\vee$ from $*$ as given in the following tables:

$*$	1	a	b	c	d
1	1	a	b	c	d
a	1	1	a	c	d
b	1	1	1	c	d
c	1	a	b	1	d
d	1	1	1	c	1

$\vee$	1	a	b	c	d
1	1	1	1	1	1
a	1	a	a	1	a
b	1	a	b	1	b
c	1	1	1	c	1
d	1	a	b	1	d

Clearly $(X,*,\vee,1)$ is a commutative BE-algebra. It is easily seen that $(a)^+ = (b)^+ = \{1,c\}$; $(c)^+ = \{1,a,b,d\}$ and $(d)^+ = \{1,c\}$. Hence X has no dual-dense element. Now, we see that $(a)^{++} = (b)^{++} = \{1,a,b,d\}$; $(c)^{++} = \{1,c\}$ and $(d)^{++} = X$. It is routinely verified that X is a generalized Stonean BE-algebra. It is clear that $\langle a\rangle = \{1,a\}$. Hence

$$\langle a\rangle \vee (a)^+ = \{1,a\} \vee \{1,b\} = \{1,a,b\} \neq X.$$

Therefore X is not a hyper Stonean BE-algebra.

In the following result, a sufficient condition is derived for every generalized Stonean BE-algebra to become a hyper Stonean BE-algebra.

Proposition 12. *Let X be a self-distributive generalized Stonean BE-algebra. If every filter of X is a dual annihilator filter, then X is a hyper Stonean BE-algebra.*

Proof. Let $x \in X$. By hypothesis, we get $(x)^{++} = \langle x\rangle$. Since X is generalized Stonean, we get $(x)^+ \vee \langle x\rangle = (x)^+ \vee (x)^{++} = X$. Therefore X is a hyper Stonean BE-algebra. □

In the following theorem, a set of equivalent conditions is given for every self-distributive and commutative BE-algebra to become a hyper Stonean BE-algebra.

Theorem 10. *Let X be a self-distributive and commutative BE-algebra. Then the following assertions are equivalent:*

(1) *X is a hyper Stonean BE-algebra;*
(2) *every prime filter is maximal;*
(3) *every prime filter is minimal.*

Proof. (1) $\Rightarrow$ (2): Assume that X is a hyper Stonean BE-algebra. Let P be a prime filter of X. Suppose P is not maximal. Then there exists a proper filter Q of X such that $P \subset Q$. Choose $x \in Q - P$. Since X is hyper Stonean, we get $\langle x\rangle \vee (x)^+ = X$. Since $x \in Q$, we get $\langle x\rangle \subseteq Q$. Since $x \notin P$ and P is prime, we get $(x)^+ \subseteq P$. Hence $X = \langle x\rangle \vee (x)^+ \subset Q \vee P = Q$, which is a contradiction. Therefore P is maximal.
(2) $\Rightarrow$ (3): Since every maximal filter is prime, it is clear.
(3) $\Rightarrow$ (1): Assume that every prime filter is minimal. Let $x \in X$. Suppose $\langle x\rangle \vee (x)^+ \neq X$. Then there exists a prime filter P such that $\langle x\rangle \vee (x)^+ \subseteq P$. Hence $x \in \langle x\rangle \subseteq P$ and

$(x)^+ \subseteq P$. Since P is minimal, by Theorem 5, we get $x \notin P$ which is a contradiction. Hence $\langle x\rangle \vee (x)^+ = X$. Therefore X is a hyper Stonean BE-algebra □

The converse of Corollary 3 is not true. For, consider the self-distributive and commutative BE-algebra given in Example 4. Note that $\{1\}$ is a Stonean filter but not a prime filter. However, in the following theorem, a set of equivalent conditions is given for every Stonean filter of a BE-algebra to become a maximal filter.

Theorem 11. *Let X be a self-distributive hyper Stonean BE-algebra and F be a proper filter of X. Then the following assertions are equivalent:*

(1) *F is a maximal filter;*
(2) *for each $x \in X$, $x \notin F$ implies $(x)^+ \subseteq F$;*
(3) *F is a prime Stonean filter.*

Proof. (1) ⇒ (2): Assume that F is a maximal filter of X. Let $x \in X$ be such that $x \notin F$. Since F is a prime filter of X, we get that $(x)^+ \subseteq F$.
(2) ⇒ (3): Assume condition (2). Let $x \in X$. Suppose $x \in F$. Then $\langle x\rangle \subseteq F$. Hence, by Proposition 1(1), we get $F^+ \subseteq (x)^+$. Thus $F^+ \subseteq (x)^+ \subseteq (x)^+ \vee (x)^{++}$. Suppose $x \notin F$. By condition (2), we get $(x)^+ \subseteq F$. Hence $F^+ \subseteq (x)^{++}$. Thus $F^+ \subseteq (x)^{++} \subseteq (x)^+ \vee (x)^{++}$. Therefore F is Stonean. We now prove that F is prime. Let $x, y \in X$ be such that $x \vee y \in F$. Suppose $x \notin F$. By condition (2), we have $(x)^+ \subseteq F$. Now,

$$\begin{aligned}
(x)^+ \vee \langle y\rangle &= X \cap \{(x)^+ \vee \langle y\rangle\} \\
&= \{(x)^+ \vee \langle x\rangle\} \cap \{(x)^+ \vee \langle y\rangle\} && \text{since } X \text{ is hyper Stonean} \\
&= (x)^+ \vee \{\langle x\rangle \cap \langle y\rangle\} = (x)^+ \vee \langle x \vee y\rangle \subseteq F && \text{since } (x)^+ \subseteq F \text{ and } x \vee y \in F
\end{aligned}$$

which gives $\langle y\rangle \subseteq (x)^+ \vee \langle y\rangle \subseteq F$. Hence $y \in F$. Therefore F is a prime Stonean filter.
(3) ⇒ (1): Assume that F is a prime Stonean filter of X. Suppose F is not maximal. Then there exists a proper filter F' of X such that $F \subset F'$. Choose $x \in F' - F$. Since F is Stonean, we get $F^+ \subseteq (x)^+ \vee (x)^{++}$. Since F is prime and $x \notin F$, we get $(x)^+ \subseteq F \subset F'$. Since $x \in F'$, we get $\langle x\rangle \subseteq F'$. Since X is hyper Stonean, we get $X = \langle x\rangle \vee (x)^+ \subseteq F'$, which is a contradiction. Therefore F is a maximal filter. □

In the following theorem, a set of equivalent conditions is given for every filter of a commutative BE-algebra to become a Stonean filter.

Theorem 12. *Let X be a commutative BE-algebra. Then the following are equivalent:*

(1) *X is a generalized Stonean;*
(2) *every filter is a Stonean filter;*
(3) *$\{1\}$ is a Stonean filter.*

Proof. (1) ⇒ (2): Assume that X is Stonean. Let F be a filter of X. Since X is Stonean, we have $F^+ \subseteq X = (x)^+ \vee (x)^{++}$ for all $x \in X$. Hence F is a Stonean filter

of X.
(2) $\Rightarrow$ (3): It is clear.
(3) $\Rightarrow$ (1): Assume that $\{1\}$ is a Stonean filter of X. Hence $X = \{1\}^+ \subseteq (x)^+ \vee (x)^{++}$. Therefore X is a generalized Stonean BE-algebra. $\square$

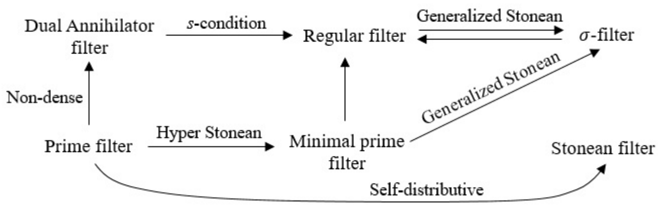
Theorem 13. *(Extension property of Stonean filters) Let F and G be two filters of a commutative BE-algebra such that $F \subseteq G$. If F is a Stonean filter, then so is G.*

Proof. If $F \subseteq G$, by Proposition 1(1), $G^+ \subseteq F^+$. Since F is a Stonean filter, then $G^+ \subseteq (x)^+ \vee (x)^{++}$ for all $x \in X$, hence G is a Stonean filter. $\square$

5. Conclusion

In this work, we have considered the Stonean BE-algebras. The notion of Stonean filter has been introduced and considered them in detail. These type of filter play a basic role. Based on these facts, we give a classification for BE-algebras. The notion of hyper Stonean BE-algebras had been introduced and we show that these structures are particular cases of commutative BE-algebras. We think such results are very useful for the further characterization of generalized Stonean BE-algebra in terms of congruences of this structure. In future, we plan to investigate the topological properties of generalized Stonean BE-algebras. Further properties of Stonean filters and their interconnections between various filters existed in BE-algebras can also be investigated.

In the following diagram, we show the relationships between some filters of BE-algebras. The notion "$A \longrightarrow B$" means A should be B.



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