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ON A MULTIDIMENSIONAL CLOSE-TO-CYCLIC SYSTEM OF DIFFERENCE EQUATIONS

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Abstract. This paper investigates the solvability and dynamic properties of the following multi-dimensional close-to-cyclic system of nonlinear difference equations

$$y_{n+1}^{(i)} = \frac{a_i y_n^{(i+1)} \left(y_{n-k}^{(i+1)}\right)^{p_{i+1}} + b_i}{\left(y_{n-k+1}^{(i)}\right)^{p_i}}; \quad n \in \mathbb{N}_0,$$

where $y_n^{(i+k)} = y_n^{(i)}, p_{i+k} = p_i, a_{i+k} = a_i, b_{i+k} = b_i; i = \overline{1,k}$, the initial values $y_{-k}^{(i)}, y_{-k+1}^{(i)}, \ldots, y_0^{(i)}$ and a_i and b_i, $i = \overline{1,k}$, are positive real numbers and p_i, $i = \overline{1,k}$, are real numbers. The system, characterized by intricate nonlinear interactions, is analyzed to derive explicit solutions and examine its asymptotic behavior. By leveraging a transformation approach, the multidimensional system is reduced to a simpler form, allowing for a comprehensive analysis of its solutions. The study demonstrates that under specific conditions, the system's equilibrium points are globally attractive, ensuring stability. The theoretical findings are supported by numerical examples that highlight the behavior of solutions under various parameter configurations, illustrating the practical applicability of the results.

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1. INTRODUCTION

Nonlinear difference equations play a fundamental role in modeling various dynamic systems across numerous scientific disciplines. Their solutions offer valuable insights into the behavior and stability of such systems, enabling us to make accurate predictions and devise effective control strategies. In this paper, we delve into the study a close-to-cyclic system of nonlinear difference equations, aiming to represent its solutions and investigate their asymptotic behavior in special cases.

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The close-to-cyclic system under consideration exhibits a rich interplay of nonlinear dynamics, where the variables in the system mutually influence each other. This intricate coupling gives rise to intricate behaviors that often defy straightforward analysis. However, by leveraging advanced mathematical techniques and rigorous reasoning, we aim to unravel the underlying dynamics and shed light on the behavior of the system's solutions.

There has been a great interest in finding solutions to system of nonlinear difference equations. Still, most of the papers published in this aspect were limited to system of two or three dimensions at most, see for example [1–24].

Overall, our aim in this paper is to provide a comprehensive understanding of the solutions and asymptotic behavior of the following close-to-cyclic system of nonlinear difference equations

$$y_{n+1}^{(i)} = \frac{a_i y_n^{(i+1)} \left(y_{n-k}^{(i+1)}\right)^{p_{i+1}} + b_i}{\left(y_{n-k+1}^{(i)}\right)^{p_i}}; \quad n \in \mathbb{N}_0, \tag{1.1}$$

where $y_n^{(i+k)} = y_n^{(i)}, p_{i+k} = p_i, a_{i+k} = a_i, b_{i+k} = b_i; i = \overline{1,k}$, the initial values $y_{-k}^{(i)}$, $y_{-k+1}^{(i)}$, $\dots, y_0^{(i)}$ and the parameters a_i and b_i, $i = \overline{1,k}$ are positive real numbers and p_i, $i = \overline{1,k}$, are real numbers.

2. Auxiliary Results

In this section we will present several results needed to prove the main results in Section 3.

Consider the following k-dimensional linear difference equations system

$$w_{n+1}^{(i)} = a_i w_n^{(i+1)} + b_i, \tag{2.1}$$

where $w_n^{(i+k)} = w_n^{(i)}$ and $w_0^{(i)}, a_i, b_i, i = \overline{1,k}$ are positive real numbers.

The following auxiliary result is used several times in the rest of the paper.

Lemma 1. *Let $(w_n^{(i)})_{n\geq 0}$ be the solution to system (2.1). Then for all $n \in \mathbb{N}_0$*

$$w_{kn+j}^{(i)} = \begin{cases} w_j^{(i)} + nT_i, & S = 1, \\ S^n w_j^{(i)} + T_i \left(\dfrac{S^n - 1}{S - 1}\right), & S \neq 1, \end{cases}$$

where, $i = \overline{1,k}$, $j = \overline{0,k-1}$ and

$$S = \prod_{l=1}^{k} a_l, \quad T_i = \sum_{r=2}^{k} \left(\prod_{l=i}^{i+r-2} a_l\right) b_{i+r-1} + b_i. \tag{2.2}$$

Proof. The systems in (2.1) immediately imply, for $i = \overline{1,k}$, the following relations

$$w_{n+k}^{(i)} = a_i w_{n+k-1}^{(i+1)} + b_i$$

$$
\begin{aligned}
&= a_i\left[a_{i+1}w^{(i+2)}_{n+k-2}+b_{i+1}\right]+b_i\\
&= a_ia_{i+1}w^{(i+2)}_{n+k-2}+a_ib_{i+1}+b_i\\
&= a_ia_{i+1}\left[a_{i+2}w^{(i+3)}_{n+k-3}+b_{i+2}\right]+a_ib_{i+1}+b_i\\
&= a_ia_{i+1}a_{i+2}w^{(i+3)}_{n+k-3}+a_ia_{i+1}b_{i+2}+a_ib_{i+1}+b_i\\
&= a_ia_{i+1}a_{i+2}a_{i+3}w^{(i+4)}_{n+k-4}+a_ia_{i+1}a_{i+2}b_{i+3}+a_ia_{i+1}b_{i+2}+a_ib_{i+1}+b_i\\
&\ \ \vdots\\
&= a_ia_{i+1}\dots a_{i+k-1}w^{(i+k)}_{n+k-k}+a_ia_{i+1}\dots a_{i+k-2}b_{i+k-1}\\
&\quad +a_ia_{i+1}\dots a_{i+k-3}b_{i+k-2}+\cdots+a_ib_{i+1}+b_i\\
&= a_ia_{i+1}\dots a_{i+k-1}w^{(i)}_{n}+a_ia_{i+1}\dots a_{i+k-2}b_{i+k-1}\\
&\quad +a_ia_{i+1}\dots a_{i+k-3}b_{i+k-2}+\cdots+a_ib_{i+1}+b_i.
\end{aligned}
$$

So, we have

$$w^{(i)}_{n+k}=\left(\prod_{l=1}^{k}a_l\right)w^{(i)}_n+\left[\sum_{r=2}^{k}\left(\prod_{l=i}^{i+r-2}a_l\right)b_{i+r-1}\right]+b_i.$$

Putting $S=\prod_{l=1}^{k}a_l$ and $T_i=\sum_{r=2}^{k}\left(\prod_{l=i}^{i+r-2}a_l\right)b_{i+r-1}+b_i$. We get

$$w^{(i)}_{n+k}=Sw^{(i)}_n+T_i,$$

for $i=\overline{1,k}$, with the initial values $w^{(i)}_j$, $j=\overline{0,k-1}$.

Consequently, instead of solving system (2.1), we will solve the following equations

$$w^{(i)}_{n+k}=Sw^{(i)}_n+T_i,\quad n\in\mathbb{N}_0, \tag{2.3}$$

where $w^{(i)}_j$, $j=\overline{0,k-1}$, are positive real numbers.

Equation (2.3) yield

$$
\begin{aligned}
w^{(i)}_k&=Sw^{(i)}_0+T_i,\\
w^{(i)}_{k+1}&=Sw^{(i)}_1+T_i,\\
&\ \ \vdots\\
w^{(i)}_{2k-1}&=Sw^{(i)}_{k-1}+T_i.\\
w^{(i)}_{2k}&=Sw^{(i)}_k+T_i=S\left(Sw^{(i)}_0+T_i\right)+T_i=S^2w^{(i)}_0+ST_i+T_i,
\end{aligned}
$$

$$
\begin{aligned}
w_{2k+1}^{(i)} &= Sw_{k+1}^{(i)} + T_i = S\left(Sw_1^{(i)} + T_i\right) + T_i = S^2 w_1^{(i)} + ST_i + T_i,\\
&\vdots\\
w_{3k-1}^{(i)} &= Sw_{2k-1}^{(i)} + T_i = S\left(Sw_{k-1}^{(i)} + T_i\right) + T_i = S^2 w_{k-1}^{(i)} + ST_i + T_i,\\
w_{3k}^{(i)} &= Sw_{2k}^{(i)} + T_i = S\left(S^2 w_0^{(i)} + ST_i + T_i\right) + T_i = S^3 w_0^{(i)} + S^2 T_i\\
&\quad + ST_i + T_i,\\
w_{3k+1}^{(i)} &= Sw_{2k+1}^{(i)} + T_i = S\left(S^2 w_1^{(i)} + ST_i + T_i\right) + T_i = S^3 w_1^{(i)} + S^2 T_i\\
&\quad + ST_i + T_i,\\
&\vdots\\
w_{4k-1}^{(i)} &= Sw_{3k-1}^{(i)} + T_i = S\left(S^2 w_{k-1}^{(i)} + ST_i + T_i\right) + T_i = S^3 w_{k-1}^{(i)} + S^2 T_i\\
&\quad + ST_i + T_i.
\end{aligned}
$$

The inductive argument proves, for $i = \overline{1,k}$, that

$$
\begin{aligned}
w_{kn}^{(i)} &= S^n w_0^{(i)} + \sum_{t=0}^{n-1} S^t T_i,\\
w_{kn+1}^{(i)} &= S^n w_1^{(i)} + \sum_{t=0}^{n-1} S^t T_i,\\
w_{kn+2}^{(i)} &= S^n w_2^{(i)} + \sum_{t=0}^{n-1} S^t T_i,\\
&\vdots\\
w_{kn+k-1}^{(i)} &= S^n w_{k-1}^{(i)} + \sum_{t=0}^{n-1} S^t T_i.
\end{aligned}
$$

More precisely, for $i = \overline{1,k}$ and $j = 0, 1, \ldots, k-1$, we obtain

$$
w_{kn+j}^{(i)} = S^n w_j^{(i)} + \sum_{t=0}^{n-1} S^t T_i.
$$

Thus, for all n$\in \mathbb{N}_0$ we obtain

$$
w_{kn+j}^{(i)} = \begin{cases} w_j^{(i)} + nT_i, & S = 1,\\ S^n w_j^{(i)} + T_i\left(\dfrac{S^n - 1}{S - 1}\right), & S \neq 1. \end{cases} \tag{2.4}
$$

Now, we will prove by induction that the relation (2.4) is true.

A simple verification shows that (2.4) holds for $n=0$. Suppose that (2.4) holds for n, that is $w_{kn+j}^{(i)} = \begin{cases} w_j^{(i)} + nT_i, & S=1, \\ S^n w_j^{(i)} + T_i\left(\frac{S^n-1}{S-1}\right), & S \neq 1. \end{cases}$

We will prove that (2.4) holds for $n+1$. We have

- If $S \neq 1$

$$w_{k(n+1)+j}^{(i)} = Sw_{kn+j}^{(i)} + T_i = S\left[S^n w_j^{(i)} + T_i\left(\frac{S^n-1}{S-1}\right)\right] + T_i$$

$$= S^{n+1} w_j^{(i)} + T_i\left[S\left(\frac{S^n-1}{S-1}\right)\right] + T_i = S^{n+1} w_j^{(i)} + T_i\left[\frac{S^{n+1}-S+S-1}{S-1}\right].$$

So

$$w_{k(n+1)+j}^{(i)} = S^{n+1} w_j^{(i)} + T_i\left(\frac{S^{n+1}-1}{S-1}\right).$$

- If $S=1$

$$w_{k(n+1)+j}^{(i)} = w_{kn+j}^{(i)} + T_i = w_j^{(i)} + nT_i + T_i$$

$$w_{k(n+1)+j}^{(i)} = w_j^{(i)} + (n+1)T_i.$$

Thus, $w_{k(n+1)+j}^{(i)} = \begin{cases} w_j^{(i)} + (n+1)T_i, & S=1, \\ S^n w_j^{(i)} + T_i\left(\frac{S^{n+1}-1}{S-1}\right), & S \neq 1. \end{cases}$

□

3. Main results

In this section we study the solvability of system (1.1) by considering changes of variables that transforms this system to the system of k linear difference equations (2.1).

3.1. *Form of solution*

Here we show that system of difference equations (1.1) is practically solvable, and following the analysis of each of the systems. By using the changes of variables

$$w_n^{(i)} = y_n^{(i)}\left(y_{n-k}^{(i)}\right)^{p_i}, \quad i=\overline{1,k}, \quad n \in \mathbb{N}_0, \tag{3.1}$$

system (1.1) is transformed to the following one

$$w_{n+1}^{(i)} = a_i w_n^{(i+1)} + b_i,$$

which is the same system studied in the previous section.

Relation (3.1), for $i=\overline{1,k}$, yield

$$y_n^{(i)} = w_n^{(i)}\left(y_{n-k}^{(i)}\right)^{-p_i}, \quad n \in \mathbb{N}_0.$$

So, for $i=\overline{1,k}$, we have

$$\begin{aligned} y_{kn}^{(i)} &= w_{kn}^{(i)}\left(y_{kn-k}^{(i)}\right)^{-p_i} = w_{kn}^{(i)}\left[w_{kn-k}^{(i)}\left(y_{kn-2k}^{(i)}\right)^{-p_i}\right]^{-p_i} \\ &= w_{kn}^{(i)}\left(w_{kn-k}^{(i)}\right)^{-p_i}\left(y_{kn-2k}^{(i)}\right)^{(-p_i)^2} = w_{kn}^{(i)}\left(w_{kn-k}^{(i)}\right)^{-p_i}\left[w_{kn-2k}^{(i)}\left(y_{kn-3k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^2}. \end{aligned}$$

Hence

$$\begin{aligned} y_{kn}^{(i)} &= w_{kn}^{(i)}\left(w_{kn-k}^{(i)}\right)^{-p_i}\left(w_{kn-2k}^{(i)}\right)^{(-p_i)^2}\left(y_{kn-3k}^{(i)}\right)^{(-p_i)^3} \\ &= w_{kn}^{(i)}\left(w_{kn-k}^{(i)}\right)^{-p_i}\left(w_{kn-2k}^{(i)}\right)^{(-p_i)^2}\left[w_{kn-3k}^{(i)}\left(y_{kn-4k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^3} \\ &= w_{kn}^{(i)}\left(w_{kn-k}^{(i)}\right)^{-p_i}\left(w_{kn-2k}^{(i)}\right)^{(-p_i)^2}\left(w_{kn-3k}^{(i)}\right)^{(-p_i)^3}\left(y_{kn-4k}^{(i)}\right)^{(-p_i)^4} \\ &= w_{kn}^{(i)}\left(w_{kn-k.1}^{(i)}\right)^{(-p_i)^1}\left(w_{kn-k.2}^{(i)}\right)^{(-p_i)^2}\cdots\left(w_{kn-k(t-1)}^{(i)}\right)^{(-p_i)^{t-1}}\left(y_{kn-kt}^{(i)}\right)^{(-p_i)^t} \\ &= w_{kn}^{(i)}\left(w_{kn-k}^{(i)}\right)^{-p_i}\left(w_{kn-2k}^{(i)}\right)^{(-p_i)^2}\left(w_{kn-3k}^{(i)}\right)^{(-p_i)^3}\cdots\left(w_{kn-kt}^{(i)}\right)^{(-p_i)^t}\cdots \\ &\quad\times\left(w_{k}^{(i)}\right)^{(-p_i)^{n-1}}\left(y_{0}^{(i)}\right)^{(-p_i)^n}. \end{aligned}$$

Hence, we obtain

$$y_{kn}^{(i)} = \left[\prod_{t=0}^{n-1}\left(w_{k(n-t)}^{(i)}\right)^{(-p_i)^t}\right]\left(y_0^{(i)}\right)^{(-p_i)^n}, \quad n\in\mathbb{N}_0. \tag{3.2}$$

By the same argument

$$\begin{aligned} y_{kn+1}^{(i)} &= w_{kn+1}^{(i)}\left(y_{kn+1-k}^{(i)}\right)^{-p_i} = w_{kn+1}^{(i)}\left[w_{kn+1-k}^{(i)}\left(y_{kn+1-2k}^{(i)}\right)^{-p_i}\right]^{-p_i} \\ &= w_{kn+1}^{(i)}\left(w_{kn+1-k}^{(i)}\right)^{-p_i}\left(y_{kn+1-2k}^{(i)}\right)^{(-p_i)^2} \\ &= w_{kn+1}^{(i)}\left(w_{kn+1-k}^{(i)}\right)^{-p_i}\left[w_{kn+1-2k}^{(i)}\left(y_{kn+1-3k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^2} \\ &= w_{kn+1}^{(i)}\left(w_{kn+1-k}^{(i)}\right)^{-p_i}\left(w_{kn+1-2k}^{(i)}\right)^{(-p_i)^2}\left(y_{kn+1-3k}^{(i)}\right)^{(-p_i)^3} \\ &= w_{kn+1}^{(i)}\left(w_{kn+1-k}^{(i)}\right)^{-p_i}\left(w_{kn+1-2k}^{(i)}\right)^{(-p_i)^2}\left[w_{kn+1-3k}^{(i)}\left(y_{kn+1-4k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^3} \\ &= w_{kn+1}^{(i)}\left(w_{kn+1-k}^{(i)}\right)^{-p_i}\left(w_{kn+1-2k}^{(i)}\right)^{(-p_i)^2}\left(w_{kn+1-3k}^{(i)}\right)^{(-p_i)^3}\left(y_{kn+1-4k}^{(i)}\right)^{(-p_i)^4}. \end{aligned}$$

Hence

$$\begin{aligned} y_{kn+1}^{(i)} &= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)}\right)^{-p_i} \left(w_{kn+1-2k}^{(i)}\right)^{(-p_i)^2} \left(w_{kn+1-3k}^{(i)}\right)^{(-p_i)^3} \cdots \\ &\quad \times \left(w_{kn+1-(t-1)k}^{(i)}\right)^{(-p_i)^{t-1}} \left(y_{kn+1-tk}^{(i)}\right)^{(-p_i)^t} \\ &= w_{kn+1}^{(i)} \left(w_{kn+1-k}^{(i)}\right)^{-p_i} \left(w_{kn+1-2k}^{(i)}\right)^{(-p_i)^2} \left(w_{kn+1-3k}^{(i)}\right)^{(-p_i)^3} \cdots \\ &\quad \times \left(w_{kn+1-tk}^{(i)}\right)^{(-p_i)^t} \cdots \left(w_{k+1}^{(i)}\right)^{(-p_i)^{n-1}} \left(y_1^{(i)}\right)^{(-p_i)^n} \\ &= w_{k(n-0)+1}^{(i)} \left(w_{k(n-1)+1}^{(i)}\right)^{-p_i} \left(w_{k(n-2)+1}^{(i)}\right)^{(-p_i)^2} \left(w_{k(n-3)+1}^{(i)}\right)^{(-p_i)^3} \\ &\quad \times \left(w_{k(n-t)+1}^{(i)}\right)^{(-p_i)^t} \cdots \left(w_{k(n-(n-1))+1}^{(i)}\right)^{(-p_i)^{n-1}} \left(y_1^{(i)}\right)^{(-p_i)^n}. \end{aligned}$$

So, we get

$$y_{kn+1}^{(i)} = \left[\prod_{t=0}^{n-1} \left(w_{k(n-t)+1}^{(i)}\right)^{(-p_i)^t}\right] \left(y_1^{(i)}\right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \tag{3.3}$$

Likewise

$$\begin{aligned} y_{kn+2}^{(i)} &= w_{kn+2}^{(i)} \left(y_{kn+2-k}^{(i)}\right)^{-p_i} = w_{kn+2}^{(i)} \left[w_{kn+2-k}^{(i)} \left(y_{kn+2-2k}^{(i)}\right)^{-p_i}\right]^{-p_i} \\ &= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)}\right)^{-p_i} \left(y_{kn+2-2k}^{(i)}\right)^{(-p_i)^2} \\ &= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)}\right)^{-p_i} \left[w_{kn+2-2k}^{(i)} \left(y_{kn+2-3k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^2} \\ &= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)}\right)^{-p_i} \left(w_{kn+2-2k}^{(i)}\right)^{(-p_i)^2} \left[w_{kn+2-3k}^{(i)} \left(y_{kn+2-4k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^3} \\ &= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)}\right)^{-p_i} \left(w_{kn+2-2k}^{(i)}\right)^{(-p_i)^2} \left(w_{kn+2-3k}^{(i)}\right)^{(-p_i)^3} \left(y_{kn+2-4k}^{(i)}\right)^{(-p_i)^4} \\ &= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)}\right)^{-p_i} \left(w_{kn+2-2k}^{(i)}\right)^{(-p_i)^2} \left(w_{kn+2-3k}^{(i)}\right)^{(-p_i)^3} \cdots \\ &\quad \times \left(w_{kn+2-(t-1)k}^{(i)}\right)^{(-p_i)^{t-1}} \left(y_{kn+2-tk}^{(i)}\right)^{(-p_i)^t}. \end{aligned}$$

Hence

$$\begin{aligned} y_{kn+2}^{(i)} &= w_{kn+2}^{(i)} \left(w_{kn+2-k}^{(i)}\right)^{-p_i} \left(w_{kn+2-2k}^{(i)}\right)^{(-p_i)^2} \left(w_{kn+2-3k}^{(i)}\right)^{(-p_i)^3} \\ &\quad \times \cdots \left(w_{kn+2-tk}^{(i)}\right)^{(-p_i)^t} \cdots \left(w_{k+2}^{(i)}\right)^{(-p_i)^{n-1}} \left(y_2^{(i)}\right)^{(-p_i)^n} \end{aligned}$$

$$
\begin{aligned}
&= w^{(i)}_{k(n-0)+2}\left(w^{(i)}_{k(n-1)+2}\right)^{-p_i}\left(w^{(i)}_{k(n-2)+2}\right)^{(-p_i)^2}\left(w^{(i)}_{k(n-3)+2}\right)^{(-p_i)^3}\cdots \\
&\quad\times\left(w^{(i)}_{k(n-t)+2}\right)^{(-p_i)^t}\cdots\left(w^{(i)}_{k(n-(n-1))+2}\right)^{(-p_i)^{n-1}}\left(y^{(i)}_2\right)^{(-p_i)^n}.
\end{aligned}
$$

So, we get

$$
y^{(i)}_{kn+2}=\left[\prod_{t=0}^{n-1}\left(w^{(i)}_{k(n-t)+2}\right)^{(-p_i)^t}\right]\left(y^{(i)}_2\right)^{(-p_i)^n},\ \ n\in\mathbb{N}_0. \tag{3.4}
$$

By the same argument

$$
\begin{aligned}
y^{(i)}_{kn+k-1} &= w^{(i)}_{kn+k-1}\left(y^{(i)}_{kn+k-1-k}\right)^{-p_i} = w^{(i)}_{kn+k-1}\left[w^{(i)}_{kn+k-1-k}\left(y^{(i)}_{kn+k-1-2k}\right)^{-p_i}\right]^{-p_i} \\
&= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left(y^{(i)}_{kn+k-1-2k}\right)^{(-p_i)^2} \\
&= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left[w^{(i)}_{kn+k-1-2k}\left(y^{(i)}_{kn+k-1-3k}\right)^{-p_i}\right]^{(-p_i)^2} \\
&= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left(w^{(i)}_{kn+k-1-2k}\right)^{(-p_i)^2} \\
&\quad\times\left[w^{(i)}_{kn+k-1-3k}\left(y^{(i)}_{kn+k-1-4k}\right)^{-p_i}\right]^{(-p_i)^3} \\
&= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left(w^{(i)}_{kn+k-1-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{kn+k-1-3k}\right)^{(-p_i)^3} \\
&\quad\times\left(y^{(i)}_{kn+k-1-4k}\right)^{(-p_i)^4} \\
&= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left(w^{(i)}_{kn+k-1-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{kn+k-1-3k}\right)^{(-p_i)^3} \\
&\quad\times\cdots\left(w^{(i)}_{kn+k-1-(t-1)k}\right)^{(-p_i)^{t-1}}\left(y^{(i)}_{kn+k-1-tk}\right)^{(-p_i)^t} \\
&= w^{(i)}_{kn+k-1}\left(w^{(i)}_{kn+k-1-k}\right)^{-p_i}\left(w^{(i)}_{kn+k-1-2k}\right)^{(-p_i)^2}\left(w^{(i)}_{kn+k-1-3k}\right)^{(-p_i)^3} \\
&\quad\times\cdots\left(w^{(i)}_{kn+k-1-tk}\right)^{(-p_i)^t}\cdots\left(w^{(i)}_{2k-1}\right)^{(-p_i)^{n-1}}\left(y^{(i)}_{k-1}\right)^{(-p_i)^n} \\
&= w^{(i)}_{k(n-0)+k-1}\left(w^{(i)}_{k(n-1)+k-1}\right)^{-p_i}\left(w^{(i)}_{k(n-2)+k-1}\right)^{(-p_i)^2}\left(w^{(i)}_{k(n-3)+k-1}\right)^{(-p_i)^3} \\
&\quad\times\cdots\left(w^{(i)}_{k(n-t)+k-1}\right)^{(-p_i)^t}\cdots\left(w^{(i)}_{k(n-(n-1))+k-1}\right)^{(-p_i)^{n-1}}\left(y^{(i)}_{k-1}\right)^{(-p_i)^n}.
\end{aligned}
$$

So, we get

$$y_{kn+k-1}^{(i)} = \left[\prod_{t=0}^{n-1}\left(w_{k(n-t)+k-1}^{(i)}\right)^{(-p_i)^t}\right]\left(y_{k-1}^{(i)}\right)^{(-p_i)^n}, \quad n \in \mathbb{N}_0. \tag{3.5}$$

From (3.2), (3.3), (3.4) and (3.5) we can conclude that for $i = \overline{1,k}$ and $j = \overline{0,k-1}$, we obtain

$$y_{kn+j}^{(i)} = \left[\prod_{t=0}^{n-1}\left(w_{k(n-t)+j}^{(i)}\right)^{(-p_i)^t}\right]\left(y_{j}^{(i)}\right)^{(-p_i)^n}, \quad n \geq 0. \tag{3.6}$$

Now, we will prove by induction that the relation (3.6) is true.

A simple verification shows that (3.6) holds for $n = 0$. Assume that (3.6) holds for n, that is

$$y_{kn+j}^{(i)} = \left[\prod_{t=0}^{n-1}\left(w_{k(n-t)+j}^{(i)}\right)^{(-p_i)^t}\right]\left(y_{j}^{(i)}\right)^{(-p_i)^n}.$$

We will prove that (3.6) holds for $n+1$. We have

$$\begin{aligned}
y_{k(n+1)+j}^{(i)} &= w_{k(n+1)+j}^{(i)}\left(y_{k(n+1)+j-k}^{(i)}\right)^{-p_i} \\
&= w_{k(n+1)+j}^{(i)}\left[w_{k(n+1)+j-k}^{(i)}\left(y_{k(n+1)+j-2k}^{(i)}\right)^{-p_i}\right]^{-p_i} \\
&= w_{k(n+1)+j}^{(i)}\left(w_{k(n+1)+j-k}^{(i)}\right)^{-p_i}\left(y_{k(n+1)+j-2k}^{(i)}\right)^{(-p_i)^2} \\
&= w_{k(n+1)+j}^{(i)}\left(w_{k(n+1)+j-k}^{(i)}\right)^{-p_i}\left[w_{k(n+1)+j-2k}^{(i)}\left(y_{k(n+1)+j-3k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^2} \\
&= w_{k(n+1)+j}^{(i)}\left(w_{k(n+1)+j-k}^{(i)}\right)^{-p_i}\left(w_{k(n+1)+j-2k}^{(i)}\right)^{(-p_i)^2} \\
&\quad\times\left[w_{k(n+1)+j-3k}^{(i)}\left(y_{k(n+1)+j-4k}^{(i)}\right)^{-p_i}\right]^{(-p_i)^3} \\
&= w_{k(n+1)+j}^{(i)}\left(w_{k(n+1)+j-k}^{(i)}\right)^{-p_i}\left(w_{k(n+1)+j-2k}^{(i)}\right)^{(-p_i)^2}\left(w_{k(n+1)+j-3k}^{(i)}\right)^{(-p_i)^3} \\
&\quad\times\left(y_{k(n+1)+j-4k}^{(i)}\right)^{(-p_i)^4}.
\end{aligned}$$

Hence

$$\begin{aligned}
y_{k(n+1)+j}^{(i)} &= w_{k(n+1)+j}^{(i)}\left(w_{k(n+1)+j-k}^{(i)}\right)^{-p_i}\left(w_{k(n+1)+j-2k}^{(i)}\right)^{(-p_i)^2}\left(w_{k(n+1)+j-3k}^{(i)}\right)^{(-p_i)^3} \\
&\quad\times\dots\left(w_{k(n+1)+j-(t-1)k}^{(i)}\right)^{(-p_i)^{t-1}}\left(y_{k(n+1)+j-tk}^{(i)}\right)^{(-p_i)^t} \\
&= w_{k(n+1)+j}^{(i)}\left(w_{k(n+1)+j-k}^{(i)}\right)^{-p_i}\left(w_{k(n+1)+j-2k}^{(i)}\right)^{(-p_i)^2}\left(w_{k(n+1)+j-3k}^{(i)}\right)^{(-p_i)^3}
\end{aligned}$$

$$\times \dots \left(w^{(i)}_{k(n+1)+j-tk}\right)^{(-p_i)^t} \dots \left(w^{(i)}_{k+j}\right)^{(-p_i)^n} \left(y^{(i)}_j\right)^{(-p_i)^{n+1}}$$

$$= w^{(i)}_{k(n+1-0)+j} \left(w^{(i)}_{k(n+1-1)+j}\right)^{-p_i} \left(w^{(i)}_{k(n+1-2)+j}\right)^{(-p_i)^2} \left(w^{(i)}_{k(n+1-3)+j}\right)^{(-p_i)^3}$$

$$\times \dots \left(w^{(i)}_{k(n+1-t)+j}\right)^{(-p_i)^t} \dots \left(w^{(i)}_{k(n+1-n)+j}\right)^{(-p_i)^n} \left(y^{(i)}_j\right)^{(-p_i)^{n+1}}.$$

So,

$$y^{(i)}_{k(n+1)+j} = \left[\prod_{t=0}^{n} \left(w^{(i)}_{k(n+1-t)+j}\right)^{(-p_i)^t}\right] \left(y^{(i)}_j\right)^{(-p_i)^{n+1}}.$$

The results below provide an explicit formula for the solution of the system (1.1).

Theorem 1. *Let $\{y^{(i)}_n\}_{n\geq -k}$ be a well-defined solution of (1.1). Then, for $i = \overline{1,k}$, $j = \overline{0,k-1}$ and $n \in \mathbb{N}_0$, we have*

- *If $S \neq 1$*

$$y^{(i)}_{kn+j} = \left[\prod_{t=0}^{n-1} \left(S^{n-t} y^{(i)}_j \left(y^{(i)}_{j-k}\right)^{p_i} + T_i \left(\frac{S^{n-t}-1}{S-1}\right)\right)^{(-p_i)^t}\right] \left(y^{(i)}_j\right)^{(-p_i)^n}.$$

- *If $S = 1$*

$$y^{(i)}_{kn+j} = \left[\prod_{t=0}^{n-1} \left(y^{(i)}_j \left(y^{(i)}_{j-k}\right)^{p_i} + (n-t)T_i\right)^{(-p_i)^t}\right] \left(y^{(i)}_j\right)^{(-p_i)^n}.$$

3.2. *Asymptotic behavior*

In this section, we will study the asymptotic behavior of the equilibrium point of the system (1.1).

The following lemma gives the equilibrium of the system (1.1).

Lemma 2.
If $\left(\overline{y^{(1)}}, \overline{y^{(1)}}, \overline{y^{(1)}}, \overline{y^{(2)}}, \overline{y^{(2)}}, \overline{y^{(2)}}, \dots, \overline{y^{(k)}}, \overline{y^{(k)}}, \overline{y^{(k)}}\right)$ is an equilibrium point of the system (1.1), then it is given by

$$\left(\left[\frac{T_1}{1-S}\right]^{\frac{1}{p_1+1}}, \left[\frac{T_1}{1-S}\right]^{\frac{1}{p_1+1}}, \left[\frac{T_1}{1-S}\right]^{\frac{1}{p_1+1}}, \left[\frac{T_2}{1-S}\right]^{\frac{1}{p_2+1}}, \left[\frac{T_2}{1-S}\right]^{\frac{1}{p_2+1}},\right.$$

$$\left.\left[\frac{T_2}{1-S}\right]^{\frac{1}{p_2+1}}, \dots, \left[\frac{T_k}{1-S}\right]^{\frac{1}{p_k+1}}, \left[\frac{T_k}{1-S}\right]^{\frac{1}{p_k+1}}, \left[\frac{T_k}{1-S}\right]^{\frac{1}{p_k+1}}\right),$$

with $S = \prod_{l=1}^{k} a_l < 1$.

Proof. Let $\left(\overline{y^{(1)}},\overline{y^{(1)}},\overline{y^{(1)}},\overline{y^{(2)}},\overline{y^{(2)}},\overline{y^{(2)}},\ldots,\overline{y^{(k)}},\overline{y^{(k)}},\overline{y^{(k)}}\right)$ is an equilibrium point of the system (1.1). So, for $i=\overline{1,k}$, we have

$$\begin{aligned}
\left(\overline{y^{(i)}}\right)^{p_i+1} &= a_i\left(\overline{y^{(i+1)}}\right)^{p_{i+1}+1}+b_i = a_i\left[a_{i+1}\left(\overline{y^{(i+2)}}\right)^{p_{i+2}+1}+b_{i+1}\right]+b_i\\
&= a_ia_{i+1}\left(\overline{y^{(i+2)}}\right)^{p_{i+2}+1}+a_ib_{i+1}+b_i\\
&= a_ia_{i+1}\left[a_{i+2}\left(\overline{y^{(i+3)}}\right)^{p_{i+3}+1}+b_{i+2}\right]+a_ib_{i+1}+b_i\\
&= a_ia_{i+1}a_{i+2}\left(\overline{y^{(i+3)}}\right)^{p_{i+3}+1}+a_ia_{i+1}b_{i+2}+a_ib_{i+1}+b_i\\
&= a_ia_{i+1}a_{i+2}a_{i+3}\left(\overline{y^{(i+4)}}\right)^{p_{i+4}+1}+a_ia_{i+1}a_{i+2}b_{i+3}\\
&\quad +a_ia_{i+1}b_{i+2}+a_ib_{i+1}+b_i\\
&= a_ia_{i+1}a_{i+2}a_{i+3}a_{i+4}\left(\overline{y^{(i+5)}}\right)^{p_{i+5}+1}+a_ia_{i+1}a_{i+2}a_{i+3}b_{i+4}\\
&\quad +a_ia_{i+1}a_{i+2}b_{i+3}+a_ia_{i+1}b_{i+2}+a_ib_{i+1}+b_i\\
&\quad \vdots\\
&= a_ia_{i+1}\ldots a_{i+k-1}\left(\overline{y^{(i+k)}}\right)^{p_{i+k}+1}+a_ia_{i+1}\ldots a_{i+k-2}b_{i+k-1}\\
&\quad +a_ia_{i+1}\ldots a_{i+k-3}b_{i+k-2}+\cdots+a_ib_{i+1}+b_i\\
&= \left(\prod_{l=1}^{k}a_l\right)\left(\overline{y^{(i+k)}}\right)^{p_{i+k}+1}+\left[\sum_{r=2}^{k}\left(\prod_{l=i}^{i+r-2}a_l\right)b_{i+r-1}\right]+b_i\\
&= \left(\prod_{l=1}^{k}a_l\right)\left(\overline{y^{(i)}}\right)^{p_i+1}+\left[\sum_{r=2}^{k}\left(\prod_{l=i}^{i+r-2}a_l\right)b_{i+r-1}\right]+b_i.
\end{aligned}$$

So

$$\left(\overline{y^{(i)}}\right)^{p_i+1}\left(1-\prod_{l=1}^{k}a_l\right)=\left[\sum_{r=2}^{k}\left(\prod_{l=i}^{i+r-2}a_l\right)b_{i+r-1}\right]+b_i,$$

consequently

$$\overline{y^{(i)}}=\left[\frac{\left[\sum_{r=2}^{k}\left(\prod_{l=i}^{i+r-2}a_l\right)b_{i+r-1}\right]+b_i}{1-\prod_{l=1}^{k}a_l}\right]^{\dfrac{1}{p_i+1}}.$$

Using notation (2.2), we get

$$\overline{y^{(i)}} = \left[\frac{T_i}{1-S}\right]^{\frac{1}{p_i+1}}, \quad i = \overline{1,k}.$$

Note that the condition $S < 1$ implies that $\overline{y^{(i)}}$ is positive whatever the values of p_i, $i = \overline{1,k}$. □

Theorem 2. *Consider system* (1.1)*. Assume, for* $i = \overline{1,k}$*, that* $S < 1$ *and* $|p_i| < 1$*. Then the equilibrium point of the system* (1.1) *is globally attractive.*

Proof. Suppose, for $i = \overline{1,k}$, that $S < 1$ and $|p_i| < 1$, so we obtain

$$\lim_{n\to+\infty} y^{(i)}_{kn+j} = \lim_{n\to+\infty}\left[\left(\prod_{t=0}^{n-1}\left(S^{n-t}y^{(i)}_j\left(y^{(i)}_{j-k}\right)^{p_i} + T_i\left(\frac{S^{n-t}-1}{S-1}\right)\right)^{(-p_i)^t}\right)\left(y^{(i)}_j\right)^{(-p_i)^n}\right]$$

$$= \prod_{t\geq 0}\left[T_i\left(\frac{-1}{S-1}\right)\right]^{(-p_i)^t} = \prod_{t\geq 0}\left[\frac{T_i}{1-S}\right]^{(-p_i)^t} = \left[\frac{T_i}{1-S}\right]^{\sum_{t\geq 0}(-p_i)^t}.$$

Moreover, we have

$$\sum_{t\geq 0}(-p_i)^t = \lim_{s\to+\infty}\sum_{t=0}^{s}(-p_i)^t = \lim_{s\to+\infty}\frac{(-p_i)^{s+1}-1}{-p_i-1} = \frac{-1}{-p_i-1} = \frac{1}{p_i+1}.$$

So

$$\lim_{n\to+\infty} y^{(i)}_{kn+j} = \left[\frac{T_i}{1-S}\right]^{\frac{1}{p_i+1}} = \overline{y^{(i)}}.$$

From where the equilibrium is globally attractive.

□

4. Numerical examples

Here, we present specific examples to illustrate the behavior of solutions for the multidimensional close-to-cyclic system of nonlinear difference equations under varying initial conditions and parameter values. Through these examples, the theoretical findings are validated, and the dynamics of the system, including stability and equilibrium points, are visually demonstrated. The examples are complemented by graphical representations, offering deeper insights into the system's characteristics and confirming the global attractiveness of equilibrium points under specified conditions.

Example 1. Let $k = 2, a_1 = 2, a_2 = \frac{1}{2}, b_1 = 2, b_2 = 3, p_1 = \frac{1}{2}$ and $p_2 = \frac{1}{3}$, and the initial values $y^{(1)}_{-2} = 4$, $y^{(1)}_{-1} = 4$, $y^{(1)}_0 = 3$, $y^{(2)}_{-2} = 8$, $y^{(2)}_{-1} = 8$, $y^{(2)}_0 = 6$ in system (1.1), then we obtain that $S=1$, and since, for $i=1,2$, we have $|\ p_i\ |<1$. So we obtain the following system

$$y_{n+1}^{(1)}=\frac{2y_n^{(2)}\left(y_{n-2}^{(2)}\right)^{\frac{1}{3}}+2}{\left(y_{n-1}^{(1)}\right)^{\frac{1}{2}}},\qquad y_{n+1}^{(2)}=\frac{\frac{1}{2}y_n^{(1)}\left(y_{n-2}^{(1)}\right)^{\frac{1}{2}}+3}{\left(y_{n-1}^{(2)}\right)^{\frac{1}{3}}},\qquad n\in\mathbb{N}_0. \tag{4.1}$$

The behavior of the solution of system (4.1) is represented in the figure (1).

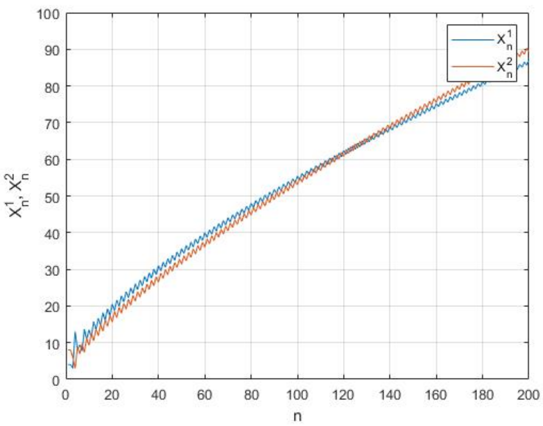

FIGURE 1. The plot of system (4.1) with $S=1$ and $|\ p_i\ |<1$

Example 2. Let $k=2$, $a_1=1, a_2=\frac{2}{3}, b_1=2, b_2=3, p_1=\frac{1}{2}$ and $p_2=\frac{1}{3}$, and the initial values $y_{-2}^{(1)}=4$, $y_{-1}^{(1)}=4$, $y_0^{(1)}=3$, $y_{-2}^{(2)}=8$, $y_{-1}^{(2)}=8$, $y_0^{(2)}=6$ in system (1.1), then we obtain that $S<1$, and since, for $i=1,2$, we have $|\ p_i\ |<1$. So we obtain the following system

$$y_{n+1}^{(1)}=\frac{y_n^{(2)}\left(y_{n-2}^{(2)}\right)^{\frac{1}{3}}+2}{\left(y_{n-1}^{(1)}\right)^{\frac{1}{2}}},\qquad y_{n+1}^{(2)}=\frac{\frac{2}{3}y_n^{(1)}\left(y_{n-2}^{(1)}\right)^{\frac{1}{2}}+3}{\left(y_{n-1}^{(2)}\right)^{\frac{1}{3}}},\qquad n\in\mathbb{N}_0. \tag{4.2}$$

The equilibrium $(\overline{y^{(1)}},\overline{y^{(1)}},\overline{y^{(1)}},\overline{y^{(2)}},\overline{y^{(2)}},\overline{y^{(2)}})=\left(15^{\frac{2}{3}},15^{\frac{2}{3}},15^{\frac{2}{3}},13^{\frac{3}{4}},13^{\frac{3}{4}},13^{\frac{3}{4}}\right)$ is globally attractive (see Figure (2), Theorem (2)).

Example 3. Let $k=6$, $a_i=b_i=\frac{1}{2}$, for $i=1,2,\dots,6$ and $p_1=\frac{1}{2}$, $p_2=\frac{1}{2}$, $p_3=\frac{3}{5}$, $p_4=\frac{9}{10}$, $p_5=\frac{-7}{10}$, $p_6=\frac{4}{5}$, and the initial values $y_{-2}^{(1)}=1$, $y_{-1}^{(1)}=2$, $y_0^{(1)}=3$,

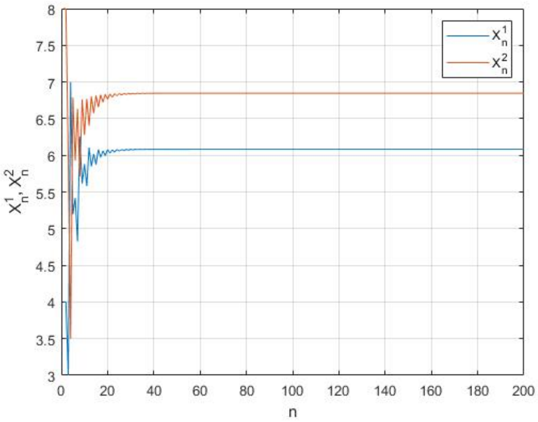

FIGURE 2. The plot of system (4.2) with $S < 1$ and $\mid p_i \mid < 1$

$y_{-2}^{(2)} = 4$, $y_{-1}^{(2)} = 5$, $y_0^{(2)} = 6$, $y_{-2}^{(3)} = 2$, $y_{-1}^{(3)} = 3$, $y_0^{(3)} = 1$, $y_{-2}^{(4)} = 2$, $y_{-1}^{(4)} = 2$, $y_0^{(4)} = 3$, $y_{-2}^{(5)} = 3$, $y_{-1}^{(5)} = 2$, $y_0^{(5)} = 2$, $y_{-2}^{(6)} = 4$, $y_{-1}^{(6)} = 2$, $y_0^{(6)} = 3$ in system (1.1), then we obtain that $S < 1$, and since, for $i = 1, 2, \ldots 6$, we have $\mid p_i \mid < 1$. Then the equilibrium $(\overline{y^{(1)}}, \overline{y^{(1)}}, \overline{y^{(1)}}, \ldots, \overline{y^{(6)}}, \overline{y^{(6)}}, \overline{y^{(6)}}) = (1, 1, 1, \ldots, 1, 1, 1)$ is globally attractive (see Figure (3), Theorem (2)).

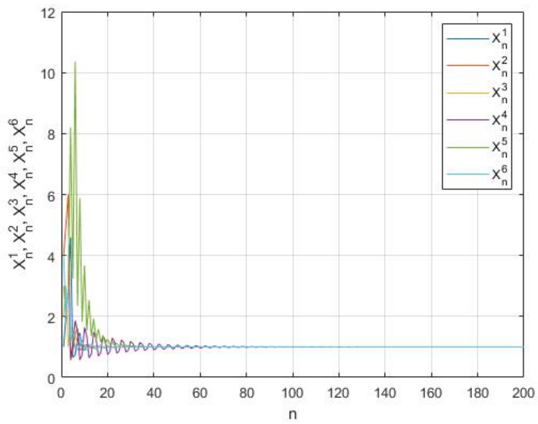

FIGURE 3. The plot of system (1.1) with $S < 1$ and $\mid p_i \mid < 1$

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REFERENCES

[1] Y. Akrour, N. Touafek, and Y. Halim, "On a system of difference equations of second order solved in closed-form." *Miskolc Math. Notes*, vol. 20, no. 2, pp. 701–717, 2019, doi: 10.18514/MMN.2019.2923.

[2] A. Allam, Y. Halim, and A. Khelifa, "Convergence of solutions of a system of recurrence equations," *J. Appl. Math. Comput.*, vol. 69, no. 2, pp. 1659–1677, 2023, doi: 10.1007/s12190-022-01807-x.

[3] E. M. Elsayed, "Solution for systems of difference equations of rational form of order two," *Comp. Appl. Math.*, vol. 33, no. 3, pp. 751–765, 2014, doi: 10.1007/s40314-013-0092-9.

[4] E. M. Elsayed and T. F. Ibrahim, "Periodicity and solutions for some systems of nonlinear rational difference equations," *Hacet. J. Math. Stat.*, vol. 44, no. 6, pp. 1361–1390, 2015, doi: 10.15672/HJMS.2015449653.

[5] M. Gümüş and R. Abo-Zeid, "Global behavior of a rational second order difference equation," *J. Appl. Math. Comput.*, vol. 62, no. 1, pp. 119–133, 2020, doi: 10.1007/s12190-019-01276-9.

[6] M. Gümüş and Ş. I. Eğilmez, "The qualitative analysis of some difference equations using homogeneous functions," *Fundam. J. Math. Appl.*, vol. 6, no. 4, pp. 218–231, 2023, doi: 10.33401/fujma.1336964.

[7] M. Gümüş, "The periodic character in a higher order difference equation with delays," *Math. Methods Appl. Sci.*, vol. 43, no. 3, pp. 1112–1123, 2020, doi: 10.1002/mma.5915.

[8] M. Gümüş and Y. Soykan, "Global character of a six-dimensional nonlinear system of difference equations," *Discrete Dyn. Nat. Soc.*, vol. 2016, p. 7, 2016, id/No 6842521, doi: 10.1155/2016/6842521.

[9] Y. Halim and M. Bayram, "On the solutions of a higher-order difference equation in terms of generalized Fibonacci sequences," *Math. Methods Appl. Sci.*, vol. 39, no. 1, pp. 2974–2982, 2016, doi: 10.1002/mma.3745.

[10] Y. Halim, A. Khelifa, and M. Berkal, "Representation of solutions of a two-dimensional system of difference equations," *Miskolc Math. Notes*, vol. 21, no. 1, pp. 203–2018, 2020, doi: 10.18514/MMN.2020.3204.

[11] Y. Halim and J. F. T. Rabago, "On the solutions of a second-order difference equation in terms of generalized Padovan sequences," *Math. Slovaca*, vol. 68, no. 3, pp. 625–638, 2018, doi: 10.1515/ms-2017-0130.

[12] Y. Halim, A. Khelifa, and N. Mokrani, "Connection of balancing numbers with solution of a system of two higher-order difference equations," *Miskolc Math. Notes*, vol. 24, no. 2, pp. 779–788, 2023, doi: 10.18514/MMN.2023.4315.

[13] M. Kara, "Investigation of the global dynamics of two exponential-form difference equations systems," *Electron. Res. Arch.*, vol. 31, pp. 6697–6724, 2023, doi: 10.3934/era.2023338.

[14] M. Kara and Y. Yazlik, "Solvability of a system of nonlinear difference equations of higher order," *Turk. J. Math.*, vol. 43, no. 3, pp. 1533–1565, 2019, doi: 10.3906/mat-1902-24.

[15] M. Kara and Y. Yazlık, "On the solutions of three-dimensional system of difference equations via recursive relations of order two and applications," *J. Appl. Anal. Comput.*, vol. 12, no. 2, pp. 736–753, 2022, doi: 10.11948/20210305.

[16] D. Karakaya, Y. Yazlik, and M. Kara, "On a solvable system of difference equations of sixth-order," *Miskolc Math. Notes*, vol. 24, no. 3, 2023, doi: 10.18514/MMN.2023.4248.

[17] A. Khelifa, Y. Halim, and M. Berkal, "Solutions of a system of two higher-order difference equations in terms of Lucas sequence," *Univers. J. Math. Appl.*, vol. 2, no. 4, pp. 202–211, 2019, doi: 10.32323/ujma.610399.

[18] A. Khelifa, Y. Halim, A. Bouchair, and M. Berkal, "On a system of three difference equations of higher order solved in terms of Lucas and Fibonacci numbers," *Math. Slovaca*, vol. 70, no. 3, pp. 641–656, 2020, doi: 10.1515/ms-2017-0378.

[19] C. Mylona, G. Papaschinopoulos, and C. J. Schinas, "Neimark–Sacker, flip, and transcritical bifurcation in a close-to-symmetric system of difference equations with exponential terms," *Math. Methods Appl. Sci.*, vol. 44, no. 13, 2021, doi: 10.1002/mma.7400.

[20] C. Mylona, N. Psarros, G. Papaschinopoulos, and C. J. Schinas, "Stability of two 3×3 close-to-cyclic systems of exponential difference equations," *Math. Methods Appl. Sci.*, vol. 41, no. 17, pp. 7936–7948, 2018, doi: 10.1002/mma.5256.

[21] D. T. Tollu, Y. Yazlik, and N. Taskara, "On the solutions of two special types of Riccati difference equation via Fibonacci numbers," *Adv. Difference Equ.*, vol. 174, no. 1, pp. 1–7, 2013, doi: 10.1186/1687-1847-2013-174.

[22] N. Touafek and E. Elsayed, "On the solutions of systems of rational difference equations," *Math. Comput. Modelling*, vol. 55, no. 7-8, pp. 1987–1997, 2012, doi: 10.1016/j.mcm.2011.11.058.

[23] N. Touafek, "On some fractional systems of difference equations," *Iran. J. Math. Sci. Inform.*, vol. 9, no. 2, pp. 73–86, 2014. [Online]. Available: www.ijmsi.ir/browse.php?slc_lang=en&sid=1&a_id=524

[24] Y. Yazlik and M. Kara, "On a solvable system of difference equations of higher-order with period two coefficients," *Commun. Fac. Sci. Univ. Ank., Sér. A1, Math. Stat.*, vol. 68, no. 2, pp. 1675–1693, 2019, doi: 10.31801/cfsuasmas.548262.

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